43 Prove that if n is an odd integer, then $n^4 \bmod 16 = 1$.

Suppose $n = 2k+1$, $k \in \mathbb{Z}$

$$n^4 = (2k+1)^4$$

$$= (2k+1)^2 (2k+1)^2$$

$$= (4k^2+4k+1)(4k^2+4k+1)$$

$$= 16k^4 + 16k^3 + 4k^2 + 16k^3 + 16k^2 + 4k + 4k^2 + 4k + 1$$

$$= 16k^4 + 32k^3 + 24k^2 + 8k + 1$$

$(2K+1)^4$

$16k^4 + \_\_ + \_\_ + \_\_ + \underline{1}$

$n = 4k+1$

$= 4k+3$

If $n = 2k+1$ continued.

$16k^4 + 32k^3 + 24k^2 + 8k + 1$

$\underbrace{16k^4 + 32k^3 + 16k^2}_{\text{mod } 16 = 0} + \underbrace{8k^2 + 8k}+1$

$16k^4 + 32k^3 + 16k^2 + 8k\underbrace{(k+1)} + 1$

$k(k+1) \text{ mod } 2 = 0$ since k and $k+1$ are consecutive integers

$\therefore\ 8k(k+1) \text{ mod } 16 = 0$

Hence $8k(k+1) = 16l$ for some $l \in \mathbb{Z}$

$n^4 = 16k^4 + 32k^3 + 16k^2 + 16l + 1$

$= 16\underbrace{(k^4 + 2k^3 + k^2 + l)}_{r} + 1$

$= 16r + 1$ $\quad r \in \mathbb{Z}$

$\therefore\ n^4 \text{ mod } 16 = 1.$

Absolute Value

For any real number x

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Lemma 1: For any real number r,

$$-|r| \leq r \leq |r|.$$

Proof:- Case 1 $(r \geq 0)$

By definition, $|r| = r$

Since $r \geq 0$, and $-|r| \leq 0$

So $-|r| \leq r \leq |r|$

Case 2 $(r < 0)$, By definition $|r| = -r$

$-|r| = r$

$\therefore -|r| \leq r$

$|r| \geq 0$ and $r < 0$

Hence $r < |r|$

$$\therefore \boxed{-|r| \leq r \leq |r|}$$

Triangle Inequality

For all real number x and y,

$$|x+y| \leq |x| + |y|$$

Proof :- Case 1 $(x+y \geq 0)$

$|x+y| = x+y$ by definition.

By Lemma 1, $x \leq |x|$, $y \leq |y|$

$x+y \leq |x|+|y|$

$\Rightarrow |x+y| \leq |x|+|y|$

Case 2 $(x+y < 0)$

$|x+y| = -(x+y) = -x-y$

$= (-x)+(-y)$

$|ab| = |a||b|$

$|-x| = |-1||x|$

$= |x|$

$-x \leq |-x| = |x|$

$-y \leq |-y| = |y|$

$(-x)+(-y) \leq |x|+|y|$

$|x+y| \leq |x|+|y|$

$\therefore \boxed{|x+y| \leq |x|+|y|}$

<u>46</u> Prove for all real number r, and $c \geq 0$, if $|r| \leq c$, then

$$-c \leq r \leq c$$

<u>Proof</u> :- $|r| \leq c$

<u>Case 1</u> $(r \geq 0)$ By definition $|r| = r$

$|r| \leq c$

$\Rightarrow r \leq c$

$c \geq 0$ and $r \geq 0$

$-c \leq 0$ and $r \geq 0$

$\therefore -c \leq r \leq c$

<u>Case 2</u> $(r < 0)$ By definition $|r| = -r$

$-r \leq c$

$r \geq -c$

$c \geq 0$

Continued from 4/30/2018

Since $r \leq |r|$ by Lemma 1

and $|r| \leq c$

Hence $r \leq c$

$\therefore -c \leq r \leq c$

4.5 Floor and Ceiling.

For any real number x

$\boxed{\text{Floor of } x}$: $\lfloor x \rfloor$ = Largest integer less than or equal to x.

