5.1 Sequences

A list of numbers related to each other.

$a_1, a_2, a_3, \ldots, a_n, \ldots$

$a_1 = 1$
$a_2 = 4$
$a_3 = 9$
$a_4 = 16$
$\vdots$

$1, 4, 9, 16, \ldots$

$\boxed{a_n = n^2}$ $\quad n \geq 1$

$4, 9, 16, 25, \ldots$

$\boxed{a_n = (n+1)^2}$, $\quad n \geq 1$

$\boxed{b_n = n^2}$, $\quad n \geq 2$

a_k read as a "sub" k.

↑
Subscript

Alternating Sequence

$1, -1, 1, -1, \ldots$

$a_n = (-1)^n \quad n \geq 0$

$1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$

$a_n = \frac{(-1)^{n+1}}{n^2}, \quad n \geq 1$

Summation Notation

$1, 4, 9, 16, 25, \ldots$

$a_n = n^2, \quad n \geq 1$

$1 + 4 + 9 + 16 + \ldots + 100$

$$\sum_{n=k}^{k+m} a_n = a_k + a_{k+1} + a_{k+2} + \ldots + a_{k+m}$$

$$\sum_{n=1}^{10} n^2$$

$$\sum_{j=1}^{m} \sum_{i=1}^{n} (i+j)^2$$

Q Express using summation notation.

$$\frac{2}{2^2+3} + \frac{3}{3^2+4} + \frac{4}{4^2+5} + \cdots + \frac{20}{20^2+21}$$

$$a_n = \frac{n}{n^2+n+1} \quad , n \geq 2$$

$$\boxed{\sum_{n=2}^{20} \frac{n}{n^2+n+1}}$$

Find the sum $\sum_{k=1}^{5} \frac{1}{k(k+1)}$

$\boxed{\text{Telescoping Series.}}$

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$\sum_{k=1}^{5} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$$\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \frac{1}{6}\right)$$

$$1 - \frac{1}{6} = \frac{5}{6}$$

$6k+1 \qquad 6k+5$

Any integer n has the form

$6k, 6k+1, 6k+2, 6k+3, 6k+4$, or $6k+5$

for some $k \in \mathbb{Z}$.

$n = 6k$ is not prime as $6 \mid 6k$

$n = 6k+2$ is not prime $2 \mid 6k+2$

$n = 6k+3$ is not prime $3 \mid 6k+3$

$n = 6k+4$ is not prime $2 \mid 6k+4$

Hence every prime number is of the form

$6k+1$ or $6k+5$.

Square root of an irrational is also irrational.

On the contrary, suppose $\exists$ an irrational number x whose square root is rational.

Let $\sqrt{x} = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$.

$$x = \frac{p^2}{q^2} \in \mathbb{Q}$$

Contradiction as x was assumed to be irrational.

Hence, the original statement is true.

Product Notation

$$\prod_{i=m}^{m+k} a_i = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_{m+k}$$

$$1^2 \cdot 2^2 \cdot 3^2 \cdots 10^2 = \prod_{i=1}^{10} i^2$$

Theorem

① $$\sum_{k=m}^{n} a_k \pm \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k \pm b_k)$$

$$\sum_{i=1}^{4} i^2 + \sum_{i=1}^{4} i^3 = \sum_{i=1}^{4} (i^2 + i^3)$$

$$(1^2+1^3)+(2^2+2^3)+(3^2+3^3)+(4^2+4^3)$$

$$\sum_{i=1}^{5} i^2 + \sum_{i=1}^{4} i^3 = \sum_{i=1}^{4} (i^2+i^3) + 5^2$$

② $$\sum_{k=m}^{n} c a_k = c \sum_{k=m}^{n} a_k$$ where c is a constant

$$\sum_{i=1}^{4} 99 i^2$$

$$99 \cdot 1^2 + 99 \cdot 2^2 + 99 \cdot 3^2 + 99 \cdot 4^2$$

$$99 \sum_{i=1}^{4} i^2$$

$$\cancel{99 i \sum_{i=1}^{4} i}$$

③ $$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k)$$

Change of variables

Q write $\sum_{k=1}^{7} \frac{1}{k+1}$ with a lower limit starting at

ⓐ 0
ⓑ 2

ⓐ $\sum_{l=0}^{6} \frac{1}{l+2}$

ⓑ $\sum_{i=2}^{8} \frac{1}{i}$

Factorial (!)

