B $\quad 10b-3 \quad b \in \mathbb{Z}$

A $\quad 5a+2 \quad a \in \mathbb{Z}$

$B \subseteq A$? Prove or disprove

let $x \in B \implies x = 10b-3 \quad b \in \mathbb{Z}$

$$= 10b - 5 + 2$$
$$= 5\underbrace{(2b-1)}_{t} + 2$$
$$= 5t + 2, \quad t \in \mathbb{Z}$$

$\therefore x \in A$. Hence $B \subseteq A$

$b = -3$

$10b-3$

$10(-3)-3$

-33

$5(-7)+2$

$$10b - 3 = 5a + 2$$
$$10b - 5a = 5$$
$$2b - a = 1 \longrightarrow$$
$$a = 2b - 1$$

$(-1, 4) \notin S$ $\quad \mathbb{R}\times\mathbb{R}$ $\quad S \subseteq \mathbb{R}\times\mathbb{R}$

$x > y$

$-1 \not> 4$

Definition :- Let R be a relation from sets A to B. The inverse relation of R, denoted by R^{-1} is a relation from B to A where

$$R^{-1} = \{(y,x) \in B\times A \mid (x,y) \in R\}$$

Q Let $A = \{2, 3, 4\}$, $B = \{2, 9, 12\}$

For all $(x, y) \in A \times B$

$$x R y \Longleftrightarrow x \mid y$$

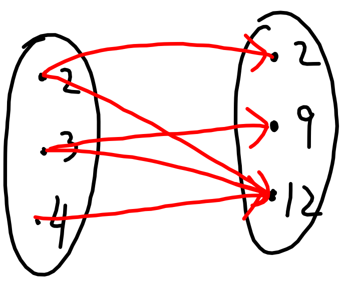
(1) Find R and R^{-1} and draw arrow diagrams

(2) Are R or R^{-1} functions?

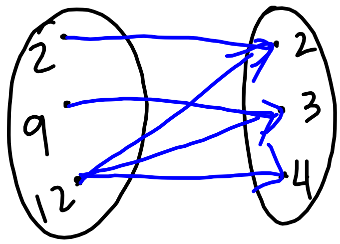
Ans:- $R = \{(2,2), (2,12), (3,9), (3,12), (4,12)\}$

$R^{-1} = \{(2,2), (12,2), (9,3), (12,3), (12,4)\}$

R



R^{-1}



Neither R or R^{-1} are functions.

as R and R^{-1} both map elements in to multiple elements.

Directed Graphs

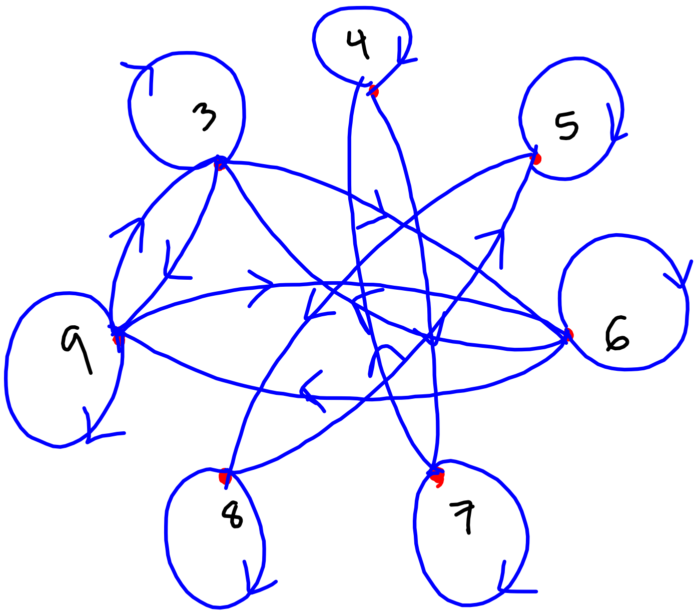
Suppose R is a relation from the set A to itself.

Q let $A = \{3, 4, 5, 6, 7, 8, 9\}$ and define a relation R on A as follows,

$$\forall x, y \in A \quad xRy \iff 3 \mid (x - y)$$

$$R = \{(3,3), (3,6), (3,9), (4,4), (4,7), (5,5), (5,8), (6,3), (6,6), (6,9), (7,4), (7,7), (8,5), (8,8), (9,3), (9,6), (9,9)\}$$

Q Draw a directed graph for R.


