1. *(5 points)* Let $A = \{2, 4\}$ and $B = \{1, 3, 5\}$ and define relations U, V, and W from A to B as follows: For all $(x, y) \in A \times B$,

$$(x, y) \in U \text{ means that } y - x > 2$$

$$(x, y) \in V \text{ means that } y - 1 = \frac{x}{2}$$

$$W = \{(2, 5), (4, 1), (2, 3)\}$$

(a) Draw arrow diagrams for U, V, and W.

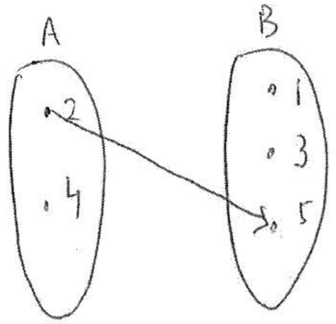
(b) Indicate whether any of the relations U, V, and W are functions.

Show your work.

$A \times B = \{(2,1), (2,3), (2,5), (4,1), (4,3), (4,5)\}$.

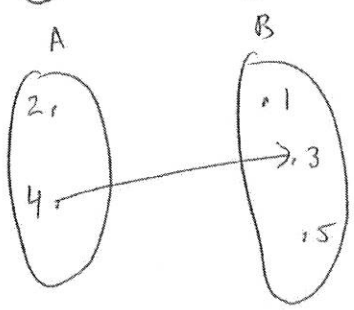
$U = \{(2,5)\}$, $V = \{(4,3)\}$, $W = \{(2,5), (4,1), (2,3)\}$

<u>U</u>



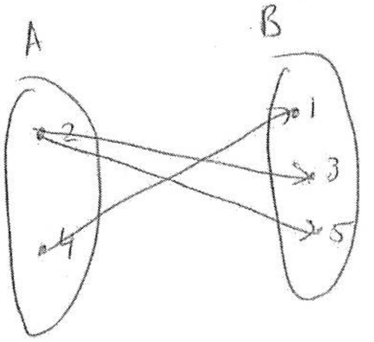
Not a function
as 4 has no image.

<u>V</u>



Not a function
as 2 has no image.

<u>W</u>



Not a function
as 2 maps to multiple elements.

2. (*5 points*) By constructing truth tables, determine whether the following statement forms are logically equivalent. Show your work.

$$p \wedge (q \vee r) \text{ and } (p \wedge q) \vee (p \wedge r)$$

p	q	r	$q \vee r$	$p \wedge q$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	0	1	1	1
1	1	0	1	1	0	1	1
1	1	1	1	1	1	1	1

Same

$$\boxed{p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)}$$

3. (*4 points*) Use De Morgan's laws to write the negations of the following statement. Show your work.

The dollar is at an all-time high and the stock market is at a record low.

(Hint: De Morgan's Law: $\sim(p \vee q) \equiv \sim p \wedge \sim q$ and $\sim(p \wedge q) \equiv \sim p \vee \sim q$)

The dollar is at an all time high $= p$.
Stock market is at a record low $= q$.

Original statement $= p \wedge q$

$\sim(p \wedge q) = \sim p \vee \sim q$

$\sim p$ = The dollar is not at an all time high.
$\sim q$ = The stock market is ~~above the lowest it has ever been.~~ not at a record low.

$\sim p \vee \sim q$ = Either The dollar is not at an all time high or the stock market is not at a record low.

4. (a) (*3 points*) Rewrite the statement in if-then form.

Having two 60° angles is a sufficient condition for this triangle to be a right triangle.

$\boxed{\text{If}}$ there are two 60° angles in a trianglee, $\boxed{\text{then}}$ the triangle is a right triangle.

(b) (*3 points*) Rewrite the statement as a conjunction of two if-then statements.

An integer is odd if, and only if, it leaves a remainder of 1 when divided by 2.

$\boxed{\text{If}}$ an integer is odd, $\boxed{\text{then}}$ it leaves a remainder of 1 when divided by 2, $\boxed{\text{and}}$, $\boxed{\text{if}}$ an integer leaves a remainder of 1 when divided by 2, $\boxed{\text{then}}$ it is odd.

