30 Basis step $n=0$

$F_0 = 1$
$F_1 = 1$
$F_2 = 2$

LHS: $F_2 F_0 - F_1^2$

$2 \cdot 1 - 1^2 = 1$ ✓

RHS: $(-1)^0 = 1$ ✓

Inductive step: Assume the statement is true for $n = k$. i.e.

$$F_{k+2} F_k - F_{k+1}^2 = (-1)^k$$

WTS
$$F_{k+3} F_{k+1} - F_{k+2}^2 = (-1)^{k+1}$$

$$F_{k+3} F_{k+1} - F_{k+2}^2$$

$$(F_{k+2} + F_{k+1}) F_{k+1} - (F_{k+1} + F_k)^2$$

$$F_{k+2} F_{k+1} + F_{k+1}^2 - (F_{k+1}^2 + 2F_{k+1}F_k + F_k^2)$$

$$F_{k+2} F_{k+1} + \cancel{F_{k+1}^2} - \cancel{F_{k+1}^2} - 2F_{k+1}F_k - F_k^2$$

$$(F_{k+1} + F_k) F_{k+1} - 2F_{k+1}F_k - F_k^2$$

$$F_{k+1}^2 - F_{k+1}F_k - F_k^2$$

$$= -(F_{k+1}F_k - F_{k+1}^2 + F_k^2)$$

$$= -(-1)^k$$

$$= (-1)^{k+1}$$

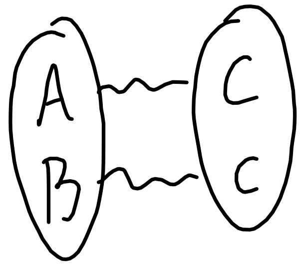
$$F_{k+2}F_k - F_{k+1}^2 = (-1)^k$$

$$(F_{k+1} + F_k)F_k - F_{k+1}^2 = (-1)^k$$

$$F_{k+1}F_k + F_k^2 - F_{k+1}^2 = (-1)^k$$

$A = B$

$A \sim B$



6. Set Theory.

Notations

another set

$$A = \{x \in S \mid P(x)\}$$

such that

Property holds.

$$A = \{x \in \mathbb{Z} \mid -3 \leq x \leq \pi\}$$

$$A = \{-3, -2, -1, 0, 1, 2, 3\}$$

Subset

$A \subseteq B \iff A$ is a subset of B

How do we show that $A \subseteq B$?

Element Method

let $x \in A$. Show that $x \in B$

Q How to show $A \subset B \iff A$ is a proper subset of B.

A ① let $x \in A$, show that $x \in B$

② Show that $\exists\, x \in B$, such that $x \notin A$.

Q $A = \{ a \in \mathbb{Z} \mid a = 4m + 8 \text{ for some } m \in \mathbb{Z} \}$

$B = \{ b \in \mathbb{Z} \mid b = 2n \text{ for some } n \in \mathbb{Z} \}$

Show that $A \subset B$.

A let $a \in A \Rightarrow a = 4m + 8$ for $m \in \mathbb{Z}$

$= 2(2m+4)$

$= 2k \quad k \in \mathbb{Z}$

$\Rightarrow a \in B$

$2 \in B$ since $2 = 2 \cdot 1$ and $1 \in \mathbb{Z}$

$2 \notin A$. Suppose on the contrary.

$2 \in A$

$\Rightarrow 2 = 4k + 8 \quad k \in \mathbb{Z}$

$-6 = 4k$

$k = -\frac{3}{2} \notin \mathbb{Z}$

$\Rightarrow$ Contradiction

Hence $2 \notin A$.

$\therefore A \subset B$.

Set Equality

How to show that for sets A and B,

$$A = B \, ?$$

A Show that $A \subseteq B$ and then show that $B \subseteq A$.

Q $A = \{ a \in \mathbb{Z} \mid a = 3m - 1, \text{ for } m \in \mathbb{Z} \}$

$B = \{ b \in \mathbb{Z} \mid b = 3n + 2, \text{ for } n \in \mathbb{Z} \}$

Show that $A = B$.

A Case 1 $A \subseteq B$

Let $x \in A \Rightarrow x = 3m - 1 \quad m \in \mathbb{Z}$

$= 3m - 3 + 2$

$= 3(m-1) + 2$

$= 3k + 2 \quad k \in \mathbb{Z}$

$\Rightarrow x \in B$.

