Negation of an Existential Statement

Existential Statement

$p: \exists x \in D, \quad Q(x)$

$\sim p: \forall x \in D, \sim Q(x)$

The negation of an existential statement is equivalent to a universal statement.

Ex:- p: $\exists$ a movie M, that is more than 3 hours long.

$\sim p$: $\forall$ movies M, M is not more than 3 hours long.

Negation of a Universal Conditional Statement

Universal Conditional Statement

$$\forall x \in D,\ P(x) \rightarrow Q(x)$$

Negation

$$\sim\left(\forall x \in D,\ P(x) \rightarrow Q(x)\right) \equiv \exists x \in D,\ \sim\left(P(x) \rightarrow Q(x)\right)$$
$$\equiv \boxed{\exists x \in D,\ P(x) \wedge \sim Q(x)}$$

Ex:- $\forall$ computer program S, if S is over 100,000 lines, then it contains a bug.

Negation: $\exists$ a computer program S, such that S is over 100,000 lines and does not contain a bug.

Variants of Universal Conditional Statement

0. $\forall x \in D, \; P(x) \to Q(x)$
1. Contrapositive: $\forall x \in D, \; \sim Q(x) \to \sim P(x)$
2. Converse: $\forall x \in D, \; Q(x) \to P(x)$
3. Inverse: $\forall x \in D, \; \sim P(x) \to \sim Q(x)$

$0 \equiv 1$ and $2 \equiv 3$

Necessary, Sufficient, Only if

Sufficient: $\forall x$, "$r(x)$ is sufficient condition for $s(x)$"

$$\boxed{\forall x \in D, \; r(x) \to s(x)}$$

Necessary: $\forall x$, "$r(x)$ is a necessary condition for $s(x)$".

$$\boxed{\forall x \in D, \; \sim r(x) \to \sim s(x)} \equiv \boxed{\forall x \in D, \; s(x) \to r(x)}$$

only if: $\forall x$, "$r(x)$ only if $s(x)$".

$$\boxed{\forall x \in D, \; \sim s(x) \to \sim r(x)}$$

$$\equiv$$

$$\boxed{\forall x \in D, \; r(x) \to s(x)}$$

24 a (1) All children in Tom's family are female

(2) All the females in Tom's family are children.

(1.1) If x is a child in Tom's family, then x is a female

(2.1) If x is a female in Tom's family, then x is a child.

(1.1) and (1.2) are converses of each other.

Write the negation

$P: \forall n \in \mathbb{Z}$, if $\underbrace{n \text{ is prime}}_{p(x)}$, then $\underbrace{n \text{ is odd}}_{q(x)}$ or $\underbrace{n=2}$.

$\sim P: \exists n \in \mathbb{Z}$, n is prime and $\sim$(n is odd or $n=2$)

$\exists n \in \mathbb{Z}$, such that n is prime and n is not odd and $n \neq 2$.

Elementary Number Theory and Methods of Proof

$\mathbb{Z}$ $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

General expression for odd | even integers.

n is an odd integer $\Rightarrow$ $n = 2k+1$ for some $k \in \mathbb{Z}$

n is an even integer $\Rightarrow$ $n = 2k$ for some $k \in \mathbb{Z}$

$7 = 2 \cdot 3 + 1$	$8 = 2 \cdot 4$
$11 = 2 \cdot 5 + 1$	

1b The average of any two odd integers is odd.

Proof:- Let n, m be odd integers.

Hence, $n = 2k+1$ for some $k \in \mathbb{Z}$

$m = 2l+1$ for some $l \in \mathbb{Z}$

Average of n and m $= \frac{n+m}{2} = \frac{(2k+1)+(2l+1)}{2}$

$= \frac{2k+2l+2}{2}$

$= k+l+1$

$\frac{7+9}{2} = 8$

$\frac{9+13}{2} = 11$

$k+l+1$ is odd when k and l have same parity.

$k+l+1$ is even when k and l have different parity.

