$\because$ Since

$\therefore$ Therefore

5.6 Defining Sequences Recursively.

$2, 4, 6, 8, \ldots$

$\boxed{a_n = 2n}$ $n \geq 1$

$\boxed{a_k = a_{k-1} + 2}$, $a_1 = 2$, $k \geq 2$

Recurrence Relation.

Goal: "Solve" the sequence i.e find an explicit formula for the nth term of the sequence a_n.

7 $u_k = k u_{k-1} - u_{k-2}$ $k \geq 3$

$u_1 = 1$

$u_2 = 1$

Find the first 4 terms

$u_1 = 1$

$u_2 = 1$

$u_3 = 3u_2 - u_1 = 3 \cdot 1 - 1 = 2$

$u_4 = 4u_3 - u_2 = 4 \cdot 2 - 1 = 7$

$1, 1, 2, 7, \ldots$

Fibonacci Sequence

$F_n = F_{n-1} + F_{n-2}$, $F_0 = 1$, $F_1 = 1$, $n \geq 2$

$F_2 = F_1 + F_0 = 1 + 1 = 2$

$F_3 = F_2 + F_1 = 2 + 1 = 3$

$F_4 = F_3 + F_2 = 3 + 2 = 5$

$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

27 Prove that

$$F_k^2 - F_{k-1}^2 = F_k F_{k+1} - F_{k+1} F_{k-1},$$

for all integers $k \geq 1$

LHS $F_k^2 - F_{k-1}^2 = (F_k - F_{k-1})(F_k + F_{k-1})$

$F_k = F_{k-1} + F_{k-2}$

$F_{k+1} = F_k + F_{k-1}$

$= (F_k - F_{k-1})(F_{k+1})$

$= F_k F_{k+1} - F_{k-1} F_{k+1}$

$=$ RHS

Hence, the result is true

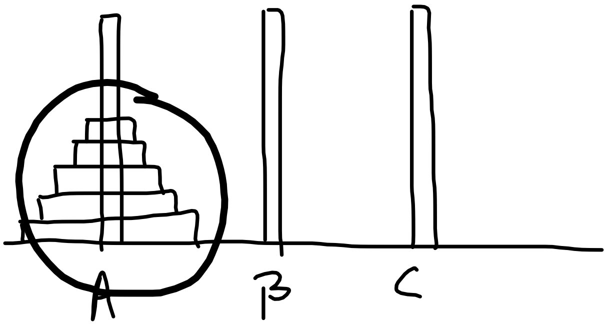
$A = B$ ⋮ $= B$

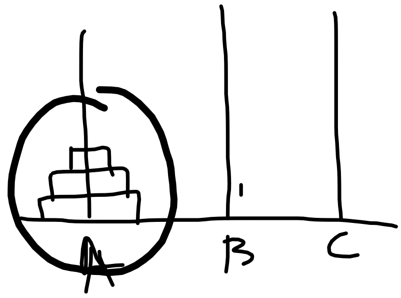
$A = B$ ⋮ $= A$

$A = B$ ⋮ C C

$A \neq B$ ⋮ $D \neq C$

Towers of Hanoi

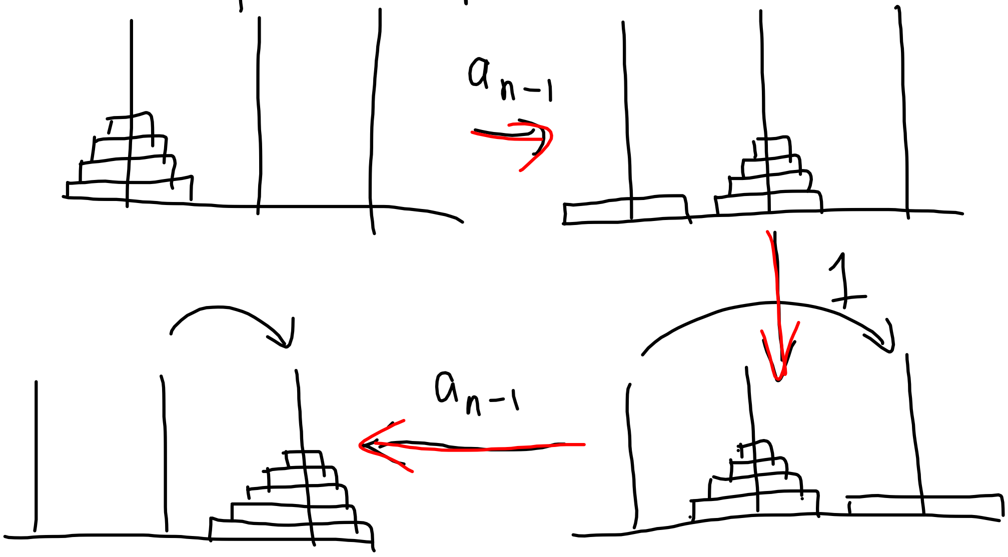




A → C
A → B
C → B
A → C
B – A
B → C

A → C

a_n = No. of steps needed to move n disks from one pillar to another.