(c) (*4 points*) Use truth tables to verify the following logical equivalence. Show your work.

$$\sim (p \to q) \equiv p \wedge \sim q$$

p	q	$p \to q$	$\sim(p \to q)$	$\sim q$	$p \wedge \sim q$
0	0	1	0	1	0
0	1	1	0	0	0
1	0	0	1	1	1
1	1	1	0	0	0

same

So, $\sim(p \to q) \equiv p \wedge \sim q$

5. (a) (*3 points*) Consider the statement.

Not every subset of a set A contains 4 elements.

Rewrite this sentence using variables and symbols such as $\exists, \forall$.

S is the set of all subsets of A. $\exists s \in S$, such that $|s| \neq 4$.

(b) (*3 points*) Consider the statement.

Every subset of a set A contains 2 elements.

Rewrite this sentence using variables and symbols such as $\exists, \forall$.

S is the set of all subsets of A. $\forall s \in S$, $|s| = 2$.

6. (a) (*3 points*) Write the negation of the following statement.

∀ integers a, b and c, if a-b is even and b-c is even then a-c is even.

∃ integers a, b, and c, such that if a−b is even and b−c is even then a−c is odd.

(b) (*3 points*) Write the contrapositive of the following statement.

If a player scores 1500 points and wins a championship, then he will be in the Hall of Fame.

If a player is not in the Hall of Fame, then he hasn't scored 1500 points or didn't win a championship.

7. (*5 points*) Prove that for all integers n and m, if $n - m$ is even then $n^3 - m^3$ is even. Show your work.

Proof:- $n - m$ is even

$\therefore$ $n - m = 2k$ for some integer k.

$\Rightarrow n = 2k + m$

$n^3 - m^3 = (2k+m)^3 - m^3$

$= 8k^3 + 12k^2m + 6km^2 + \cancel{m^3} - \cancel{m^3}$

$= 2(4k^3 + 6k^2m + 3km^2)$

Since $k \in \mathbb{Z}$, $m \in \mathbb{Z}$, $4k^3 + 6k^2m + 3km^2 \in \mathbb{Z}$

$\therefore$ $n^3 - m^3 = 2r$ for some integer r.

$\therefore$ $n^3 - m^3$ is even

9. (*4 points*) A student has answered the following question by giving a proof. Is the proof right? **If so, say Yes. If not, then why not and then correct the proof.** Show your work.

Question : The difference between any odd integer and any even integer is odd.

Proof: Suppose n is any odd integer, and m is any even integer. By definition of odd, $n = 2k + 1$ where k is an integer, and by definition of even, $m = 2k$ where k is an integer. Then

$$n - m = (2k + 1) - 2k = 1.$$

Since 1 is odd, therefore, the difference between any odd integer and any even integer is odd.

The ~~stu~~ Proof given by the student is incorrect. First, he chooses his odd and even integers as $2k+1$ and $2k$. which are consecutive integers since ~~the~~ the student chooses the same k.

Thus $n-m=1$. due to this selection. However not every difference of any odd and even integer is 1. For eg. $7-2=5$ which is not 1.

<u>Correct Proof</u>

Let $n = 2k+1$ for some $k \in \mathbb{Z}$

let $m = 2l$ for som $l \in \mathbb{Z}$

$$n-m = 2k+1-2l.$$
$$= 2(k-l)+1$$

Since $k, l \in \mathbb{Z}$, therefore $k-l \in \mathbb{Z}$

$\therefore n-m = 2r+1$ where $r = k-l$.

$\therefore n-m$ is odd.

8. (a) (*5 points*) Prove that the difference of squares of any two consecutive integers is odd. Show your work.

Proof :- Let the integers be n and $n+1$

Difference of their squares

$= (n+1)^2 - n^2$

$= \cancel{n^2} + 2n + 1 - \cancel{n^2}$

$= 2n + 1$

Since $n \in \mathbb{Z}$ $\therefore$ $2n+1$ is an odd integer.

Hence the difference of their squares is odd.