$\therefore A \subseteq B$

<u>Case 2</u> <u>$B \subseteq A$</u>

Let $x \in B \Rightarrow x = 3n+2 \qquad n \in \mathbb{Z}$

$= 3n+3-1$

$= 3(n+1)-1$

$= 3k-1 \qquad k \in \mathbb{Z}$

$\Rightarrow x \in A$

$\therefore B \subseteq A$

Combining the two cases we conclude that

$A = B$

Operations on Sets

U = Universal Set

Union $\cup$

$$A \cup B = \{ x \in U \mid x \in A \text{ or } x \in B \}$$

Intersection $\cap$

$$A \cap B = \{ x \in U \mid x \in A \text{ and } x \in B \}$$

Difference $-$

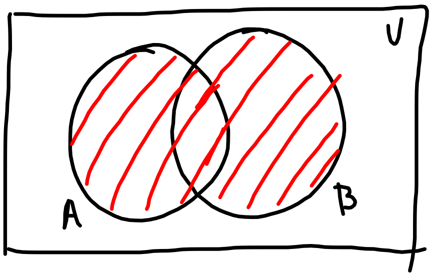
$$A - B = \{ x \in U \mid x \in A \text{ and } x \notin B \}$$

Complement ^{c}

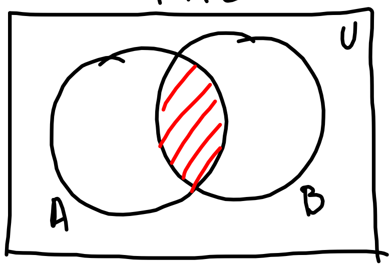
$$A^c = \{ x \in U \mid x \notin A \}$$

Venn Diagrams

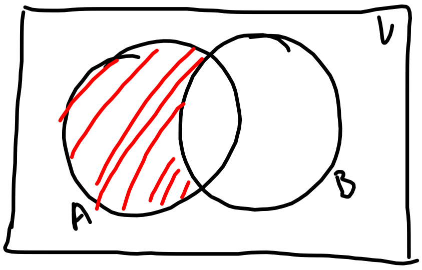
$A \cup B$



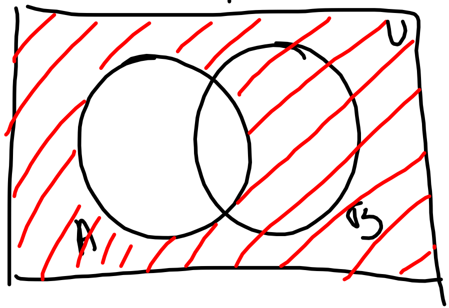
$A \cap B$



$A - B$

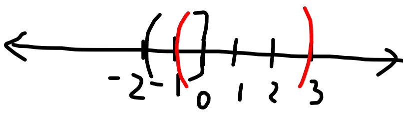


A^c



Q $U = \mathbb{R}$ $A = (-2, 0]$, $B = (-1, 3)$

$A \cup B = (-2, 3)$



$A \cap B = (-1, 0]$

$A - B = (-2, -1]$

$A^c = (-\infty, -2] \cup (0, \infty)$

Union and Intersection of an indexed collection of sets

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

<u>Empty Set</u> → set with no elements

ϕ "phi"

$\{ \}$

$A = (-2, 4]$ $B = (5, 7]$

$A \cap B = \phi$

Disjoint Sets

Definition :- Two sets A and B are disjoint if $A \cap B = \phi$.

Mutually Disjoint Sets

Definition :- Sets $A_1, A_2, \dots, A_n$ are mutually disjoint if

$$A_i \cap A_j = \phi \quad \text{for any } i \neq j$$

Ex :- $A = \{1, 2\}$, $B = \{7, 8, 9\}$, $C = \{5, 6\}$

A, B, C are mutually disjoint sets

as $A \cap B = \phi$

$B \cap C = \phi$

$A \cap C = \phi$

Partition of a Set

Definition :- A collection of sets $\{A_1, A_2, \ldots\}$ is a partition of a set A iff,

1. $A = \bigcup A_i$
2. The set A_i's are mutually disjoint.

Ex :- $A = \{1, 2, 3, 4\}$, $A_1 = \{1\}$, $A_2 = \{2, 4\}$, $A_3 = \{3\}$

Does A_1, A_2, A_3 partition A?

Ans Yes, as $A_1 \cup A_2 \cup A_3 = A$ and A_1, A_2, A_3 are mutually disjoint.