The factorial of a non-negative integer n is defined to be

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

$1! = 1$
$2! = 2$
$3! = 6$
$4! = 24$

$5! = 120$
$6! = 720$
$7! = 7\cdot 6! = 5040$
$\vdots$
$n! = n(n-1)!$

$\boxed{0! = 1}$

Important $\frac{8!}{4!} = \frac{8\cdot 7\cdot 6\cdot 5\cdot \cancel{4!}}{\cancel{4!}}$

$= 56\cdot 30 = 1680$

$\binom{n}{r} = {}^{n}C_{r}$ n "Choose" r

${}^{n}C_{r}$ is the number of ways of selecting r items from n distinct items where the order of the items don't matter.

3 tickets 10 friends

$${}^{10}C_3 = \frac{10!}{3!\,7!}$$

$$= \frac{\cancel{10}^{5}\cdot \cancel{9}^{3}\cdot 8\cdot \cancel{7!}}{\cancel{6}_{\cancel{3}}\cdot \cancel{7!}} = 15\cdot 8 = 120$$

$$\boxed{{}^{n}C_{r} = \frac{n!}{r!(n-r)!}}$$

40 $\sum_{i=1}^{k} i^3 + (k+1)^3$

$1^3+2^3+3^3+\cdots+k^3+(k+1)^3$

$\sum_{i=1}^{k+1} i^3$

$^{100}C_{98} = \frac{100!}{98!\,2!} = \frac{100\cdot 99\cdot \cancel{98!}}{\cancel{98!}\,2!}$

$= \frac{{}^{50}\cancel{100}\cdot 99}{\cancel{2}\cdot 1} = 4950$

The number of ways of choosing n items from n distinct items = 1.

Hence, $^{n}C_{n} = 1$

$\frac{\cancel{n!}}{\cancel{n!}\,0!} = 1 \Rightarrow \boxed{0! = 1}$

77 Prove that $n!+2$ is divisible by 2 for all integers $n \geq 2$.

Proof :- $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

If $n \geq 2$ then $2 \mid n!$

Hence $n! = 2k \quad k \in \mathbb{Z}$

$n!+2 = 2k+2$

$= 2(k+1)$

$\Rightarrow 2 \mid (n!+2)$ as $k+1 \in \mathbb{Z}$.

Q Show that the product of any k consecutive positive integers is divisible by $k!$.

5.2 Mathematical Induction

Show that $1+2+3+4+\cdots+n = \frac{n(n+1)}{2}$, $n \geq 1$

Proof:- ① Basis Step

Verify that the given result is true for the initial value of n (in this problem $n=1$)

LHS $= 1$ for $n=1$. ✓

RHS $= \frac{1(1+1)}{2} = \frac{1\cdot\cancel{2}}{\cancel{2}} = 1$ ✓

Step 2 Inductive Step.

Assume the statement is true for $n=k$. i.e for every integer less than k.

i.e $1+2+3+\cdots+k = \frac{k(k+1)}{2}$

Step 3 Use the inductive step to show that the statement is true for $n=k+1$.

$$1+2+3+\cdots+k = \frac{k(k+1)}{2}$$

$$1+2+3+\cdots+k+k+1 = \frac{k(k+1)}{2}+(k+1)$$

$$= (k+1)\left[\frac{k}{2}+1\right]$$

$$= (k+1)\left(\frac{k+2}{2}\right)$$

$$= \frac{(k+1)((k+1)+1)}{2}$$

$$\boxed{1+2+3+\cdots+k+1 \overset{?}{=} \frac{(k+1)(k+2)}{2}}$$

This is of the form $\frac{n(n+1)}{2}$.

Hence, by induction the result is true.

Q Show that

$$1+r+r^2+r^3+\cdots+r^{n-1} = \frac{r^n-1}{r-1}, \quad r \neq 1, \; n \geq 1$$

Proof :- Basis Step $(n=1)$

$\text{LHS} = r^{1-1} = r^0 = 1$ ✓

$\text{RHS} = \dfrac{r^1-1}{r-1} = \dfrac{r-1}{r-1} = 1$ ✓

Inductive Step Assume the result is true for $n=k$

i.e., $1+r+r^2+\cdots+r^{k-1} = \dfrac{r^k-1}{r-1}$

we show that the result is also true for $n=k+1$.

$$1+r+r^2+\cdots+r^k \stackrel{?}{=} \frac{r^{k+1}-1}{r-1}$$

$$1+r+r^2+\cdots+r^{k-1}+r^k = \frac{r^k-1}{r-1}+r^k$$

$$= \frac{r^k-1+r^k(r-1)}{r-1}$$

$$= \frac{\cancel{r^k}-1+r^{k+1}-\cancel{r^k}}{r-1}$$

$$= \frac{r^{k+1}-1}{r-1} = \frac{r^{(k+1)}-1}{r-1}$$

Hence, by induction the result is true.

