$n^2 = 1, 4, 9, 16, 25, 36, 49, \ldots$

$n^2 \bmod 4 = 1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots$

1.1 Variables

x

Q Is there a number with the property that tripling it and adding 4 gives the square of the number?

A Variable $= x$

Let the number be x.

$$3x + 4 = x^2$$

$$x^2 - 3x - 4 = 0$$

$$(x-4)(x+1) = 0$$

$$\Rightarrow x = 4, -1$$

Q Given any real number, its square is non-negative.

A For $x \in \mathbb{R}$, $x^2 \geq 0$.

$\in$: element

$\mathbb{R}$: set of real numbers

Mathematical Statements

Universal Statement → Says that a certain property is true for all elements in a set.

Ex:- Every positive real number is greater than zero.

② Conditional Statement → Says that if one thing is true, then some other thing also has to be true.

Ex:- If an integer is divisible by 4, then it is also divisible by 2.

③ Existential Statement → Says that there is at least one thing for which the property is true.

Ex:- There exists at least one even integer in the set of all integers.

Universal Conditional Statement

Both Universal and Conditional

Ex:- For all integers a, if a is divisible by 2, then a is an even integer.

Universal Existential Statement

Both Universal and Existential

Ex:- For every prime number p, there exists an even prime number.

Existential Universal Statement

Both existential and Universal.

Ex:- There exist an integer that is less than or equal to every positive integer.

Keywords

Universal → For all, For every

Conditional → If, then

Existential → There exists, There is a

1.2 The language of Sets

A Set is a collection of objects.

$x \in S \Rightarrow x$ is an element of S.

$x \notin S \Rightarrow x$ is not an element of S.

Ex: $A = \{1, 2, 3\}$, $B = \{1, 1, 2, 3, 3\}$

$B = \{1, 2, 3\}$

$\boxed{A = B}$

Q $A = \{3\}$

3

Is $3 = \{3\}$ ✗

$3 \in \{3\}$ ✓

$3 \in A$ ✓

$A = \{2, \{2\}, 3, a\}$

$2 \in A \quad \{2\} \in A, \quad 3 \in A$

$\{3\} \notin A$

<u>Standard Sets</u>

$\mathbb{R} \rightarrow$ Set of real numbers $\left\{\pi, e, -\frac{422}{711}, \pi^e, \frac{1}{2}, 209, \ldots\right\}$

$\mathbb{Q} \rightarrow$ Set of rational numbers

$\left\{\frac{p}{q} \text{ where } p, q \in \mathbb{Z}, q \neq 0\right\}$

$\pi \notin \mathbb{Q}$

$\pi = 3.1415926 \ldots$

$e = 2.718 \ldots$

Irrational numbers

$\left\{-\frac{1}{2}, \frac{5}{3}, \frac{422}{711}, \frac{1}{2}\right\}$

$-0.5, 1.\overline{6}66\ldots$

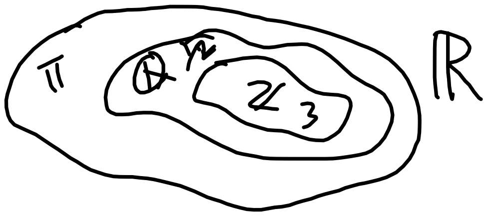
Be real

Be rational

π $\quad$ i

$\mathbb{Z}$ = Set of integers.

$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$



Zahlen

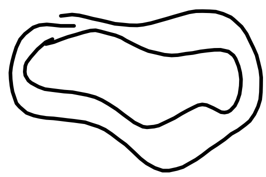
Subsets :- If A and B are sets, then A is a subset of B if every element of A is also an element of B.