4
3
5
9
6
8
7

1.2

15 If $A \subseteq B$ then show that $\boxed{B^c \subseteq A^c}$

Let $x \in B^c$

$\Rightarrow x \notin B$

$\Rightarrow x \notin A$ since $A \subseteq B$

$\Rightarrow x \in A^c$ Hence $B^c \subseteq A^c$

17 Suppose $A \subseteq C$ and $B \subseteq C$

Show that $\boxed{A \cup B \subseteq C}$

Let $x \in A \cup B$

$\Rightarrow x \in A$ or $x \in B$

Case 1 Suppose $x \in A$

$\Rightarrow x \in C$

$\Rightarrow A \cup B \subseteq C$

Case 2 Suppose $x \in B$

$\Rightarrow x \in C$

$\Rightarrow A \cup B \subseteq C$

Hence combining Cases 1 and 2

$A \cup B \subseteq C$.

Show that if $B \subseteq A^c$ then $A \cap B = \phi$.

Proof :- On the contrary suppose $\exists$ sets A, B
Such that if $B \subseteq A^c$ then $A \cap B \neq \phi$

Let $x \in A \cap B$
$\Rightarrow x \in A$ and $x \in B$
$\Rightarrow x \in A$ and $x \in A^c$ since $B \subseteq A^c$
$\Rightarrow$ Contradiction as x cannot be in both A and A^c

Hence, the original statement is true.

8.2 Reflexivity, Symmetry, and Transitivity

Let R be a relation on a set A.

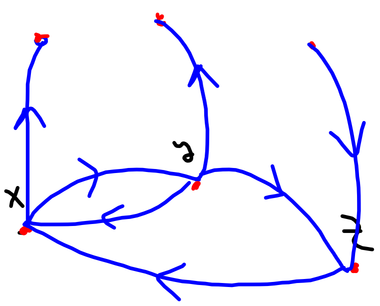
1. R is reflexive iff $\forall x \in A$, xRx
   i.e. $(x,x) \in R$

2. R is symmetric iff $\forall x, y \in A$, if xRy then yRx, i.e. if $(x,y) \in R$, then $(y,x) \in R$.

$p \to q$

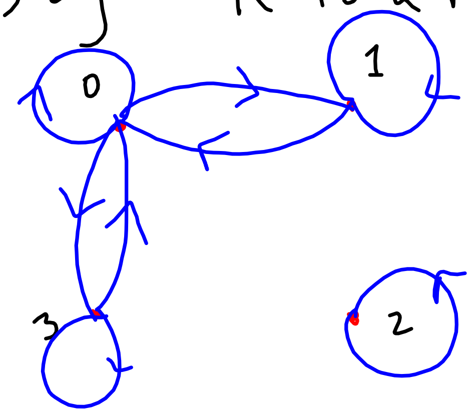
p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

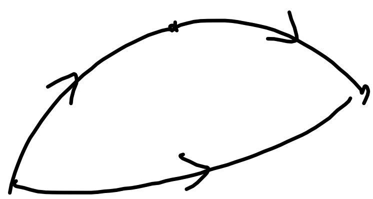
3. R is transitive iff $\forall x, y, z \in A$, if xRy and yRz, then xRz, i.e., if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$.



$$R = \{(0,0), (0,1), (0,3), (1,0), (1,1), (2,2), (3,0), (3,3)\}$$

$A = \{0, 1, 2, 3\}$ R is a relation on A.



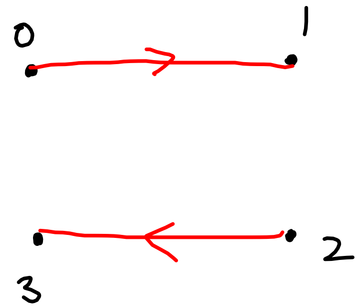


R is reflexive.

R is symmetric

R is not transitive since $(3,0) \in R$ and $(0,1) \in R$ but $(3,1) \notin R$.

Q



$R = \{(0,1),(2,3)\}$

R is not reflexive since $(0,0) \notin R$

R is not symmetric since $(0,1) \in R$ but $(1,0) \notin R$

R is transitive since $(0,1) \in R$ and $(2,3) \in R$ and the hypothesis is false. Hence, the conclusion that R is transitive is vacuously true.

