Chapter 2 - Sets and Relations

Sets

A set is a collection of objects.

$$A = \{1, 5, 8, \text{Adam}, 1/3\}$$

1, 5, 8, Adam, 1/3 are elements of the set A.

$5 \in A \Rightarrow$ Means that 5 is an element of the set A.

$8 \in A$

$2/3 \notin A \Rightarrow$ Means that 2/3 is not an element of the set A.

Set - Builder Notation

$A \leftarrow$ Universal Set.

$$S = \{ \underbrace{x \in A}_{} \mid P(x) \}$$

such that

Some property P is satisfied by x.

Set of even integers.

$$S = \{ x \in \mathbb{Z} \mid 2 \mid x \}$$

set of integers. divides

$$\mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$$

Zahlën

$a \mid b \iff$ a "divides" b

$3 \mid 9 \quad 4 \nmid 7$

Some standard sets

$\mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$

$\uparrow$

set of integers.

$\mathbb{R}$ = Set of real numbers.

$\mathbb{Q}$ = Set of rational numbers.

$= \left\{ \frac{m}{n} \;\middle|\; m, n \in \mathbb{Z},\ n \neq 0 \right\}$

$-\frac{1}{3} \in \mathbb{Q}$, $\frac{5}{1} \in \mathbb{Q}$, $-3/2 \in \mathbb{Q}$

$\pi \notin \mathbb{Q}$, $e \notin \mathbb{Q}$ | $1/3 \in \mathbb{Q}$

$0.3333\ldots$

$0.\overline{3}$

$0.48272727\ldots$

$0.48\overline{27}$

$x = 0.482727\ldots$

$10000x = 4827.2727\ldots$

$-\quad 100x = 48.2727\ldots$

$9900x = 4779.0000\ldots$

$x = \frac{4779}{9900}$

$\pi = 3.141592\ldots$

$\mathbb{N}$ - Set of Natural Numbers

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

$\mathbb{Z}_+$ $\mathbb{Z}_-$

Equality of Sets

Two sets A and B are considered equal when they have the same elements and not any more.

$A = B$ means that every element in A is in B, and every element in B is in A.

$$A = \{1, 2, \{1, 2\}\}$$

Is $\{1,2\} \in A$? Yes.

$1 \in A$ $\{3\} \notin A$

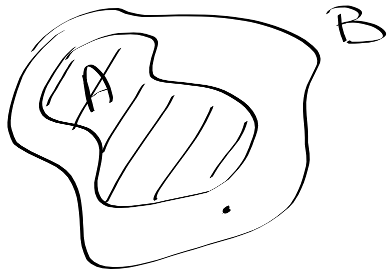
$2 \in A$

Subset

A set A is a subset of a set B if every element in A is also an element of B.
We write it as $\boxed{A \subseteq B}$

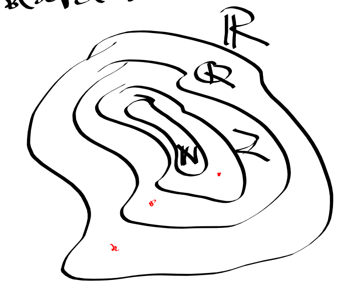
Moreover, if there exists at least one element in B that is not in A, then A is a proper subset of B, denoted by

$\boxed{A \subset B}$



Relations between standard sets

$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$



$\mathbb{Z} \subseteq \mathbb{Q}$

$A = \{1, 2, \{1,2\}\}$

Is ① $1 \subseteq A$ False $\{1\} \subseteq A$ ✓

② $\{1,2\} \subseteq A$ True

③ $\{\{1,2\}\} \subseteq A$ True

Empty set

An empty set is a set with no elements.
It is denoted by ϕ or $\{ \}$.

$$A = \{ n \in \mathbb{Z} \mid n^2 + 1 = 0 \} \qquad n^2+1=0$$
$$\qquad\qquad\qquad\qquad\qquad\qquad n^2 = -1$$
$$= \phi$$

Axiom: The empty set is a subset of every set.

Proposition: A set A is a subset of itself.

Power Set

The Power Set of a set A denoted by $\mathcal{P}(A)$ is the set of all subsets of A.

$A = \{1, 2\}$

$\mathcal{P}(A) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$

$A = \{\underbrace{a, b, c}\}$

$\mathcal{P}(A) = \{\phi, \{a,b,c\}, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}$
$, \{b,c\}\}$

If a set A has N elements, then its power set $\mathcal{P}(A)$ has 2^N elements.

