$a, b, c \in A$

$(a,b) \in R, \ (b,c) \in R \implies (a,c) \in R$

If $(b,a) \in R^{-1}, (c,b) \in R^{-1} \overset{?}{\implies} (c,a) \in R^{-1}$

True

$(a,b), (b,c), (a,c) \in R$ as $(a,c) \in R$

$(1,2) \in R \ (2,3) \in R, (1,3) \in R$

$R^{-1} = \{ \dots, (2,1), (3,2), (3,1), \dots \}$

8.3 Equivalence Relation

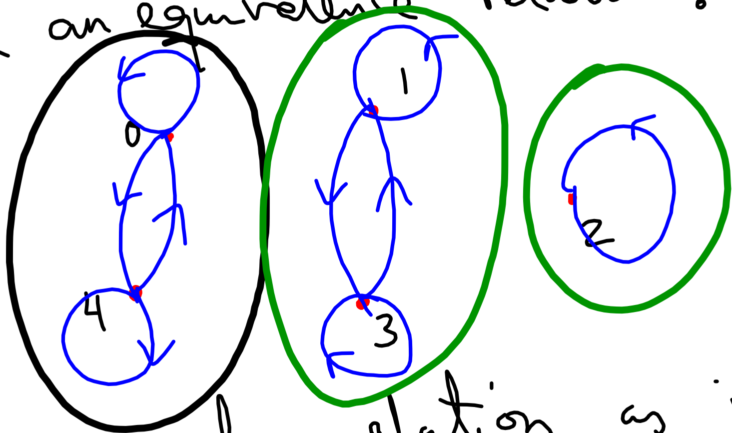
Definition:– Let A be a set and R a relation on A. R is an equivalence relation iff R is reflexive, symmetric, and transitive.

Q $A = \{0,1,2,3,4\}$

$R = \{(0,0),(0,4),(1,1),(1,3),(2,2),(3,1),(3,3),(4,0),(4,4)\}$

Is R an equivalence relation?

Ans



R is an equivalence relation as it is reflexive, symmetric and transitive.

$0R4$ $\quad 1R3$ $\quad 2R2$

$4R0$ $\quad 3R1$

$\{0,4\}$ $\quad \{1,3\}$ $\quad \{2\}$

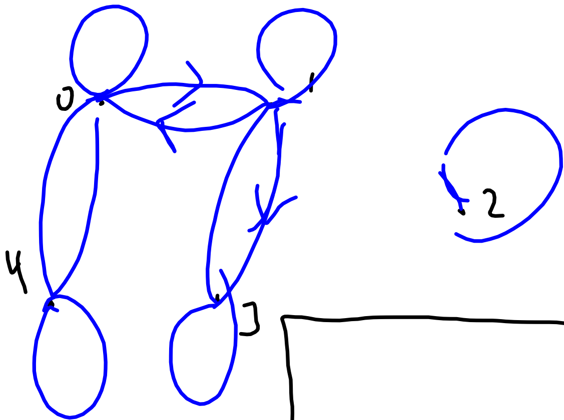
$0R4$ $\quad 1R3$

$0R1$

$1R2$

$\{$

$2R2$



<u>Definition</u>:- Suppose A is a set and R is an equivalence relation on A. For each element $a \in A$, the <u>equivalence class</u> of a denoted by $[a]$ is defined as the set of all elements $x \in A$ that are related to a.

i.e.

$$[a] = \{x \in A \mid xRa\}$$

In our previous example,

$[0] = \{0, 4\}$ $[3] = \{1, 3\}$

$[4] = \{0, 4\}$ $[2] = \{2\}$

$[1] = \{1, 3\}$

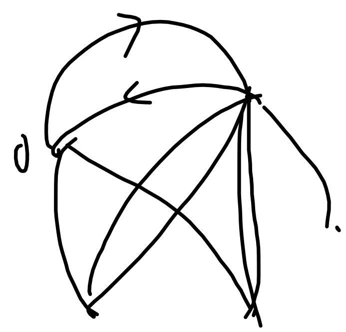
$[0] \cup [1] \cup [2] = A$

$[0] = A$

$[0] \cap [1] = \phi$

$[1] \cap [2] = \phi$

$[2] \cap [0] = \phi$



Theorem :- If A is a set and R is an equivalence relation on A, then the distinct equivalence classes of R form a partition of the set A, i.e the union of the distinct equivalence classes is all of A and the intersection of any two distinct equivalence classes is the empty set.

$$R = \{(0,0), (1,2), (0,1), (1,1)\} \quad \{0,1,2\}$$

$$[0] = \{0,1\}$$

$$[1] = \{1,2\}$$

$$[2] = \phi$$

$\mathbb{Q}$ <u>Rational numbers</u>.

$$\frac{2}{6} = \frac{1}{3} \qquad \frac{4}{8} = \frac{1}{2}$$

$$\frac{a}{b} = \frac{c}{d} \qquad a, b \in \mathbb{Z} \quad b \neq 0$$
$$c, d \in \mathbb{Z}, \quad d \neq 0$$

$$ad = bc$$

Let $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$. $\frac{a}{b} R \frac{c}{d} \iff ad = bc$

