Different Types of Relations

① Reflexive - A binary relation R on a set A is reflexive iff for every $a \in A$, $(a,a) \in R$.

Ex:- $A = \mathbb{Z}$

$R = \{x, y \in A \mid xRy \text{ iff } x \leq y\}$

Let $x \in A$, Now $x \leq x$

$\Rightarrow xRx$

$\therefore R$ is a reflexive relation.

② Symmetric

A binary relation on a set A is symmetric iff

For $a, b \in A$, if $(a,b) \in R$, then $(b,a) \in R$.

$A = \mathbb{Z}$

$R = \{x, y \in A \mid xRy \text{ iff } x \leq y\}$

Is R symmetric?

$(1,2) \in R$ since $1 \leq 2$

but $(2,1) \notin R$ as $2 \nleq 1$

$\therefore R$ is not symmetric.

(3) Antisymmetric

A binary relation R on a set A is antisymmetric iff

For $a, b \in A$, and if both $(a,b) \in R$ and $(b,a) \in R$, then $a = b$.

Important: Anti-symmetric $\neq$ Not symmetric.

Ex: $A = \mathbb{Z}$
$R = \{x, y \in A \mid xRy \text{ iff } x \leq y\}$

Is R anti-symmetric?

let $(a,b), (b,a) \in R$

$\Rightarrow a \leq b$ and $b \leq a$

$\Rightarrow a = b$

$\therefore$ R is anti-symmetric

(4) Transitive

A binary relation R on a set A is transitive iff

For $a, b, c \in A$, if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$.

Ex: $A = \mathbb{Z}$
$R = \{x, y \in A \mid xRy \text{ iff } x \leq y\}$

Is R transitive?

let $(a,b) \in R$, $(b,c) \in R$

$\Rightarrow a \leq b$ and $b \leq c$

$\Rightarrow a \leq c$

$\Rightarrow (a,c) \in R$

$\therefore$ R is transitive.

9(e) $A = \mathbb{R}$

$R = \{a, b \in A \mid aRb \text{ iff } a-b \leq 3\}$

Reflexive

let $a \in \mathbb{R}$

$a - a = 0 \leq 3 \Rightarrow (a,a) \in R$

$\therefore R$ is reflexive.

Symmetric

let $(a,b) \in R$

$\Rightarrow a-b \leq 3$ $\quad b-a \geq -3$

$(-5,-1) \in R$ since $-5-(-1) = -5+1 = -4 \leq 3$

However, $(-1,-5) \notin R$ as $-1-(-5) = -1+5 = 4 \not\leq 3$

$\therefore R$ is not symmetric

Anti-symmetric

let $(a,b), (b,a) \in R$

$\Rightarrow a-b \leq 3$ and $b-a \leq 3$

$(2,3) \in R$ since $2-3 = -1 \leq 3$

$(3,2) \in R$ since $3-2 = 1 \leq 3$

However, $2 \neq 3$

$\therefore R$ is not-antisymmetric.

Transitive

Let $(a,b), (b,c) \in R$

$\Rightarrow a-b \leq 3$ and $b-c \leq 3$

$(-5,2)$ $(2,$

$a-c \leq 3$?

$(-1,-3)$ $(-3,-6)$

$(-1,-3) \in R$ as $-1-(-3) = 2 \leq 3$

$(-3,-6) \in R$ as $-3-(-6) = 3 \leq 3$

However, $(-1,-6) \notin R$ as $-1-(-6) = 5 \nleq 3$.

$\therefore R$ is not transitive.