In other words,
$\mathcal{P}(A) = \{ B \mid B \subseteq A \}$

Q How do we show that $A \subseteq B$?

So how do we show that $\mathbb{Z} \subseteq \mathbb{Q}$

- Suppose $x \in A$. Now show that x is also in B.

Let $n \in \mathbb{Z}$

Hence $n = \frac{n}{1}$ $\quad n \in \mathbb{Z}$, $1 \in \mathbb{Z}$

$\therefore n \in \mathbb{Q}$

$\therefore \mathbb{Z} \subseteq \mathbb{Q}$

Q How to show that $\mathbb{Z} \subsetneq \mathbb{Q}$?

A First show $\mathbb{Z} \subseteq \mathbb{Q}$.
Then find one specific $x \in \mathbb{Q}$ which is not in $\mathbb{Z}$.

Let $x = \frac{1}{2} \in \mathbb{Q}$ but $\frac{1}{2} \notin \mathbb{Z}$.

$\therefore \mathbb{Z} \subsetneq \mathbb{Q}$.

Proposition: For sets A and B,

$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

$$A = B \iff A \subseteq B \text{ and } B \subseteq A$$

If A then B

$(\Rightarrow)$ Suppose $A = B$

By definition, every element of A is an element of B and vice versa.

Hence $A \subseteq B$ and $B \subseteq A$

$(\Leftarrow)$ Suppose $A \subseteq B$ and $B \subseteq A$.

Since every element of A is in B and vice versa, hence $A = B$ by definition.

$\therefore$ The proposition is true.

Operations on sets

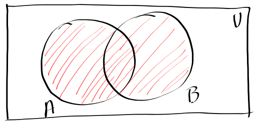
① Union of sets

Given sets A and B, the union of A and B, denoted by $A \cup B$, is the set of all elements that are in either A or B.

$A = \{1, 2, 3, 4\}$

$B = \{3, 4, 7\}$

$A \cup B = \{1, 2, 3, 4, 7\}$



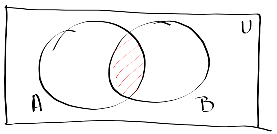
② Intersection of sets

Given sets A and B, the intersection of A and B, denoted by $A \cap B$, is the set of all elements that are in both A and B.

$A = \{1, 2, 3, 4\}$

$B = \{3, 4, 7\}$

$A \cap B = \{3, 4\}$



$A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n$

$= \bigcup_{i=1}^{n} A_i$

$A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n$

$= \bigcap_{i=1}^{n} A_i$

$A \cup (B \cap C)$

$\uparrow$ has to done first.

Q For sets A and B, prove that
$A \cup B = B$ iff $A \subseteq B$
$\uparrow$ if and only if

Ans: ($\Rightarrow$) Assume $A \cup B = B$

Let $x \in A$
$\Rightarrow x \in A \cup B$ [$x \in A \cup$ anything]
$\Rightarrow x \in B$
$\therefore A \subseteq B$

($\Leftarrow$) Suppose $A \subseteq B$.

Let $x \in A$ $\Rightarrow x \in B$
$\Rightarrow x \in A \cup B$
$\Rightarrow A \cup B \subseteq B$.

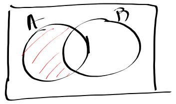
Let $y \in B$
$\Rightarrow y \in B \cup A$
$\Rightarrow B \subseteq A \cup B$

Combining both inclusions we have
$A \cup B = B$.

$\therefore$ The original statement is true.

③ Set Difference

Given sets A and B, the difference of A and B, denoted by $A \setminus B$, is the set of all elements in A that are not in B.



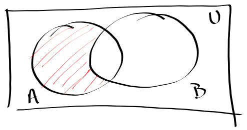
$A = \{1, 2, 3, 4\}$

$B = \{3, 4, 7\}$

$A \setminus B = \{1, 2\}$

$B \setminus A = \{7\}$

$A \cap B = \phi \Rightarrow A \setminus B = A$

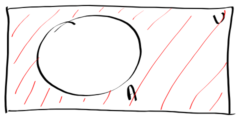


④ Set complement

Given a set A, the complement of A denoted by A^c is the set of all elements that are not in A.