<u>Reflexive</u> :- Let $\frac{a}{b} \in \mathbb{Q}$

$$ab = ab$$
$$\Rightarrow \frac{a}{b} R \frac{a}{b}$$

<u>Symmetric</u> :- Let $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$

and $\frac{a}{b} R \frac{c}{d} \Rightarrow ad = bc$

$$\Rightarrow bc = ad$$

Is $\frac{c}{d} R \frac{a}{b}$

$$\Rightarrow \frac{c}{d} R \frac{a}{b}$$

Transitive :- let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$

and $\frac{a}{b} R \frac{c}{d}$, $\frac{c}{d} R \frac{e}{f}$

$\Rightarrow ad = bc$ $\Rightarrow cf = de$

$\Rightarrow c = \frac{de}{f}$

Is $\frac{a}{b} R \frac{e}{f}$?

$\Rightarrow ad = b\left(\frac{de}{f}\right)$ $af = be$?

$\Rightarrow a = \frac{be}{f}$ since $d \neq 0$

$\Rightarrow af = be$

$\Rightarrow \frac{a}{b} R \frac{e}{f}$

$\therefore$ R is transitive

$$\left[\frac{1}{3}\right] = \left\{ \ldots\ldots, \frac{-2}{-6}, \frac{-1}{-3}, \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots \right\}$$

29 $(w,x), (y,z) \in \mathbb{R}\times\mathbb{R}$

$$(w,x) R (y,z) \iff w = y$$

A **Reflexive**:- Let $(w,x) \in \mathbb{R}\times\mathbb{R}$

$(w,x) R (w,x)$ since $w = w$.

Symmetry:- let $(w,x), (y,z) \in \mathbb{R}\times\mathbb{R}$

$$(w,x) R (y,z) \implies w = y$$
$$\implies y = w$$
$$\implies (y,z) R (w,x)$$

Transitive:- let $(w,x), (y,z), (a,b) \in \mathbb{R}\times\mathbb{R}$

$(w,x) R (y,z)$ and $(y,z) R (a,b)$

$\implies w = y$ $\qquad\qquad$ $\implies y = a$

$$\Downarrow$$

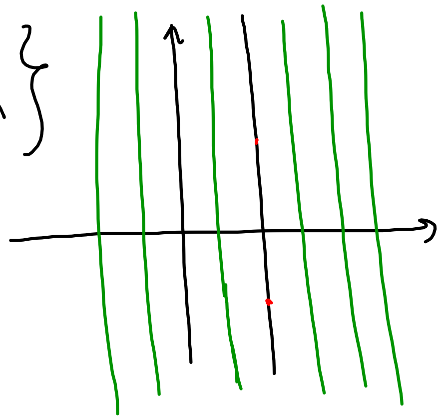
$$w = a$$

$\implies (w,x) R (a,b)$ $\qquad$ Hence transitive.

$\therefore R$ is an equivalence relation.

Equivalence classes.

$$[x] = \{(x,a) \mid a \in \mathbb{R}\}$$



44 $b_k = \frac{2}{b_{k-1}}$ integers $k \geq 2$ $b_1 = 1$.

$$b_2 = \frac{2}{b_1} = \frac{2}{1} = 2$$

$$b_3 = \frac{2}{b_2} = \frac{2}{2} = 1$$

$$b_4 = \frac{2}{b_3} = \frac{2^2}{2} = 2$$

$$b_5 = \frac{2}{b_4} = \frac{2}{\frac{2^2}{2}} = \frac{2^2}{2^2} = 1$$

$$b_6 = \frac{2}{b_5} = \frac{2}{\frac{2^2}{2^2}} = \frac{2^3}{2^2} = 2$$

$\vdots$

$$b_n = \begin{cases} 1 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$$

Basis step $n = 1$, $b_1 = 1$ ✓ ← Initial condition

$b_1 = 1$ ✓

Inductive step Suppose the result is true for $n = k$.

$$\Rightarrow b_k = \begin{cases} 1 & k \text{ odd} \\ 2 & k \text{ even} \end{cases}$$

$$b_{k+1} = \frac{2}{b_k} = \begin{cases} 2 & k \text{ odd or } (k+1) \text{ even} \\ 1 & k \text{ even or } (k+1) \text{ odd} \end{cases}$$

let $t = k+1$

$$b_t = \begin{cases} 1 & \text{when } t \text{ odd} \\ 2 & \text{when } t \text{ is even} \end{cases}$$

Hence the result is true for $n = k+1$.
and thus the result is true for all n
using induction.