Hence the average could be either even or odd.

even + even = even

$$2k + 2l = 2(\underbrace{k+l})$$

even + odd = odd

$$2k + 2l+1 = 2\underbrace{(k+l)}_{2m+1} + l$$

odd + odd = even

$$(2k+1)+(2l+1) = 2\underbrace{(k+l+1)}_{m} = 2m$$

Prime Number

An integer $n > 1$ is a prime number if the only factors of n are 1 and itself.

i.e If $n = rs$, then either $r = 1$ or n
or $s = n$ or 1

Ex: 2, 3, 5, 7, 11, 13, 17, 19, 23, . . .

Composite Number

A positive integer is a composite number if it is not a prime number.

i.e. if $n = rs$, then $1 < r, s < n$.

Universal Statement

$\forall x \in D, P(x)$

To prove the predicate $P(x)$ has to be true for all $x \in D$.

To disprove we just need one counterexample.
i.e. find an $x \in D$ such that $\sim P(x)$.

Prove or Disprove

$\forall$ real numbers a and b, if $a^2 = b^2$ then $a = b$.

Disprove If $a = -4$ and $b = 4$
then $a^2 = 16 = b^2$
however, $a \neq b$.

12 Disprove
For all integers n, if n is odd then $\frac{n-1}{2}$ is odd.

A $n = 9$, $\frac{n-1}{2} = \frac{9-1}{2} = 4$ is not odd

$2k+1$

$n = 4k+1$
$n = 4k+3$
$k \in \mathbb{Z}$

$\frac{n-1}{2} = \frac{4k+\cancel{1}-\cancel{1}}{2} = \frac{4k}{2} = 2k$ even

$\frac{n-1}{2} = \frac{4k+3-1}{2} = \frac{4k+2}{2} = 2k+1$ odd.

31 Prove the statement

If k is any odd integer and m is any even integer, then k^2+m^2 is odd.

Proof:- Let $k = 2l+1$ for some $l \in \mathbb{Z}$
let $m = 2n$ for some $n \in \mathbb{Z}$

$$k^2+m^2 = (2l+1)^2 + (2n)^2$$
$$= 4l^2+4l+1+4n^2$$
$$= 2\underbrace{(2l^2+2l+2n^2)}_{r}+1$$
$$= 2r+1 \text{ for an integer } r.$$

Hence k^2+m^2 is odd.

36 Prove that the statement is false.

There exists an integer n such that $6n^2+27$ is prime.

Prove $\forall$ integer n, $6n^2+27$ is not a prime

$$6n^2+27 = 3\underbrace{(2n^2+9)}$$

$2n^2+9 \geq 9$

so $2n^2+9 \neq 1$

3 is always a factor of $6n^2+27$.

Hence $6n^2+27$ is not a prime for any integer n.

$a+b=c$

$a^2+b^2=c^2$ $\quad 3^2+4^2=5^2$

$a^3+b^3=c^3$ $\quad 12^2+5^2=13^2$

$a^4+b^4=c^4$ $\quad a^n+b^n=c^n$ $\quad n \in \mathbb{Z}_+$, ≥ 3

$\vdots$

$a^7+b^7=c^7$

Millenium Problem

---

16 The average of any two odd integers is odd!

$\frac{3+5}{2}=4$ $\qquad \frac{5+5}{2}=5$ ✓

Let $n=2k+1$ for some $k \in \mathbb{Z}$

let $m=2l+1$ for some $l \in \mathbb{Z}$

$$\text{Avg.}=\frac{n+m}{2}=\frac{2k+1+2l+1}{2}$$

$$=\frac{2k+2l+2}{2}=k+l+1$$

Let $n=4k+1$

$m=4l+1$

$$\text{Avg.}=\frac{n+m}{2}=\frac{4k+1+4l+1}{2}$$

$$=\frac{4k+4l+2}{2}=2(k+l)+1 \text{ odd}$$

$n=4k+1$

$m=4l+3$

$$\frac{n+m}{2}=\frac{4k+4l+4}{2}=2(2k+2l+2)$$

even

$n=4k+3$

$m=4l+3$

$$\frac{n+m}{2}=\frac{4k+4l+6}{2}=\underbrace{2(2k+2l+2)+1}_{\text{odd}}$$