$A = \{1, 2, 3, 4\}$ $U = \mathbb{Z}$

$A^c = \{\ldots, -3, -2, -1, 0, 5, 6, \ldots\}$



De Morgan's Law

Given sets A and B,

① $(A \cup B)^c = A^c \cap B^c$

② $(A \cap B)^c = A^c \cup B^c$

$(A^c)^c = A$

9. Let $a, b \in \mathbb{R}$ with $a < b$. Find

$(a,b)^c$, $[a,b)^c$, $(-\infty, b]^c$

$(a,b)^c = (-\infty, a] \cup [b, \infty)$

$[a,b)^c = (-\infty, a) \cup [b, \infty)$

$(-\infty, b]^c = (b, \infty)$

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10. M = set of math majors

CS = set of CS majors

T = set of students who had a test on Friday

P = set of students who ate pizza last Thursday

① CS majors had a test on Friday.

$CS \subseteq T$

② No math majors ate pizza last Thursday.

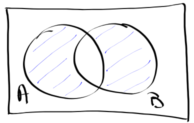
$M \cap P = \phi$.

$A \setminus B^c$

$(A \setminus B)^c$

Symmetric Difference

The symmetric difference of two sets A and B, denoted by $A \oplus B$, is the set of all elements that are in A or B, but not in both.



$A \oplus B = (A \cup B) \setminus (A \cap B)$

$A = \{1, 2, 3, 4\}$
$B = \{3, 4, 7\}$
$A \oplus B = \{1, 2, 7\}$

The Cartesian product of sets

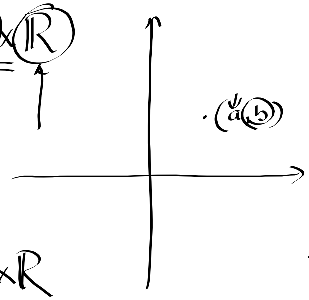
Given sets $A_1, A_2, \ldots, A_n$, the Cartesian product of these sets, denoted by

$$A_1 \times A_2 \times A_3 \times \cdots \times A_n$$

is the set of n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, 1 \leq i \leq n\}$$

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$



$\mathbb{R}^2 = \{(2, 5), (\frac{1}{3}, 10^{50}) \ldots$

$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

(a, b, c)

$\mathbb{R}^2 \times \mathbb{Z}$

(a, b, c)

Two points $(a_1, a_2, \ldots a_n)$ and $(b_1, b_2, \ldots, b_n)$ are the same, iff

$a_1 = b_1, \; a_2 = b_2, \ldots, a_n = b_n$

$A = \{1, 2\}$

$A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$

$A \times A \times A = \{(1,1,1), (1,1,2), (1,2,1), \ldots$

$B = \{a\}$

$B \times B = \{(a, a)\}$

$A = \{1, 2\} \quad B = \{x, y\}$

$A \times B = \{(1, x), (1, y), (2, x), (2, y)\}$

28(a) $A \subseteq C$, $B \subseteq D \rightarrow A \times B \subseteq C \times D$

Want to show $A \times B \subseteq C \times D$

let $(a, b) \in A \times B$

$\Rightarrow a \in A$ and $b \in B$

$\Rightarrow a \in C$ and $b \in D$ (since $A \subseteq C$ and $B \subseteq D$)

$\Rightarrow (a, b) \in C \times D$

$\therefore$ The result is true.

De Morgan's Law

$$(A \cup B)^c = A^c \cap B^c$$

let $x \in (A \cup B)^c$

$\Leftrightarrow x \notin A \cup B$

$\Leftrightarrow x \notin A$ and $x \notin B$

$\Leftrightarrow x \in A^c$ and $x \in B^c$

$\Leftrightarrow x \in A^c \cap B^c$

$\therefore \boxed{(A \cup B)^c = A^c \cap B^c}$

2.3 Binary Relations

$S =$ All students in USA

$R =$ $a \in S$ $b \in S$ are related if they are siblings.

$S \times S = \{(a, b), \mid a \in S, b \in S\}$

Relation

$R \subseteq S \times S$

$M =$ Set of all male students

$F =$ Set of all female students

$M \times F = \{(a, b) \mid a \in M, b \in F\}$

$R \subseteq M \times F$

Definition :- Let A and B denote sets. A binary relation from A to B is a subset of $A \times B$. A binary relation on A is a subset of $A \times A$.

Ex: $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5\}$

$x \in A$, $y \in B$, $xRy \iff x < y$

$R = \{(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5)\}$