Notation :- $A \subseteq B$

$\mathbb{Q} \subseteq \mathbb{Q}$

$<$

$>$



$A = \{1, 2, \{3\}\}$

$1 \in A$ True

$3 \in A$ False

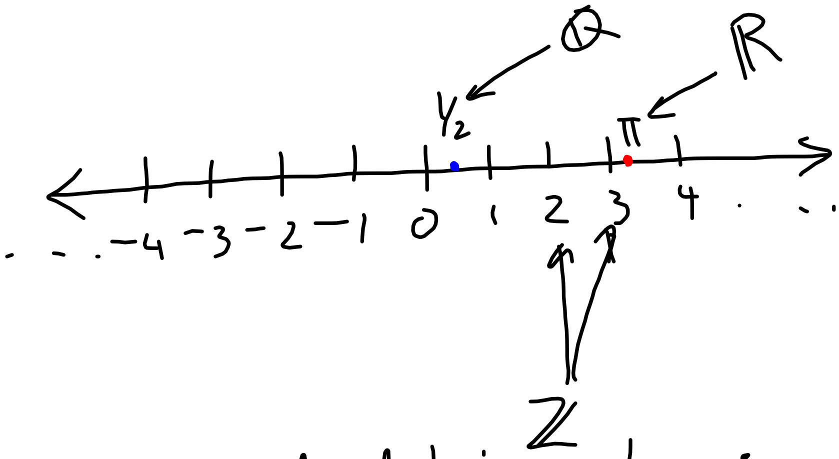
$\{3\} \in A$ True

$\{3\} \subseteq A$ False

$\{2\} \subseteq A$ True

$\{\{3\}\} \subseteq A$ True

Real Number Line



$\mathbb{Z}_+$ Set of all positive integers
$\mathbb{R}_+$ Set of all positive real numbers.

Set Builder Notation

Let $P(x)$ be a property that an element x of a set S may or may not satisfy.

$$\{ x \in S \mid P(x) \}$$

$\mid$: such that

Ex:- $\{ x \in \mathbb{Z} \mid x \bmod 2 = 0 \}$

$$\{ \ldots, -4, -2, 0, 2, 4, 6, \ldots \}$$

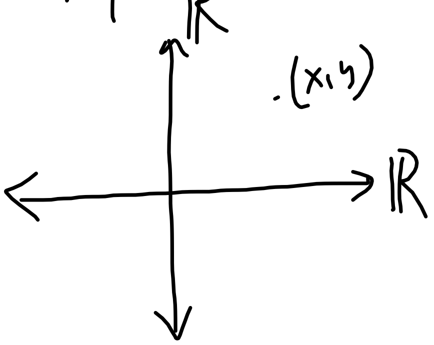
Proper Subset :- If A and B are sets, then A is a proper subset of B if every element of A is an element of B, and there exists an element of B that is not an element of A.

Notation :- $A \subset B$ | $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

Cartesian Product: Let A and B be two sets. Then the cartesian product of A and B denoted by $A \times B$ is set consisting of ordered pairs of elements (a, b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}$$

$$\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R},\ y \in \mathbb{R}\}$$



Ex :- $A = \{1, 2, 3\}$ $B = \{2, 4\}$

$$A \times B = \{(1,2), (1,4), (2,2), (2,4), (3,2), (3,4)\}$$

$$B \times A = \{(2,1), (2,2), (2,3), (4,1), (4,2), (4,3)\}$$

$$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

1.3 The language of Relations and Functions.

Relations :- Let S be a set and $x, y \in S$. Then x is related to y if there is some property that "relates" both x and y. This is denoted by xRy

Ex:- let $A = \{1, 2, 3\}$ $B = \{2, 3, 4, 5\}$

Ex 1 xRy if $x > y$ $x \in A$, $y \in B$

$1 \not R 2$

$3 R 2$

Ex 2 xRy if $x < y$

$1R2$ $2R3$ $3R4$
$1R3$ $2R4$ $3R5$
$1R4$ $2R5$
$1R5$

Relation Set : $R = \{(a, b) \mid aRb, a \in A, b \in B\}$

For Ex 1 $R = \{(3, 2)\}$

For Ex 2 $R = \{(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5)\}$

The relation set of A and B is a subset of the product set $A \times B$, i.e $\boxed{R \subseteq A \times B}$

Definition:- Let A and B be sets. Given an ordered pair $(x,y) \in A \times B$, x is related to y via R, i.e

$$xRy \quad \text{iff} \quad (x,y) \in R$$

if and only if

The set A is called the domain of R, and the set B is called the co-domain of R.

For Ex 1:- $\text{domain}(R) = \{1,2,3\}$

$\text{codomain}(R) = \{2,3,4,5\}$

$\text{Range}(R) = \{2\}$

$$\text{Range}(R) = \{y \in B \mid xRy \text{ for some } x \in A\}$$

Alternative representation of a relation

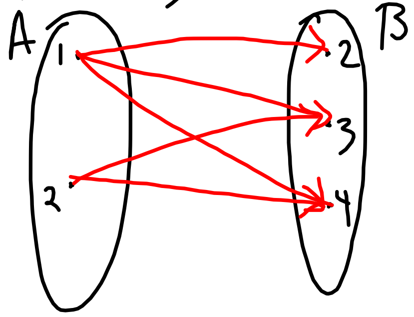
Arrow diagram

$A = \{1, 2\}$ $B = \{2, 3, 4\}$

$x R y$ if $x < y$ $x \in A$, $y \in B$

$R = \{(1,2), (1,3), (1,4), (2,3), (2,4)\}$

Arrow diagram



Function → A special type of relation.