$$\therefore\ a_n = a_{n-1} + 1 + a_{n-1}$$

$$a_n = 2a_{n-1} + 1 \qquad a_1 = 1$$

$$a_2 = 2a_1 + 1 = 3$$

$$a_3 = 2a_2 + 1 = 2 \cdot 3 + 1 = 7$$

$$a_4 = 2a_3 + 1 = 15$$

$$a_5 = 2a_4 + 1 = 31$$

1, 3, 7, 15, 31, 63, ...

? $a_n = 2^n - 1$

$$a_{64} = 2^{64} - 1 \approx 1.8 \times 10^{19}$$

585 billion year

5.7 Solving Recurrence Relations

$$a_n = a_{n-1} + 2, \quad a_1 = 2$$

$$a_1 = 2$$

$$a_2 = a_1 + 2 = 2 + 2$$

$$a_3 = a_2 + 2 = 2 + 2 + 2$$

$$a_4 = a_3 + 2 = 2 + 2 + 2 + 2$$

$\vdots$

$$a_n = \underbrace{2 + 2 + 2 \cdots + 2}_{n \ 2\text{'s}.}$$

$$a_n = 2n, \quad n \geq 1$$

We prove that $a_n = 2n$ for $n \geq 1$
we will use Mathematical Induction to verify the result.

$a_n = 2n \quad , \quad n \geq 1$

Basis Step $n = 1$

LHS: $a_1 = 2$ ✓

RHS: $2 \cdot 1 = 2$ ✓

Inductive Step

Suppose that the result is true for $n = k$

i.e. $a_k = 2k$

$$\begin{aligned} \text{Now, } a_{k+1} &= a_k + 2 \\ &= 2k + 2 \\ &= 2(k+1) \end{aligned}$$

$\therefore$ a_{k+1} is in the same form as $a_n = 2n$

Hence, $a_n = 2n$ for $n \geq 1$ is the solution of the recurrence relation.

$2, 4, 6, 8, \ldots$

Arithmetic Sequence.

a ↑ Initial Term

d ↑ Common difference

$a = 2$
$d = 2$

$2, 4, 6, 8, \ldots$

$a_n = a_{n-1} + d$

$a_1 = a$

$a_1 = a$
$a_2 = a + d$
$a_3 = a + d + d$
$a_4 = a + d + d + d$
$\vdots$

$\boxed{a_n = a + (n-1)d}$ ← Verify by Induction.

