$p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

XOR $\oplus$ (Exclusive OR)

p or q but not both

$$p \oplus q \equiv (p \vee q) \wedge \sim(p \wedge q)$$

p	q	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$p \oplus q$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Q When are two statements logically equivalent?

A When their truth tables match.

$$P \equiv Q$$

Ex:- $p \wedge q \equiv q \wedge p$

Ex:- $\sim(p \wedge q) \not\equiv \sim p \wedge \sim q$

De Morgan's Laws

1. $\sim(p \wedge q) \equiv \sim p \vee \sim q$
2. $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Let R : $\underbrace{\text{Jim is tall}}_{p}$ $\underset{\wedge}{\text{and}}$ $\underbrace{\text{Jim is thin}}_{q}$.

$\sim R$: Jim is not tall or Jim is not thin.

Tautology

A statement that is always true. Denoted by t.

Ex:- $p \vee \sim p$

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

Tautology.

Contradiction

A statement that is always false. Denoted by c.

Ex:- $p \wedge \sim p$

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Contradiction

$p \wedge t \equiv p$

$p \wedge c \equiv c$

30 R: The dollar is at an all-time high and the stock market is at a record low.

$\sim R$: The dollar is not at an all-time high or the stock market is not at a record low.

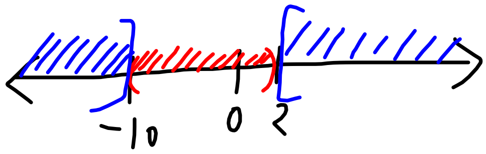
---

33 R: $-10 < x < 2$

$-10 < x$ and $x < 2$

$\sim R$: $-10 \geq x$ or $x \geq 2$

$x \leq -10$ or $x \geq 2$



2.2 Conditional Statements

Ex:- If $\underbrace{x \text{ is divisible by } 6}_{p}$, then $\underbrace{x \text{ is divisible by } 3}_{q}$.

$p \to q$

Q If you eat dinner, then you will get dessert.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Order of operation

$(\quad)$ Parantheses

$\sim$ Not

$\vee , \wedge$ Or , And

$\rightarrow$ Conditional

Q Show that $p \vee q \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$

p	q	r	$p \vee q$	$p \rightarrow r$	$q \rightarrow r$	$p \vee q \rightarrow r$	$(p \rightarrow r) \wedge (q \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

The columns match. Hence $p \vee q \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$

$$\boxed{p \to q \equiv \sim p \vee q}$$

Check their truth tables.

r: Either you get to work on time or you are fired.

$\sim p$: You get to work on time

q: You are fired $\qquad r \equiv \sim p \vee q$

p: You do not get to work on time

$p \to q$: If you do not get to work on time then you are fired.

Negation of a conditional statement

$\sim(r \vee s) \equiv \sim r \wedge \sim s$

$\sim(p \to q)$

Since $p \to q \equiv \sim p \vee q$

$$\begin{aligned} \sim(p \to q) &\equiv \sim(\sim p \vee q) \\ &\equiv \sim(\sim p) \wedge \sim q \\ &\equiv p \wedge \sim q \end{aligned}$$

Ex:- r : If $\underbrace{\text{I wake up late}}_{p}$, then $\underbrace{\text{I cannot get to class on time}}_{q}$.

$\sim r$: $\underbrace{\text{I wake up late}}_{p}$ and $\underbrace{\text{I get to class on time}}_{\sim q}$.

Contrapositive, Inverse, and converse of a conditional statement

Conditional Statement : $p \to q$

Contrapositive : $\sim q \to \sim p$

$\boxed{p \to q \equiv \sim q \to \sim p}$ $\boxed{\text{Check}}$

Ex:- If $\underbrace{\text{today is Friday}}_{p}$, then $\underbrace{\text{tomorrow is a holiday}}_{q}$.

Contrapositive: If $\underbrace{\text{tomorrow is not a holiday}}_{\sim q}$, then $\underbrace{\text{today is not a Friday}}_{\sim p}$.

Conditional Statement : $p \to q$

Converse : $q \to p$

Inverse : $\sim p \to \sim q$

Converse $\equiv$ Inverse

$q \to p \equiv \sim p \to \sim q$

however Converse / Inverse $\not\equiv$ Original conditional Statement.

Converse: If tomorrow is a holiday, then today is a Friday.

Inverse: If today is not a Friday, then tomorrow is not a holiday.

Only if

p only if q

p happens only if q happens.

If not q then not p.

Ex:- John will pass the class only if he gets an A in the final exam.

Conditional statement: If John does not get an A is the final exam then he will not pass the class.

Biconditional Statement

p if and only if q $\qquad p \longleftrightarrow q$

True when both p and q are true or both are false.

p	q	$p \longleftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

order of operations

$(\quad)$

$\sim$

$\wedge, \vee$

$\longleftrightarrow \quad \longrightarrow$

$$\boxed{p \longleftrightarrow q \equiv (p \to q) \wedge (q \to p)}$$

Check

Q A student gets an A iff the student's final score is above 90%.

p is the blue-underlined part, q is the red-underlined part.

Write this sentence using two if-then statements.

If a student gets an A, then the student's final score is above 90% and If the final score is above 90%, then the student gets an A.

Necessary and sufficient conditions.

Sufficient condition:- "If p then q"

p is sufficient for q.

Necessary condition:- "If not p then not q"

If p does not occur, then q does not occur.

So p is a necessary condition for q.

