Cheat Sheet for Integrals

1. **u-Substitution**

   For u-substitution, we *usually* look for a function (which we substitute as u), whose derivative is also present there. For example if the integrand (the function to be integrated) is $\cos^3 x \sin x$, then the derivative of $\cos x$ which is $-\sin x$ is also present (ignore that " $-$ " as it is just the constant -1). So, we will substitute $u = \cos x$, and continue from there.

   - $\displaystyle\int \frac{\cos(\frac{1}{x})}{x^2}\,dx$ - Here notice that the derivative of $\frac{1}{x}$ is $-\frac{1}{x^2}$ which is also present in the function. Hence, the right substitution is $u = \frac{1}{x}$ and NOT $u = \cos(\frac{1}{x})$.
   - $\displaystyle\int \frac{x}{x+9}\,dx$ - Even though this does not look like a problem where substitution could be used, however, sometimes it's a good idea to experiment. Substituting $u = x + 9$ will give $du = dx$ and then the integrand becomes $\dfrac{u-9}{\sqrt{u}}$. This look very similar to the original integrand except we can now split this up by dividing each term in the numerator by $\sqrt{u}$ and then integrating them separately.

2. **Integration by Parts**

   $$\int u\,dv = uv - \int v\,du$$

   We *usually* look for an integrand which is the product of an **algebraic function** like $x, x^2, 3x^3 - 1$ etc. and a **transcendental function** like $e^x, \ln x, \sin x, \sin^{-1} x$ etc to use integration by parts. The goal is to make sure that the choices of u and dv are made correctly, so that du and v can be found out quickly, and $v\,du$ can be integrated without much trouble.

   We make choices for u and dv and then find du by differentiating u and find v by integrating dv. So make sure the function dv can be integrated easily.

   - $\displaystyle\int x \ln x\,dx$ - The right choices are $u = \ln x$ and $dv = x\,dx$. This is because finding $du = \frac{1}{x}\,dx$ and $v = \frac{x^2}{2}$ is easier than the other choice of $u = x$ and $dv = \ln x\,dx$. With the wrong choice we would need to integrate $\ln x$ to get v which is not an easy task.
   - $\displaystyle\int x \sin x\,dx$ - Here the right choices are $u = x$ and $dv = \sin x\,dx$. This will lead to $du = dx$ and $v = -\cos x$. Integrating $v\,du$ is easier in this case. However, the other choice of $u = \sin x$ and $dv = x\,dx$ leads us to $du = \cos x dx$ and $v = \frac{x^2}{2}$. Even though we could find du and v easily, the integral of $v\,du$ is not that easy to find out.

3. **Trigonometric Integral**

   Any integrand which is a power of sine, cosine, tangent, and secant (also cosecant and cotangent) is a trigonometric integral. In addition, products of powers of sine and cosine and products of powers of tangent and secant are also considered trigonometric integrals.

   - Integrals of $\sin^n x, \cos^n x, \tan^n x, \sec^n x$ can be evaluated using the reduction formulas for each. The "base" cases are important. If we start with an even power of sine, cosine, tangent, or secant, then the base case is $\displaystyle\int dx$ which is $x + C$. If we start with an odd power of sine and cosine then the base cases are $\displaystyle\int \sin x\,dx$ and $\displaystyle\int \cos x\,dx$ which are standard integrals. However, if we start with an odd power of tangent, then the base case is $\int \tan x\,dx$ which is not a standard integral. $\int \tan x\,dx = \ln|\sec x| + C$. On the other hand, if we start with an odd power of secant, then the base case is $\int \sec x\,dx$ which is again not a standard integral. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$. Check the lecture notes on how we evaluate these base cases.

- $\int \sin^m x \cos^n x\,dx$ - We divide these into cases.

  (a) Case 1 - **At least one of m or n is odd**. In this case take out one sine or cosine from the function that is raised to the odd power. Then use u-substitution. If the integrand is $\sin^4 x \cos^3 x$, then write it as $\sin^4 x \cos^2 x \cos x$. Then replace $\cos^2 x$ with $1 - \sin^2 x$, and then substitute $u = \sin x$ and continue with u-substitution.

  (b) Case 2 - **Both m and n are even**. In this case choose the lower of the two even m and n. Use the identity $\sin^2 x + \cos^2 = 1$, to write that power of sine or cosine in terms of the other. Then the integrand will be a power of a single trigonometric function and reduction formulas can be used. For example, if the integrand is $\sin^2 x \cos^4 x$, then replace $\sin^2 x$ with $1 - \cos^2 x$ which leads to $(1 - \cos^2 x)\cos^4 x = \cos^4 x - \cos^6 x$. Each of these powers can now be integrated using the reduction formula.