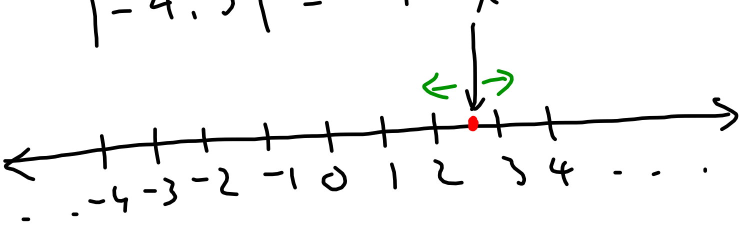
$\lfloor 1.2 \rfloor = 1$ $\quad \lfloor -2.1 \rfloor = -3$

$\lfloor 2 \rfloor = 2$

$\boxed{\text{Ceiling of } x}$: $\lceil x \rceil$ = Smallest integer greater than or equal to x.

Ex: $\lceil 2.1 \rceil = 3$ $\quad \lceil -4 \rceil = -4$

$\lceil -4.3 \rceil = -4$



Formal definition

$\lfloor x \rfloor = n$ iff $n \le x < n+1$

$\lceil x \rceil = n$ iff $n-1 < x \le n$

Theorem:- For all real numbers x, and integers m,

$\lfloor x+m \rfloor = \lfloor x \rfloor + m$

Proof:- Let $\lfloor x \rfloor = n \Rightarrow n \le x < n+1$

$n+m \le x+m < n+m+1$

$\Rightarrow \lfloor x+m \rfloor = n+m$

$= \lfloor x \rfloor + m$

Prove or disprove

$\forall x, y \in \mathbb{R}$

$\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$

$x = 1.5$	$\lfloor x \rfloor = 1$
$y = 1.5$	$\lfloor y \rfloor = 1$
$x+y = 3$	$\lfloor x \rfloor + \lfloor y \rfloor$
$\lfloor x+y \rfloor = 3$	$= 2$

<u>21</u> For all odd integers n, show that

$$\left\lceil \frac{n}{2} \right\rceil = \frac{n+1}{2}$$

<u>Proof</u> :- Let $n = 2k+1$ for $k \in \mathbb{Z}$

$$\left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{2k+1}{2} \right\rceil = \left\lceil k + \frac{1}{2} \right\rceil$$

since $k \leq k + \frac{1}{2} < k+1$

$$\Rightarrow \left\lceil k + \frac{1}{2} \right\rceil = k+1$$

$$\frac{n+1}{2} = \frac{2k+1+1}{2} = \frac{2k+2}{2} = k+1$$

Hence, $\left\lceil \frac{n}{2} \right\rceil = \frac{n+1}{2}$.

Theorem :- For any integer n,

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} n/2 & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof :- Case 1 n is even.

$n = 2k$ for $k \in \mathbb{Z}$

$$\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2k}{2} \right\rfloor = \lfloor k \rfloor = k$$

$\frac{n}{2} = \frac{2k}{2} = k$. Hence $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}$ when n is even.

Case 2 n is odd, $n = 2k+1$ for $k \in \mathbb{Z}$

$$\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2k+1}{2} \right\rfloor = \left\lfloor k + \frac{1}{2} \right\rfloor$$

Since $k \le k + \frac{1}{2} < k+1$

Hence, $\left\lfloor k + \frac{1}{2} \right\rfloor = k$

$\frac{n-1}{2} = \frac{2k+1-1}{2} = \frac{2k}{2} = k$. Therefore, $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2}$ if n is odd.