$R = \{\ \}$ $\quad A = \{0\}$

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

Case 1 WTS $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

Let $(x, y) \in A \times (B \cup C)$

$\Rightarrow x \in A$ and $y \in B \cup C$

$\Rightarrow x \in A$ and $(y \in B$ or $y \in C)$

Case 1(a) Suppose $x \in A$ and $y \in B$

$\Rightarrow (x, y) \in A \times B$

$\Rightarrow (x, y) \in (A \times B) \cup (A \times C)$

Case 1(b) Suppose $x \in A$ and $y \in C$

$\Rightarrow (x, y) \in A \times C$

$\Rightarrow (x, y) \in (A \times C) \cup (A \times B)$

$\therefore A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

Case 2: WTS $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$

Let $(x, y) \in (A \times B) \cup (A \times C)$

$\Rightarrow (x, y) \in A \times B$ or $(x, y) \in A \times C$

Case 2(a) Suppose $(x, y) \in A \times B$

$\Rightarrow x \in A$ and $y \in B$

$\Rightarrow x \in A$ and $y \in B \cup C$

$\Rightarrow (x, y) \in A \times (B \cup C)$

Case 2(b) Suppose $(x, y) \in A \times C$

$\Rightarrow x \in A$ and $y \in C$

$\Rightarrow x \in A$ and $y \in B \cup C$

$\Rightarrow x \in A \times (B \cup C)$

$\therefore (A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$

Combining the two cases, the result is true.

#12 E is the congruence modulo 2 relation on $\mathbb{Z}$.

For all $m, n \in \mathbb{Z}$, $m \, E \, n \iff 2 \mid (m-n)$

A Reflexive :- Let $m \in \mathbb{Z}$, $2 \mid (m-m)$

$\Rightarrow E$ is reflexive.

Symmetry :- let $m, n \in \mathbb{Z}$. suppose $m \, E \, n$

$\Rightarrow 2 \mid (m-n)$

$\Rightarrow m - n = 2k$

$\Rightarrow n - m = 2(-k)$ for $k \in \mathbb{Z}$

$\Rightarrow 2 \mid (n-m)$ as $-k \in \mathbb{Z}$

$\Rightarrow n \, E \, m$

Hence, E is symmetric

Transitive :- let $m, n, p \in \mathbb{Z}$, suppose mEn and nEp.

$\Rightarrow 2 \mid (m-n)$ and $2 \mid (n-p)$

$\Rightarrow m-n = 2k$, $k \in \mathbb{Z}$ and $n-p = 2l$, $l \in \mathbb{Z}$

$\Rightarrow (m-\cancel{n}) + (\cancel{n}-p) = 2k + 2l$

$m - p = 2(k+l)$

$\Rightarrow 2 \mid (m-p)$

$\Rightarrow mEp$

Hence, E is transitive.

#22 $X = \{a, b, c\}$

$P(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$

A relation on N is defined on $P(X)$

For all $A, B \in P(X)$, # of elements

$A \, N \, B \iff |A| \neq |B|$

Ans Reflexive :- Let $A \in P(X)$

$|A| = |A|$ hence

$A \not{N} A$

$\therefore N$ is not reflexive.

Symmetry :- Let $A, B \in P(X)$

Suppose $A \, N \, B \Rightarrow |A| \neq |B|$

$\Rightarrow |B| \neq |A|$

$\Rightarrow B \, N \, A$

$\therefore N$ is symmetric.

Transitive :- Let $A = \{a\}$, $B = \{a, b\}$, $C = \{c\}$

$A \, N \, B$ as $|A| \neq |B|$

$B \, N \, C$ as $|B| \neq |C|$

However, $A \not N C$ since $|A| = |C|$

$\therefore$ N is not transitive.

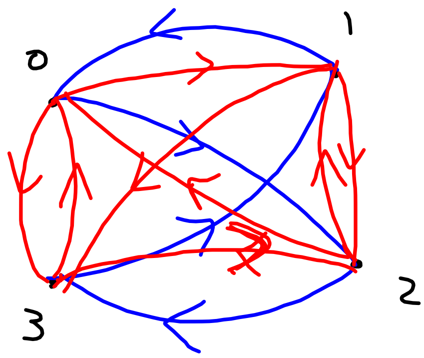
Transitive closure of a relation

Definition :- Let A be a set and R be a relation on A. The transitive closure of R is the relation R^t on A that satisfies the following properties.

1. R^t is transitive
2. $R \subseteq R^t$
3. If S is any other transitive relation that contains R, then $R^t \subseteq S$ i.e. R^t is the smallest transitive relation that contains R.

#53 $T = \{(0,2), (1,0), (2,3), (3,1)\}$

Find the transitive closure of T.



$T^t = \{ \quad - \quad - \quad - \}$