Power Set

Given a set A, the power set of A denoted by $P(A)$ is the set of all subsets of A.

Ex:- $A = \{a, b, c\}$

$$P(A) = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}, \phi\}$$

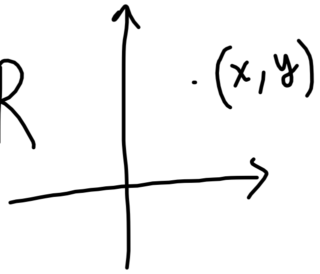
If the set A has n elements, then the power set of A, $P(A)$ has 2^n elements.

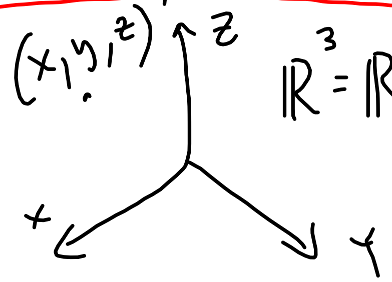
Cartesian Product

Given sets $A_1, A_2, \ldots, A_n$, the Cartesian Product of $A_1, A_2, \ldots, A_n$ denoted by $A_1 \times A_2 \times \cdots \times A_n$ is the set of all ordered n tuples $(a_1, a_2, \ldots, a_n)$ with $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$

i.e $\boxed{A_1 \times A_2 \times \cdots \times A_n = \{ (a_1, a_2, \ldots, a_n) \mid a_i \in A_i \}}$

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$





$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ copies}}$ $(x_1, x_2, \ldots, x_n)$

__Ex:__ $A = \{a, b\}$ $B = \{1, 2, 3\}$

$$A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$$

If the set A has n elements and B has m elements, then $A \times B$ has nm elements.

6.2 Properties of Sets

Some relations on Sets

1 (a) $A \cap B \subseteq A$

Let $x \in A \cap B$

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow A \cap B \subseteq A$

(b) $A \cap B \subseteq B$

$A \cap B \subseteq B$

2 (a) $A \subseteq A \cup B$

Let $x \in A$

$\Rightarrow x \in A \cup B$

$\Rightarrow A \subseteq A \cup B$

(b) $B \subseteq A \cup B$

let $x \in B$

$\Rightarrow x \in B \cup A$

$x \in S$

$x \in S \cup$

1 (c) If $A \subseteq B$, and $B \subseteq C$, then $A \subseteq C$.

Let $x \in A$

$\Rightarrow x \in B$ as $A \subseteq B$

$\Rightarrow x \in C$ as $B \subseteq C$

$\Rightarrow A \subseteq C$

Look at Pg. 355 for a list of properties of sets.

$\underline{Q}$ Show that $A \cup (A \cap B) = A$

$\underline{\text{Case 1}}$ WTS $A \cup (A \cap B) \subseteq A$

Let $x \in A \cup (A \cap B)$

$\Rightarrow x \in A$ or $x \in A \cap B$

$\underline{\text{Case 1(a)}}$ Suppose $x \in A \Rightarrow A \cup (A \cap B) \subseteq A$

$\underline{\text{Case 1(b)}}$ Suppose $x \in A \cap B$

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in A \Rightarrow A \cup (A \cap B) \subseteq A$

<u>Case 2</u> : WTS $A \subseteq A \cup (A \cap B)$

Let $x \in A$

$\implies x \in A \cup (A \cap B)$

Hence, $A \subseteq A \cup (A \cap B)$

<u>Note</u> : Always remember:

$$\boxed{A \subseteq A \cup \underbrace{\text{anything}}}$$

Combining Case 1 and 2 we see that

$A \cup (A \cap B) = A$

Distributive Law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Case 1: WTS $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Let $x \in A \cup (B \cap C)$

$\Rightarrow x \in A$ or $x \in B \cap C$

Case 1(a) Suppose $x \in A$

$\Rightarrow x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\therefore A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Case 1(b) Suppose $x \in B \cap C$