Q $a_n = 3a_{n-1} \quad n \geq 1, \quad a_0 = 2$

$a_0 = 2$

$a_1 = 3a_0 = 3 \cdot 2$

$a_2 = 3a_1 = 3 \cdot 3 \cdot 2$

$a_3 = 3a_2 = 3 \cdot 3 \cdot 3 \cdot 2$

$\vdots$

$a_n = \underbrace{3 \cdot 3 \cdot \cdots 3}_{n\ 3\text{'s}} \cdot 2 = 3^n \cdot 2$

Verification $a_n = 3^n \cdot 2 \quad n \geq 0$

Basis Step $n = 0$

LHS: $a_0 = 2$ ✓

RHS: $3^0 \cdot 2 = 2$ ✓

Inductive step

Assume the result is true for $n = k$.

i.e. $a_k = 3^k \cdot 2$

Now, $a_{k+1} = 3a_k$ by recurrence relation

$= 3 \cdot 3^k \cdot 2$

$= 3^{k+1} \cdot 2$

a_{k+1} is of the same form as $a_n = 3^n \cdot 2$

Hence, the result is true by induction.

Geometric Sequence

$a \neq 0$ — Initial Term

$r \neq 0$ — Common ratio

$a_n = r \cdot a_{n-1} \quad n \geq 1, \; a_0 = a$

$a_0 = a$

$a_1 = r \cdot a$

$a_2 = r \cdot r \cdot a$

$a_3 = r \cdot r \cdot r \cdot a$

$\vdots$

$a_n = \underbrace{r \cdot r \cdot \ldots \cdot r}_{n \; r\text{'s}} \cdot a = r^n \cdot a$

$\boxed{a_n = r^n \cdot a}$ ← Verify by induction.

Towers of Hanoi

$a_n = 2a_{n-1} + 1 \qquad a_1 = 1$

$a_1 = 1$

$a_2 = 2a_1 + 1 = 2 \cdot 1 + 1 = 2 + 1$

$a_3 = 2a_2 + 1 = 2(2 \cdot 1 + 1) + 1 = 2^2 + 2 + 1$

$a_4 = 2a_3 + 1 = 2(2^2 + 2 + 1) + 1 = 2^3 + 2^2 + 2 + 1$

$a_5 = 2a_4 + 1 = 2(2^3 + 2^2 + 2 + 1) + 1 = 2^4 + 2^3 + 2^2 + 2 + 1$

$\vdots$

$$a_n = \underbrace{1 + 2 + 2^2 + \cdots + 2^{n-2} + 2^{n-1}}_{\text{Geometric Series.}}$$

we know that $1 + r + r^2 + \cdots + r^{n-1} = \dfrac{r^n - 1}{r - 1}$

Hence, $a_n = \dfrac{2^n - 1}{2 - 1} = 2^n - 1$

Verification

Basis Step $n = 1$

LHS: $a_1 = \underline{1}$ ✓

RHS: $2^1 - 1 = 1$ ✓

Inductive step

Suppose the result is true for $n = k$.

i.e $a_k = 2^k - 1$

Now, $a_{k+1} = 2a_k + 1$

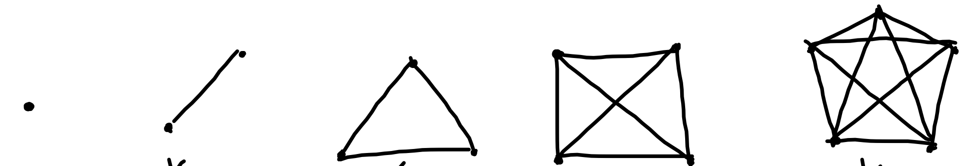
$= 2(2^k - 1) + 1$

$= 2^{k+1} - 2 + 1$

$= 2^{k+1} - 1$

a_{k+1} is of the same form as $a_n = 2^n - 1$.

Hence, the result is true for $n = k+1$, and thus the result is true in general.



K_1 K_2 K_3 K_4 K_5

K_n = Complete Graph in n vertices.

Let e_n denote the number of edges of K_n.

$e_1 = 0$

$e_2 = 1$

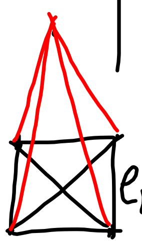
$e_3 = 3$

$e_4 = 6$

$e_5 = 10$

$e_n = ?$

$e_n = e_{n-1} + (n-1)$



$e_n = e_{n-1} + (n-1)$