$(-3, 3]$

$\{-3, -1, 0, 1, 2, 3\}$

$\left\{ \frac{x}{y} \mid x, y \in \mathbb{R},\ x^2 + y^2 = 25 \right\}$

$\frac{0}{5}, \frac{3}{4}, \frac{\sqrt{24}}{1}, \ldots$

<u>29</u> $(A \cap B) \times C = (A \times C) \cap (B \times C)$

<u>Case 1</u> WTS $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$

let $(x, y) \in (A \cap B) \times C$

$\Rightarrow x \in A \cap B$ and $y \in C$

$\Rightarrow x \in A$ and $x \in B$ and $y \in C$

$\Rightarrow (x, y) \in A \times C$ and $(x, y) \in B \times C$

$\Rightarrow (x, y) \in (A \times C) \cap (B \times C)$

$\therefore (A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$

<u>Case 2</u>: WTS $(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$

let $(x, y) \in (A \times C) \cap (B \times C)$

$\Rightarrow (x, y) \in A \times C$ and $(x, y) \in B \times C$

$\Rightarrow x \in A$ and $y \in C$ and $x \in B$

$\Rightarrow x \in A \cap B$ and $y \in C$

$\Rightarrow (x, y) \in (A \cap B) \times C$

$\therefore (A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$

$\therefore$ The sets are equal.

2.1 Equivalence Relations

$R = \{(a,b) \mid aRb$ if a and b have the same parents $\}$

Reflexive? ✓

aRa as a and have the same parents.

Symmetric?

Let $(a,b) \in R \Rightarrow$ a and b have the same parents
$\Rightarrow$ b and a have the same parents
$\Rightarrow (b,a) \in R$

R is symmetric

Transitive?

Let $(a,b), (b,c) \in R$.
$\Rightarrow$ a and b have the same parents.
$\Rightarrow$ b and c have the same parents
$\Rightarrow$ a and c have the same parents
$\Rightarrow (a,c) \in R$.
$\therefore$ R is transitive.

R is an $\boxed{\text{equivalence}}$ relation

Definition :- An equivalence relation on a set A is a binary relation R on A that is reflexive, symmetric, and transitive.

<u>Notations</u>

a is related to b via the relation R.

① aRb

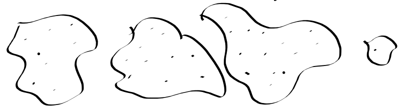
② $(a,b) \in R$

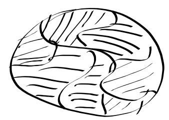
③ $a \underset{R}{\sim} b$

<u>Q</u> $R = \{ \underset{\in A \times A}{(a,b)} \mid$ a and b are related if a and b are from the same state in the US $\}$

Check that R is an equivalence relation.

$a \sim b$





<u>Definition</u> :- Given a set A, a <u>partition</u> of this set into subsets $A_1, A_2, A_3, \dots, A_n$ satisfies the properties

① $A_1 \cup A_2 \cup \dots \cup A_n = A$

② $A_i \cap A_j = \emptyset$ for any $i \neq j$

An equivalence relation R <u>partitions</u> the set A into <u>equivalence classes</u>.

$R \subseteq A \times A$

Equivalence Class.

Given a relation R on a set A, the equivalence class of $a \in A$ is the set of elements of A that are related to a., i.e.

$$\bar{a} = \left\{ x \in A \;\middle|\; x \sim a \right\}$$

$\overline{OR} \cup \overline{CA}$ = All the students

$\overline{OR} \cap \overline{CA} = \phi$

$A = \{1, 2, 3, 4\}$

$A_1 = \{1\}$

$A_2 = \{2, 3\}$ $\quad A_1, A_2, A_3$ partitions A.

$A_3 = \{4\}$

Proposition :- Let $\sim$ denote an equivalence relation on a set A. Let $a \in A$. Then, for any $x \in A$, $x \sim a$ iff $\bar{x} = \bar{a}$

Proposition :- Suppose $\sim$ denotes an equivalence relation on A, and let $a, b \in A$. Then the equivalence classes $\bar{a}$ and $\bar{b}$ are either the same or disjoint, i.e. either $\bar{a} = \bar{b}$ or $\bar{a} \cap \bar{b} = \phi$.

Proofs by contradiction

$\bar{a} \cap \bar{b} \neq \phi$

Let $x \in \bar{a} \cap \bar{b}$

$x \in \bar{a}$ and $x \in \bar{b}$

$\Rightarrow x \sim a$ and $x \sim b$

$\Rightarrow a \sim x$ and $x \sim b$

$\Rightarrow a \sim b$ contradiction

2(a) $S=\{1,2,3\}$

$R=\{(1,1),(1,2),(3,2),(3,3),(2,3),(2,1)\}$

Why is R not an equivalence relation on S?

R is not reflexive as $(2,2)\notin R$

S = Set of all students

$a\sim b$ iff a and b are from the same state.

