MATH 327
DISCRETE MATHEMATICS
EXAMINATION 2

May 14, 2018

Name (Last, First) Key

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Directions

- There are 7 problems on the test.
- The total points for the test is 50.
- You have a maximum of 50 minutes for this exam.
- You will only get partial credit on problems depending on your solution.
- You must show your work in every problem.

TOTAL POINTS SCORED____________________

1. (*6 points*) Show that for all integers n, if $n \bmod 5 = 3$, then $n^2 \bmod 5 = 4$. Show your work.

$n \bmod 5 = 3 \Rightarrow n = 5k+3$ for $k \in \mathbb{Z}$

$\Rightarrow n^2 = (5k+3)^2$

$= 25k^2 + 30k + 9$

$= 25k^2 + 30k + 5 + 4$

$= 5\underbrace{(5k^2+6k+1)}_{t} + 4$

$= 5t + 4$ for $t \in \mathbb{Z}$

$\therefore \boxed{n^2 \bmod 5 = 4}$

2. (*8 points*) Prove by **contradiction** that for all integers m and n, if mn is even then m is even or n is even. Show your work.

   Mention clearly what you plan to prove and then show a contradiction.

On the contrary suppose $\exists\ m, n \in \mathbb{Z}$ such that mn is even and m is odd and n is odd.

$\therefore\ m = 2k+1 \quad k \in \mathbb{Z}$

$n = 2l+1 \quad l \in \mathbb{Z}$

$$mn = (2k+1)(2l+1)$$
$$= 4kl + 2k + 2l + 1$$
$$= 2\underbrace{(2kl+k+l)}_{t} + 1$$
$$= 2t+1, \quad t \in \mathbb{Z}$$

$\therefore$ mn is odd $\Rightarrow$ Contradiction.

Thus, the original statement is true.

3. (*8 points*) Prove by **contraposition** that for all integers m and n, if mn is even then m is even or n is even. Show your work.

Mention clearly what you plan to prove and then do your work.

Contraposition :- If m and n are odd integers then mn is an odd integer.

Proof :- let $m = 2k+1 \quad k \in \mathbb{Z}$
$n = 2l+1 \quad l \in \mathbb{Z}$

$$mn = (2k+1)(2l+1)$$
$$= 4kl + 2k + 2l + 1$$
$$= 2(\underbrace{2kl + k + l}_{t}) + 1$$
$$= 2t+1 \quad t \in \mathbb{Z}$$

$\therefore$ mn is odd

4. (*6 points*) How many zeros are at the end of $\mathbf{45^8 \cdot 88^5}$? Explain how you can answer this question without actually computing the number. You will **not** get any points if you use the calculator to find the answer. (Hint: $10 = 2 \cdot 5$.)

$$45^8 \cdot 88^5 = (9 \cdot 5)^8 \cdot (8 \cdot 11)^5$$
$$= 9^8 \cdot 5^8 \cdot (2^3 \cdot 11)^5$$
$$= 9^8 \cdot 5^8 \cdot 2^{15} \cdot 11^5$$
$$= (5 \cdot 2)^8 \cdot 9^8 \cdot 2^7 \cdot 11^5$$
$$= 10^8 \cdot 9^8 \cdot 2^7 \cdot 11^5$$

$\therefore 10^8 \mid 45^8 \cdot 88^5$ but $10^9 \nmid 45^8 \cdot 88^5$

$\therefore$ There are 8 zeroes at the end of $45^8 \cdot 88^5$.

5. (*8 points*) Prove that $n! + k$ is divisible by k, for all integers $n \geq 2$ and $k = 2, 3, \ldots, n$. Show your work.

Note that $k \leq n$

$\therefore$ Since $k, n \in \mathbb{Z}$

$\therefore$ $k \mid n!$

and $k \mid k$

$\therefore$ $k \mid n! + k$ for $n \geq 2$ and $k = 2, 3, \ldots n$.

6. (*6 points*) Find **two** different explicit formulas for the following sequence using **two** different variables. Show your work.

$$\frac{1}{3}, \frac{9}{9}, \frac{25}{27}, \frac{49}{81}, \frac{81}{243}, \frac{121}{729}$$

$$a_n = \frac{(2n+1)^2}{3^{n+1}}, \quad 0 \leq n \leq 5$$

$$b_m = \frac{(2m-1)^2}{3^m}, \quad 1 \leq m \leq 6$$

7. (*8 points*) Use the **principle of mathematical induction** to prove the following.

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}, \qquad \text{for all integers } n \geq 2.$$

Show your work and show each step.

Basis step $n=2$ LHS: $\sum_{i=1}^{1} i(i+1) = 1(1+1)$
$= 2$ ✓

RHS: $\frac{2(2-1)(2+1)}{3} = \frac{2 \cdot 1 \cdot \cancel{3}}{\cancel{3}} = 2$ ✓

Inductive step Suppose the result is true for $n=k$.

$$\therefore \sum_{i=1}^{k-1} i(i+1) = \frac{k(k-1)(k+1)}{3}$$

$$\sum_{i=1}^{k} i(i+1) = \sum_{i=1}^{k-1} i(i+1) + k(k+1) = \frac{k(k-1)(k+1)}{3} + k(k+1)$$

$$= k(k+1)\left[\frac{k-1}{3} + 1\right]$$

$$= \frac{k(k+1)(k+2)}{3}$$

$$\sum_{i=1}^{(k+1)-1} i(i+1) = \sum_{i=1}^{k} i(i+1) = \frac{(k+1)((k+1)-1)((k+1)+1)}{3}$$

Thus the result is true for $n=k+1$.

Hence, by the induction the result is true for all $n \geq 2$.