$m D n \Longleftrightarrow m \mid n$

$m \mid m \Longrightarrow m D m$

$a_n = \left(\frac{3}{2}\right) + (-1)^n \left(\frac{1}{2}\right)$

$a_n = \left(\frac{a+b}{2}\right) + (-1)^n \left(\frac{a-b}{2}\right)$

$a, b, a, b, \ldots$

$3, 7, 3, 7, 3, 7, \ldots$

$a_n = 5 + (-1)^{n+1}(-2)$

$a_n = (n \bmod 2) + 1$

$1, 2, 1, 2, \ldots$

$4, 5, 4, 5, \ldots$

$A \subseteq B$

Let $x \in A$ WTS $x \in B$

Show that $A \cup B \subseteq B^c \cap A^c$

Let $x \in A \cup B$

$A \times (B \cup C)$

Let $(x, y) \in A \times (B \cup C)$

$\Rightarrow x \in A$ and $y \in B \cup C$

ϕ "Contradiction"

If $A \subseteq B$ show $A \cap B^c = \phi$

On the contrary, suppose $A \cap B^c \neq \phi$

Let $x \in A \cap B^c$

$\Rightarrow x \in A$ and $x \in B^c$

$\Rightarrow x \in A$ and $x \notin B$

$\Rightarrow$ Contradiction as $x \in A$ so $x \in B$

47 $t_k = k - t_{k-1} \qquad k \geq 1 \qquad t_0 = 0$

$t_1 = 1 - t_0 = 1 - 0 = 1$

$t_2 = 2 - t_1 = 2 - 1 = 1 = 2 - (1 - 0) = 2 - 1 + 0$

$t_3 = 3 - t_2 = 3 - 1 = 2 = 3 - (2 - (1 - 0))$

$= 3 - 2 + 1 - 0$

$t_4 = 4 - t_3 = 4 - 2 = 2 = 4 - (3 - (2 - (1 - 0)))$

$= 4 - 3 + 2 - 1 + 0$

$t_5 = 5 - t_4 = 3$

$t_6 = 6 - t_5 = 3$

$t_7 = 7 - t_6 = 4$

$t_8 = 8 - t_7 = 4$

$t_9 = 9 - t_8 = 5$

$$t_n = \begin{cases} \frac{n}{2} & n \text{ even} \\ \frac{n+1}{2} & n \text{ odd} \end{cases}$$

$$t_n = \left\lceil \frac{n}{2} \right\rceil$$

① <u>Basis Step</u>

$n = 0$

$t_0 = 0$ ✓

$t_0 = \frac{0}{2} = 0$ ✓

② <u>Inductive Step</u>

Assume the result is true for $n = k$.

i.e.

$$t_k = \begin{cases} \frac{k}{2} & k \text{ even} \\ \frac{k+1}{2} & k \text{ odd} \end{cases}$$

$$t_{k+1} = (k+1) - t_k$$

<u>Case 1</u> $(k+1)$ even $\Rightarrow k$ odd $\Rightarrow t_k = \frac{k+1}{2}$

$$\therefore t_{k+1} = (k+1) - \frac{k+1}{2} = \frac{k+1}{2}$$

$$t_{(k+1)} = \frac{(k+1)}{2}$$

<u>Case 2</u> $(k+1)$ odd $\Rightarrow k$ even $\Rightarrow t_k = \frac{k}{2}$

$$\therefore t_{k+1} = (k+1) - \frac{k}{2} = 1 + \frac{k}{2}$$

$$= \frac{k+2}{2}$$

$$t_{(k+1)} = \frac{(k+1)+1}{2}$$

As t_{k+1} is of the same form of our formula for the n^{th} term, hence the formula is true for all $n \geq 0$ by mathematical induction.

15 $y_k = y_{k-1} + k^2 \qquad k \geq 2 \qquad y_1 = 1$

$y_2 = y_1 + 2^2 = 1 + 2^2$

$y_3 = y_2 + 3^2 = 1 + 2^2 + 3^2$

$y_4 = y_3 + 4^2 = 1 + 2^2 + 3^2 + 4^2$

$\vdots$

$y_n = 1 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Basis Step $\boxed{n=1}$ $y_1 = 1$ ✓

$y_1 = \dfrac{1(2)(3)}{6} = 1$ ✓

Inductive Step Assume the formula is true for $n = k$, i.e.,

$$y_k = \frac{k(k+1)(2k+1)}{6}$$

$y_{k+1} \stackrel{?}{=} \dfrac{(k+1)(k+2)(2k+3)}{6}$

Now $y_{k+1} = y_k + (k+1)^2$

$= \dfrac{k(k+1)(2k+1)}{6} + (k+1)^2$

$= (k+1)\left[\dfrac{k(2k+1)}{6} + (k+1)\right]$

$= (k+1)\left[\dfrac{2k^2 + k + 6k + 6}{6}\right]$

$= (k+1)\left[\dfrac{2k^2 + 7k + 6}{6}\right]$

$(k+1)\left[\dfrac{(k+2)(2k+3)}{6}\right]$

$2k^2 + 7k + 6$
$2k^2 + 4k + 3k + 6$
$2k(k+2) + 3(k+2)$
$(k+2)(2k+3)$

$x^2 + 5x + 6$
$x^2 + 2x + 3x + 6$
$x(x+2) + 3(x+2)$
$(x+2)(x+3)$

$y_{k+1} \quad \dfrac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

Since y_{k+1} is of the same form, hence the result is true by induction.