$S/\sim = \{\underbrace{\overline{AK}}, \underbrace{\overline{OR}}, \underbrace{\overline{CA}} \ldots \}$

17 $(x,y), (u,v) \in \mathbb{R}^2$

$(x,y)\sim(u,v)$ iff $x^2-y^2=u^2-v^2$

Show that $\sim$ is an equivalence relation

① $x^2-y^2=x^2-y^2 \Rightarrow (x,y)\sim(x,y)$

$\sim$ is reflexive.

② suppose $(x,y)\sim(u,v)$

then, $x^2-y^2=u^2-v^2$

then $u^2-v^2=x^2-y^2$

$\Rightarrow (u,v)\sim(x,y)$ $\therefore \sim$ is symmetric

③ Suppose $(x,y)\sim(u,v)$ and $(u,v)\sim(r,s)$

where $(x,y),(u,v),(r,s)\in\mathbb{R}^2$

$\therefore x^2-y^2=u^2-v^2$ and $u^2-v^2=r^2-s^2$

$\Rightarrow x^2-y^2=r^2-s^2$

$\Rightarrow (x,y)\sim(r,s)$ $\therefore \sim$ is transitive.

$\therefore \sim$ is an equivalence relation.

Find $\overline{(0,0)}$, $\overline{(1,0)}$

$$\overline{(0,0)} = \{ (x,y) \in \mathbb{R}^2 \mid (x,y) \sim (0,0) \}$$

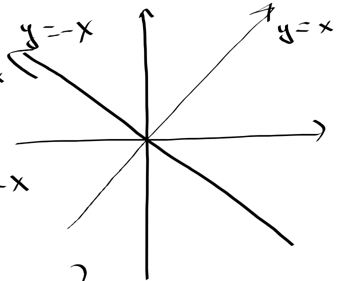
$\Rightarrow x^2 - y^2 = 0^2 - 0^2 = 0$

$\Rightarrow x^2 - y^2 = 0$

$x^2 - y^2 = 0$

$y^2 = x^2$

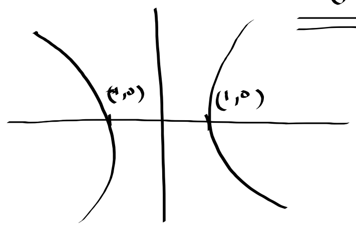
$y = \pm x$ — $y = x$, $y = -x$

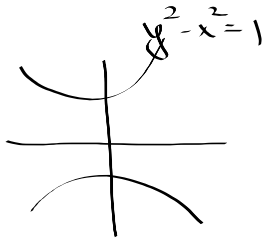


$$\overline{(1,0)} = \{ (x,y) \in \mathbb{R}^2 \mid (x,y) \sim (1,0) \}$$

$\Rightarrow x^2 - y^2 = 1^2 - 0^2 = 1$

$\Rightarrow \underline{\underline{x^2 - y^2 = 1}}$





7. $a, b \in \mathbb{R}$

$a \sim b$ iff $a - b \in \mathbb{Z}$

(a) $\sim$ is an equivalence relation on $\mathbb{Z}$

(b) $\overline{5} = \{x \in \mathbb{R} \mid x \sim 5\}$

$\Rightarrow x - 5 \in \mathbb{Z}$

$\Rightarrow x - 5 = n$ where $n \in \mathbb{Z}$

$\Rightarrow x = n + 5$

$\Rightarrow x \in \mathbb{Z}$

$\overline{a} = \mathbb{Z}$ iff $a \in \mathbb{Z}$

$\overline{5\frac{1}{2}} = \{x \in \mathbb{R} \mid x \sim 5\frac{1}{2}\}$

$x - \frac{11}{2} \in \mathbb{Z}$

$x - \frac{11}{2} = n$ for some $n \in \mathbb{Z}$

$x = n + \frac{11}{2}$ for some $n \in \mathbb{Z}$

$= \underbrace{(n+5)}_{m} + \frac{1}{2}$

$\underline{m + \frac{1}{2}}$

$\overline{5\frac{1}{2}} = \{m + \frac{1}{2} \mid m \in \mathbb{Z}\}$