$p \leftrightarrow q$ means p is necessary and sufficient for q.

$$p \leftrightarrow q \equiv \underbrace{(p \to q)}_{\text{Sufficient}} \wedge \underbrace{(q \to p)}_{\text{Necessary condition}}$$

Ex:- The sky being cloudy is a necessary condition for it to rain.

Q Write this as a conditional statement.

A If the sky is not cloudy, then it will not rain.

Ex:- John's age over 18 is a sufficient condition for him to vote.

Q Write this as a conditional statement.

A If John's age is over 18, then he can vote.

3 The logic of quantified statements

3.1 Predicates and Quantified statements I

Definition:- A predicate is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variable. The domain of a predicate variable is the set of all values that may be substituted in place of the variable.

Ex:- $P(x)$ is the predicate "$x^3 > x$" with domain $\mathbb{R}$.

$$\left.\begin{array}{lll} P(1): & 1^3 > 1 & \text{False} \\ P(\frac{1}{2}): & (\frac{1}{2})^3 > \frac{1}{2} & \text{False} \\ P(2): & 2^3 > 2 & \text{True} \end{array}\right\}$$

Truth Set of a predicate

Definition:- If $P(x)$ is a predicate and x has domain D, then the truth set of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x.

The truth set is written as

$$\{x \in D \mid P(x)\}$$

Ex: $P(x): \; x \leq 6$

$D = \mathbb{Z}_+$ Truth Set $= \{1, 2, 3, 4, 5, 6\}$

$D = \mathbb{R}$ Truth Set $= (-\infty, 6]$

Universal Quatifier

$\forall$ "for all" / "for every"

Definition :- Let $P(x)$ be a predicate and D its domain. A universal statement is a statement of the form "$\forall x \in D, P(x)$". It is defined to be true iff $P(x)$ is true for all x in D. It is defined to be false, iff $P(x)$ is false for at least one x in D. A value of x for which $P(x)$ is false is called a contraexample to the universal statement.

Ex:- $P(x): \forall x \in D, \ x^3 > x \quad D = \{2, 3, 4\}$

$P(2): \quad 2^3 > 2$ True

$P(3): \quad 3^3 > 3$ True

$P(4): \quad 4^3 > 4$ True

Ex:- $Q(x): \forall x \in \mathbb{R}, \ x^3 > x$. Is this true?

$Q(\frac{1}{2}): \quad \left(\frac{1}{2}\right)^3 \not> \frac{1}{2}$ False

Hence, $Q(x)$ is false

Existential Quantifier

$\exists$ "there exists / there is a"

Ex:- There is a person who likes ice-cream.

$\exists$ a person p, such that p likes ice-cream

<u>Definition</u>:- Let $Q(x)$ be a predicate and D the domain of x. An existential statement is a statement of the form "$\exists\, x \in D$ such that $Q(x)$". It is defined to be true iff $Q(x)$ is true for at least one x in D. It is false iff $Q(x)$ is false for all x in D.

<u>Ex</u>:- $\exists\, n \in \mathbb{Z}$ such that $n < 5$.

True for $n = 3$.

<u>Ex</u>:- $\exists\, n \in \mathbb{Z}$ such that $n^2 < 0$

False since $\forall n,\ n \in \mathbb{Z}$
$n^2 \geq 0$

Formal and Informal Languages

Ex:- $\forall x \in \mathbb{R},\ x^2 \geq 0$ ← Formal language.

Every real number has a non-negative square.

↑

Informal language.

Ex:- All rectangles have four sides

↑

Informal statement

$\forall r \in \{\text{rectangles}\}$, r has 4 sides.

↑

Formal statement.

Universal Conditional Statement

$\forall x$, if $P(x)$ then $Q(x)$

$\forall x$, $P(x) \rightarrow Q(x)$

Ex:- All oranges are fruits. (Informal)

$\forall x \in \{\text{Foods}\}$, if x is an orange, then x is a fruit.

(Formal)

$\forall x$, if x is an orange, then x is a fruit.

$$\Rightarrow \qquad \underbrace{\forall x,\ P(x) \rightarrow Q(x)}_{P(x) \Rightarrow Q(x)}$$

$$\Leftrightarrow \qquad \underbrace{\forall x,\ P(x) \leftrightarrow Q(x)}_{P(x) \Leftrightarrow Q(x)}$$

Ex:- $n \in \mathbb{Z}_+$: $Q(n)$: "n is a factor of 8"

$n \in \mathbb{Z}_+$: $R(n)$: "n is a factor of 4"

$n \in \mathbb{Z}_+$: $S(n)$: "n is a factor of 2".

Truth Set $Q(n)$: $\{1, 2, 4, 8\}$

Truth set $R(n)$: $\{1, 2, 4\}$

Truth Set $S(n)$: $\{1, 2\}$

$S(n) \Rightarrow R(n)$

$S(n) \Rightarrow Q(n)$

$R(n) \Rightarrow Q(n)$

$S(n) \Rightarrow R(n) \Rightarrow Q(n)$

3.2 Predicates and Quantified Statements II

Negation of a Universal Statement

Universal Statement: $p: \forall x \in D, \ Q(x)$

Negation of p : $\sim p: \exists x \in D, \ \sim Q(x)$

The negation of a universal statement is logically equivalent to an existential statement.

Ex:- $p: \forall x \in \{\text{Even integers}\}$, x is divisible by 2.

$\sim p: \exists x \in \{\text{Even integers}\}$, x is not divisible by 2.