27 For all real numbers x, if $x - \lfloor x \rfloor \geq \frac{1}{2}$, then

$$\lfloor 2x \rfloor = 2\lfloor x \rfloor + 1$$

Proof :- let $\lfloor x \rfloor = n \Rightarrow n \leq x < n+1$

$$2n \leq 2x < 2n+2$$

$$\Rightarrow \lfloor 2x \rfloor = 2n \text{ or } 2n+1$$

$$x - \lfloor x \rfloor \geq \frac{1}{2}$$

$$x - n \geq \frac{1}{2}$$

$$2x - 2n \geq 1$$

$$2x \geq 2n+1$$

$$2n+1 \leq 2x < 2n+2$$

$$\Rightarrow \lfloor 2x \rfloor = 2n+1$$

$$= 2\lfloor x \rfloor + 1$$

$a, b, c \in \mathbb{Z}$

$a-b$ odd $\quad a-b=2k+1 \quad k \in \mathbb{Z}$

$b-c$ even $\quad b-c=2l \quad l \in \mathbb{Z}$

$$a-c=(a-b)+(b-c)$$
$$2k+1+2l$$
$$2(k+l)+1$$
$$2t+1 \quad t \in \mathbb{Z}$$

40 $\quad n(n^2-1)(n+2)$ is always divisible by 4.

$n \in \mathbb{Z}$

$$n(n^2-1)(n+2)$$
$$n(n-1)(n+1)(n+2)$$
$$(n-1)n(n+1)(n+2)$$

n odd

$n+1$ and $n-1$ are even

$n+1=2k \quad n-1=2l$

$4 \mid n(n^2-1)(n+2)$

$a \mid b = c$ /

$\frac{b}{a} = k$

$a \mid b \quad a$ "divides" b

$b = ak \quad k \in \mathbb{Z}$

4.6 Indirect Arguments

Proofs by Contradiction

Proofs by Contraposition.

Proofs by Contradiction

Question: Prove a statement p.

Step 1 :- Construct the negation of the statement $\sim p$.

Step 2 :- Assume $\sim p$ is true and try to prove it. In the process, arrive at a contradiction.

Step 3 :- Due to the contradiction $\sim p$ is not true, hence it's negation which is p is true.

Q Use proof by contradiction to show that the sum of any rational number and any irrational number is an irrational number.

Proof :- On the contrary, suppose $\exists\ a \in \mathbb{Q}$ and $b \in \mathbb{R} \setminus \mathbb{Q}$, such that $a+b \in \mathbb{Q}$

$a = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$

$a+b = \frac{r}{s}$, $r, s \in \mathbb{Z}$, $s \neq 0$

$\frac{p}{q} + b = \frac{r}{s}$

$b = \frac{r}{s} - \frac{p}{q} = \frac{rq - ps}{sq} \in \mathbb{Q}$ $\quad rq - ps \in \mathbb{Z}$, $sq \in \mathbb{Z}$, $sq \neq 0$

Contradiction since $b \in \mathbb{R} \setminus \mathbb{Q}$

Therefore, the original statement is true.

Q Prove that for all integers n, if n^2 is odd then n is odd.

Proof :- On the contrary, suppose $\exists\, n \in \mathbb{Z}$, such that if n^2 is odd then n is even.

Let $n^2 = 2k+1$ $\quad k \in \mathbb{Z}$

$n^2 - 1 = 2k$

$(n-1)(n+1) = 2k$

suppose $n = 2l$

$(n-1)(n+1) = (2l-1)(2l+1)$

$= 4l^2 - 1$

$= 2t+1 \quad t \in \mathbb{Z}$

$\Rightarrow 2t+1 = 2k$

$1 = 2(k-t)$

$\Rightarrow 2 \mid 1$

Contradiction.

Hence, the original statement is true.

$\sqrt{2} = 1.414\ldots$

Q Show that $\sqrt{2}$ is an irrational number.

Proof :- On the contrary, assume $\sqrt{2} \in \mathbb{Q}$.