Q Show using mathematical induction that

$$1+3+5+7+\cdots+(2n-1)=n^2, \quad n\geq 1$$

A <u>Basis Step</u> $\underline{n=1}$

$\underline{LHS}$: 1

$\underline{RHS}$: $1^2 = 1$

<u>Inductive Step</u>

Suppose that the result is true for $n=k$.

i.e. $1+3+5+7+\cdots+(2k-1)=k^2$

$1+3+5+\cdots+(2k-1)+(2k+1)=k^2+(2k+1)$

$1+3+5+\cdots+(2k+1)=(k+1)^2$

$1+3+5+\cdots+(2k+1) \overset{?}{=} (k+1)^2$

Hence, this result is true for $n=k+1$.

Therefore, by induction the statement is true for all n.

14 Use induction to show that

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2 \quad \text{for all integers } n \geq 0.$$

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + (n+1) \cdot 2^{n+1} = n \cdot 2^{n+2} + 2$$

① Basis Step.

$n=0$ LHS $\sum_{i=1}^{1} i \cdot 2^i = 1 \cdot 2^1 = 2$ ✓

RHS $0 \cdot 2^{0+2} + 2 = 2$ ✓

② Inductive Step

Suppose the result is true for $n = k$. i.e.

$$\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+2} + 2$$

$$\boxed{\sum_{i=1}^{k+2} i \cdot 2^i \overset{?}{=} (k+1) \cdot 2^{k+3} + 2}$$

$$\underbrace{\sum_{i=1}^{k+1} i \cdot 2^i + (k+2) \cdot 2^{k+2}} = k \cdot 2^{k+2} + 2 + (k+2) \cdot 2^{k+2}$$

$$\sum_{i=1}^{k+2} i \cdot 2^i = k \cdot 2^{k+2} + k \cdot 2^{k+2} + 2^{k+3} + 2$$

$$= 2k \cdot 2^{k+2} + 2^{k+3} + 2$$

$$= k \cdot 2^{k+3} + 2^{k+3} + 2$$

$$= (k+1) \cdot 2^{k+3} + 2$$

$$= (k+1) \cdot 2^{(k+1)+2} + 2$$

Therefore, by induction the result is true.

$$S = 1 + 2 + 3 + \cdots + n$$
$$\underline{S = n + (n-1) + (n-2) + \cdots + 1}$$
$$2S = (n+1) + (n+1) + (n+1) + \cdots + (n+1)$$
$$2S = n(n+1)$$
$$S = \frac{n(n+1)}{2}$$

$4k \quad 4k+1 \quad 4k+2 \quad 4k+3$

$7k \quad 7k+1 \; \cdots\cdots \quad 7k+6$

Product of any four consecutive integer is divisible by 8

$n(n+1)(n+2)(n+3)$

$\underline{n=2k}$ $\quad 2k(2k+1)(2k+2)(2k+3)$

$$4k(k+1)\left[(2k+1)(2k+3)\right]$$

k and $k+1$ are consecutive integers. Hence one of them is even. Therefore $2 \mid k(k+1)$

$$\Rightarrow 8 \mid 4k(k+1)$$

$$\Rightarrow 8 \mid n(n+1)(n+2)(n+3)$$

$\underline{n = 2k+1}$

$$n(n+1)(n+2)(n+3) = (2k+1)(2k+2)(2k+3)(2k+4)$$
$$= 4\underbrace{(k+1)(k+2)}\left[(2k+1)(2k+3)\right]$$

$k+1$ and $k+2$ are consecutive integers.
Hence, one of them is even. Thus $2 \mid (k+1)(k+2)$

$$\Rightarrow 8 \mid 4(k+1)(k+2)$$
$$\Rightarrow 8 \mid n(n+1)(n+2)(n+3)$$