$\Rightarrow x \in B$ and $x \in C$

$\Rightarrow x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\therefore A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Case 2 :- WTS $(A\cup B)\cap(A\cup C) \subseteq A\cup(B\cap C)$

let $x \in (A\cup B)\cap(A\cup C)$

$\Rightarrow x \in A\cup B$ and $x \in A\cup C$

$\Rightarrow (x\in A \text{ or } x\in B)$ and $(x\in A \text{ or } x \in C)$

Case 2(a) Suppose $x\in A$

$\Rightarrow x \in A\cup(B\cap C)$

$\therefore (A\cup B)\cap(A\cup C) \subseteq A\cup(B\cap C)$

Case 2(b) Suppose $x \in A$ and $x\in C$

$\Rightarrow x\in A\cup(B\cap C)$

$\therefore (A\cup B)\cap(A\cup C) \subseteq A\cup(B\cap C)$

Case 2(c) Suppose $x\in B$ and $x\in A$

$\Rightarrow x\in A\cup(B\cap C)$

$\therefore (A\cup B)\cap(A\cup C)\subseteq A\cup(B\cap C)$

Case 2(d) Suppose $x\in B$ and $x\in C$

$\Rightarrow x\in B\cap C$

$\Rightarrow x \in (B\cap C)\cup A$

$\therefore (A\cup B)\cap(A\cup C)\subseteq A\cup(B\cap C)$

Hence, combining both cases, the given result is true.

9 For all sets A, B, and C,

$$(A-B)\cup(C-B)=(A\cup C)-B$$

Proof :- Case 1 WTS $(A-B)\cup(C-B)\subseteq(A\cup C)-B$

Let $x\in(A-B)\cup(C-B)$

$\Rightarrow x\in A-B$ or $x\in C-B$

Case 1(a) Suppose $x\in A-B$

$\Rightarrow x\in A$ and $x\notin B$

$\Rightarrow x\in A\cup C$ and $x\notin B$

$\Rightarrow x\in(A\cup C)-B$

$\therefore (A-B)\cup(C-B)\subseteq(A\cup C)-B$

Case 1(b) Suppose $x\in C-B$

$\Rightarrow x\in C$ and $x\notin B$

$\Rightarrow x\in A\cup C$ and $x\notin B$

$\Rightarrow x\in(A\cup C)-B$

$\therefore (A-B)\cup(C-B)\subseteq(A\cup C)-B$

Case 2 WTS $(A \cup C) - B \subseteq (A - B) \cup (C - B)$

Let $x \in (A \cup C) - B$

$\Rightarrow x \in A \cup C$ and $x \notin B$

$\Rightarrow (x \in A$ or $x \in C)$ and $x \notin B$

Case 2(a) Suppose $x \in A$ and $x \notin B$

$\Rightarrow x \in A - B$

$\Rightarrow x \in (A - B) \cup (C - B)$

$\therefore (A \cup C) - B \subseteq (A - B) \cup (C - B)$

Case 2(b) Suppose $x \in C$ and $x \notin B$

$\Rightarrow x \in C - B$

$\Rightarrow x \in (A - B) \cup (C - B)$

$\therefore (A \cup C) - B \subseteq (A - B) \cup (C - B)$

$\therefore$ Combining all the cases, the result is true.

De Morgan's Law

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

Theorem :- If E has no elements and A is any set, then $E \subseteq A$.

Proof :- On the contrary, suppose $\exists$ a set E with no elements, such that $E \not\subseteq A$.

This implies $\exists\ x \in E$ such that $x \notin A$.

But this is a contradiction as E contains no elements.

Hence, the original statement is true.

Note :- Usually problems involving complements or the empty set are solved with <u>proofs by contradiction</u>

<u>Theorem</u> :- There is only one set with no element.

<u>Proof</u> :- On the contrary, suppose there are two sets E_1 and E_2 with no elements.

By previous theorem, $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$, hence $E_1 = E_2$ which is a contradiction as E_1 and E_2 were assumed to be different sets.

Hence, the original statement is true.

30 For all sets A and B, show that if
$A \subseteq B$ then $A \cap B^c = \phi$

Ans On the contrary, suppose $\exists$ sets A and B
such that $A \subseteq B$ and $A \cap B^c \neq \phi$

Let $x \in A \cap B^c$
$\Rightarrow x \in A$ and $x \in B^c$
$\Rightarrow x \in A$ and $x \notin B$
$\Rightarrow$ Contradiction as $A \subseteq B$ so any element in A is also in B.

Hence, the original statement is true.

8.1 Relations on Sets

Definition :- let A and B be sets. A relation R from A to B is a subset of $A \times B$. Given an ordered pair $(x, y) \in A \times B$, x is related to y via R written $x R y$ iff $(x, y) \in R$.

Ex: Suppose S is relation from $\mathbb{R}$ to $\mathbb{R}$

$$x S y \iff x > y$$