A function F from a set A to a set B is a relation with domain A and co-domain B that satisfies:-

1. For every element $x \in A$, there is an element $y \in B$ such that $(x,y) \in F$, i.e. xFy.
2. For all $x \in A$ and $y, z \in B$ if $(x,y) \in F$ and $(x,z) \in F$ then $y = z$.

All functions are relations, but all relations are not functions.

Ex:- $A = \{1, 2, 3\}$, $B = \{1, 9\}$

$(x,y) \in A \times B$ xRy iff $x^2 = y$

$R = \{(1,1), (3,9)\}$

Is R a function?

No, since 2 is not related to any element of B.

Domain(R) = $\{1, 2, 3\}$ | Codomain(R) = $\{1, 9\}$

Range(R) = $\{1, 9\}$

11 Define a relation P from $\mathbb{R}^+$ to $\mathbb{R}$ as follows.

For all $x, y \in \mathbb{R}$ with $x > 0$

$$(x, y) \in P \text{ iff } x = y^2$$

Is P a function? Explain.

A P is not a function. Suppose $x = 4$ and $y = \pm 2$

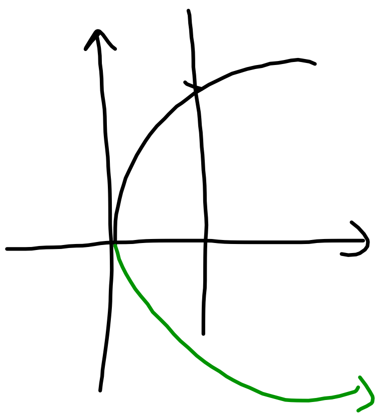
then $x = y^2$

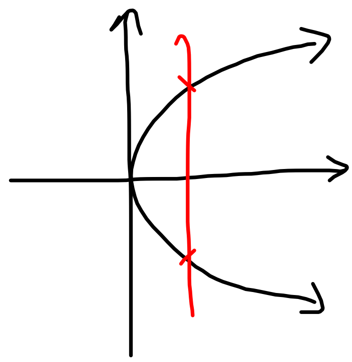
Hence $(4, 2)$ and $(4, -2)$ are both elements of P.

Therefore, P is not a function as 4 maps to 2 and -2

$y = \sqrt{x}$

$y = -\sqrt{x}$





2.1 Logical Form and Logical Equivalence

Statements will be denoted by p, q, r etc.

Ex:- If x is a real number, $\underbrace{\text{such that } x^2 < 9}_{p}$,
$\underbrace{\text{then } -3 < x < 3}_{q}$

If p then q.

Contrapositive

$x^2 < 9$

$x^2 - 9 < 0$

$(x-3)(x+3) < 0$

Equivalent statement

If not q then not p.

If $x \geq 3$ or $x \leq -3$, then $x^2 \geq 9$.

Definition :- A statement (or proposition) is a sentence that is true or false but not both.

Compound statement

Negation of p $(\sim p)$ not p

Conjunction of p and q $(p \wedge q)$ p and q

Disjunction of p and q $(p \vee q)$ p or q

Order of operation

Parantheses ( )
Negation $\sim$
Conjunction, Disjunction $\wedge, \vee$

But p but $q \Leftrightarrow p$ and q $p \wedge q$

Neither p nor q $\Rightarrow$ Not p and Not q $\sim p \wedge \sim q$

Ex:- $\underbrace{\text{It is not hot}}_{p}$ but $\underbrace{\text{it is sunny}}_{q}$.

$p \text{ but } q \equiv p \wedge q$ It is not hot and it is sunny.

Ex:- It is neither $\underbrace{\text{hot}}_{p}$ nor $\underbrace{\text{sunny}}_{q}$

It is not hot and it is not sunny.

Truth Tables

$\sim p$

p	$\sim p$
T	F
F	T

$0 = F$

$1 = T$

$p \wedge q$ True only when both p and q are true

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F