$\therefore \sqrt{2} = \frac{p}{q}$ $\quad p, q \in \mathbb{Z}, q \neq 0$

$\gcd(p, q) = 1$

$2 = \frac{p^2}{q^2}$

$2q^2 = p^2 \Rightarrow p^2$ is even $\Rightarrow p$ is even

$\Rightarrow p = 2k \quad k \in \mathbb{Z}$.

$\Rightarrow 2q^2 = (2k)^2$

$\Rightarrow 2q^2 = 4k^2$

$\Rightarrow q^2 = 2k^2 \Rightarrow q^2$ is even $\Rightarrow q$ is even

$\Rightarrow q = 2t \quad t \in \mathbb{Z}$

$\Rightarrow \gcd(p, q) \geq 2$

Contradiction as the assumption was $\gcd(p, q) = 1$.

Therefore, $\sqrt{2}$ is irrational.

Proof by Contraposition.

Given a statement which we have to prove.

Step 1 - Suppose the statement is $p \to q$.

Step 2 - Construct the contrapositive $\sim q \to \sim p$ of the statement.

Step 3 Prove the contrapositive.

Q. Prove that for all integers n, if n^2 is odd, then n is odd.

Proof: Contrapositive - If n is not odd i.e. n is even, then n^2 is not odd, i.e. n^2 is even.

Suppose $n = 2k$ $\quad k \in \mathbb{Z}$

$$n^2 = 4k^2 = 2(2k^2)$$
$$= 2t \quad t \in \mathbb{Z}$$

$\therefore n^2$ is even.

Hence the original statement is true.

26 For all integers a, b, and c, if $a \nmid bc$ then $a \nmid b$.

Proof by Contraposition:

Contrapositive - For all integers a, b, and c, if $a \mid b$, then $a \mid bc$.

$$a \mid b \implies b = ak \quad , k \in \mathbb{Z}$$

$$bc = akc$$

$$bc = a(kc) \quad \text{where } t = kc \in \mathbb{Z}$$

$$bc = at$$

$$\implies a \mid bc$$

Hence the statement is true.

Proof by Contradiction :-

On the contrary suppose $\exists$ integers a, b, and c, such that if $a \nmid bc$, then $a \mid b$.

$$a \nmid bc$$
$$bc = ak \quad \text{has no solutions for any } k \in \mathbb{Z}$$
$$b = a\left(\frac{k}{c}\right)$$

Claim :- $\frac{k}{c} \notin \mathbb{Z}$. On the contrary if $\frac{k}{c}$ is an integer then

$$\frac{k}{c} = d \in \mathbb{Z}$$
$$k = dc \in \mathbb{Z}$$

Contradiction since $k \notin \mathbb{Z}$

Since $\frac{k}{c} \notin \mathbb{Z}$ hence $a \nmid b$. which is a contradiction.

Therefore, the original statement is true.

11, 13

5, 7

29, 31

Any even number is the sum of two primes.

$10 = 7 + 3$

$12 = 7 + 5$

17 Prove by contradiction

For all integers a, if $a \bmod 6 = 3$ then $a \bmod 3 \neq 2$.

Proof :- On the contrary, suppose $\exists\, a \in \mathbb{Z}$, such that if $a \bmod 6 = 3$, then $a \bmod 3 = 2$.

$$a \bmod 6 = 3 \implies a = 6k+3 \quad , k \in \mathbb{Z}$$
$$= 3(2k+1)$$
$$\implies a \bmod 3 = 0$$

Contradiction.

Hence, the original statement is true.

19 Prove by contraposition

If a product of two positive real numbers is greater than 100, then at least one of the numbers is greater than 10.

Proof :- Contrapositive :- If both real numbers are less than or equal to 10, then their product is less than or equal to 100.

Suppose $m, n \in \mathbb{R}$. $0 < m \leq 10$ $0 < n \leq 10$

$0 < mn \leq 100$

Therefore, the statement is true.