- $\int \tan^m x \sec^n x\,dx$ - We again divide this into cases.

- Case 1 - m **odd**. In this case take out $\sec x \tan x$ and then write the remaining integrand in terms of $\sec x$. Then use u-substitution with $u = \sec x$. For example if the integrand is $\tan^3 x \sec^2 x$, then write it as $\tan^2 x \sec x(\sec x \tan x)$. Now use the identity $1 + \tan^2 x = \sec^2 x$ to re-write the integrand as $(\sec^2 x - 1)\sec x(\sec x \tan x)$, and then substitute $u = \sec x$ to continue. The derivative of $\sec x$ is $\sec x \tan x$, so that part will go away with u-substitution.

- Case 2 - n **even**. In this case take out $\sec^2 x$ nd then write the remaining integrand in terms of $\tan x$. Then use u-substitution with $u = \tan x$. For example if the integrand is $\tan^3 x \sec^3 x$, then write it as $\tan^3 x \sec^2 x(\sec^2 x)$. Now use the identity $1 + \tan^2 x = \sec^2 x$ to re-write the integrand as $\tan^3 x(1 + \tan^2 x)(\sec^2 x)$, and then substitute $u = \tan x$ to continue. The derivative of $\tan x$ is $\sec^2 x$, so that part goes away with u-substitution, and what's left is an algebraic function of u that can be easily integrated.

- Case 3 - m **is even,** n **is odd**. In this case write the even power of tangent in terms of secant using the identity $1 + \tan^2 x = \sec^2 x$. Then use the reduction formula. For example, if the integrand is $\tan^2 \sec^3 x$, then re-write it as $(\sec^2 - 1)\sec^3 x = \sec^5 x - \sec^3 x$. Reduction formulas can now be used to evaluate the integral of $\sec^5 x$ and $\sec^3 x$.

4. **Trigonometric Substitution**

We *usually* use trigonometric substitution when certain types of functions are present in the integrand. These functions are $a^2 - x^2, a^2 + x^2$, and $x^2 - a^2$, where a is a constant.

- Case 1 - $a^2 - x^2$. When a function of this type is present, use the substitution $x = a\sin\theta$, and continue with finding $dx = a\cos\theta\,d\theta$ and simplifying the integrand.
- Case 2 - $a^2 + x^2$. When a function of this type is present, use the substitution $x = a\tan\theta$, and continue with finding $dx = a\sec^2\theta\,d\theta$ and simplifying the integrand.
- Case 3 - $x^2 - a^2$. When a function of this type is present, use the substitution $x = a\sec\theta$, and continue with finding $dx = a\sec\theta\tan\theta\,d\theta$ and simplifying the integrand.

5. **Partial Fractions**

We *usually* use the method of partial fractions when the integrand is a rational function, i.e. a quotient of two polynomials such as $\dfrac{x-1}{x^2+x-6}$. Here are the steps involved in this process.

Let $I(x) = \dfrac{P(x)}{Q(x)}$ be the integrand.

(a) If $\deg P(x) \geq \deg Q(x)$, then use long division to write $\dfrac{P(x)}{Q(x)} = Q_1(x) + \dfrac{R(x)}{Q(x)}$. Now use partial fraction method on $\dfrac{R(x)}{Q(x)}$.

(b) If $\deg P(x) < \deg Q(x)$, then factor $Q(x)$ completely. If no factor in $Q(x)$ is repeated then split $\dfrac{R(x)}{Q(x)}$ into partial fractions with the denominator of each partial fraction being one of the factors. The corresponding numerator is the most general polynomial of one degree less than the denominator. So, for example

$$\frac{x+1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

This is because the factor $x-1$ is linear, so one degree less than that is a constant polynomial which we denote by A. The factor x^2+1 has degree two, hence, one degree lees than that is a linear polynomial. The most general linear polynomial is $Bx+C$. **It is important that we use different letters for the constants.**

(c) If factors of $Q(x)$ are repeated, then every repetition introduces a new partial fraction with each denominator of those partial fractions being a different power of that factor. For example,

$$\frac{x+1}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Since x^2+1 is repeated twice, hence we have two partial fractions coming out of this. Each of their denominators have a different power of that factor. The important thing here to note is that the numerator depends on just the polynomial inside the parentheses. This means the numerator is a polynomial of one degree less than the polynomial in the denominator INSIDE the parentheses. Since the polynomial inside the parentheses is a degree two polynomial, therefore, the numerators are still the most general degree one polynomial. Note that we have used different letters for every constant. This is important.