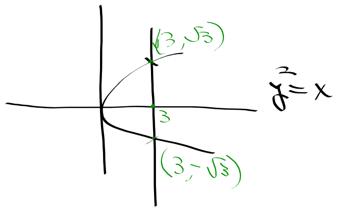
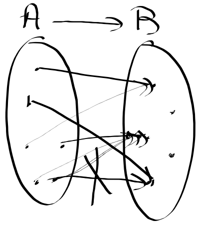
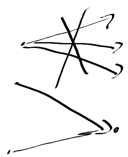
Chapter 3
Functions





Definition :- A function from a set A to a set B is a binary relation f from A to B with the property that for every $a \in A$, there is exactly one $b \in B$ such that $(a, b) \in f$.



$f = \{(1, x), (2, y), (2, z)\}$

f is not a function because 2 maps to both y and z.

$f : \mathbb{R} \to \mathbb{R}^+$

$f(x) = x^2$

$f(1) = 1 \quad (1, 1)$

Domain

The domain of a function which maps elements in set A to set B, is the set A.

Range

The range of a function which maps elements in set A to elements in set B, is the subset of set B whose elements have been mapped by some elements from A.

$$\text{Range} = \{ y \in B \mid f(x) = y \text{ for some } x \in A \}$$

$\therefore$ $y \in \text{Range}(f)$ iff $f(x) = y$ for some $x \in A$.

iff $(x, y) \in f$ for some $x \in A$.

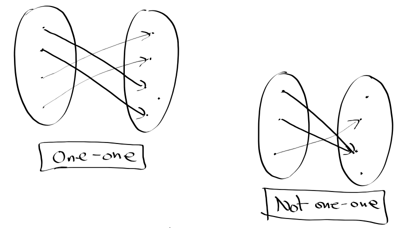
Image of an element

Given a function $f: A \to B$, and an element $x \in A$. The image of x denoted by $\text{im}(x)$ is the element $f(x)$.

$$f: \mathbb{R} \to \mathbb{R}$$
$$x \to x^2$$

$\text{Domain}(f) = \mathbb{R}$

$\text{Range}(f) = [0, \infty)$



Definition: A function $f: A \to B$ is a one-one function if every element $x \in A$ maps to a unique $y \in B$.

How do we show a function is one-one?

A function is one-one iff whenever $f(x_1) = f(x_2)$ then $x_1 = x_2$ for any $x_1, x_2 \in A$.

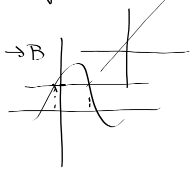
$f(x_1) = f(x_2)$

$\vdots$

$x_1 = x_2$

Q Determine if the function $f(x) = 3x+4$ is one-one or not.

$f: A \to B$



Ans Let $x_1, x_2 \in A$.

Suppose $f(x_1) = f(x_2)$

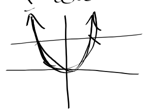
$\Rightarrow 3x_1 + \cancel{4} = 3x_2 + \cancel{4}$

$\Rightarrow 3x_1 = 3x_2$

$\Rightarrow x_1 = x_2$

$\therefore f$ is one-one.

Q What about the function $f(x) = x^2$



Let $x_1, x_2 \in A$

Suppose $f(x_1) = f(x_2)$

$x_1^2 = x_2^2$

$x_1 = \pm x_2$

$\Rightarrow x_1 = x_2$ and $x_1 = -x_2$

As $x_1 = -x_2$ hence f is not one-one.

Counterexample
$f(3) = f(-3) = 9$
but $3 \neq -3$
hence f is not one-one

$a \sim b$ iff $a^2 - b^2$ is divisible by 3.

① Reflexive $a^2 - a^2 = 0$

and $3 | 0$

$\therefore a \sim a$ $\therefore R$ is reflexive.

② Symmetric Suppose $a \sim b$

$\Rightarrow 3 | (a^2 - b^2)$

$3 | b^2 - a^2$

$\Rightarrow a^2 - b^2 = 3k$ for some $k \in \mathbb{Z}$

$\Rightarrow b^2 - a^2 = -3k$

$\Rightarrow b^2 - a^2 = 3(-k)$

$\Rightarrow 3 | (b^2 - a^2)$ as $-k \in \mathbb{Z}$.

$\Rightarrow b \sim a$. Hence R is symmetric

③ Transitive : Suppose $a \sim b$ and $b \sim c$

$\Rightarrow 3 | (a^2 - b^2)$ and $3 | (b^2 - c^2)$

$\Rightarrow a^2 - b^2 = 3k$ and $b^2 - c^2 = 3m$ for some $k, m \in \mathbb{Z}$

$\Rightarrow (a^2 - \cancel{b^2}) + (\cancel{b^2} - c^2) = 3k + 3m$

$\Rightarrow a^2 - c^2 = 3\underbrace{(k+m)}$

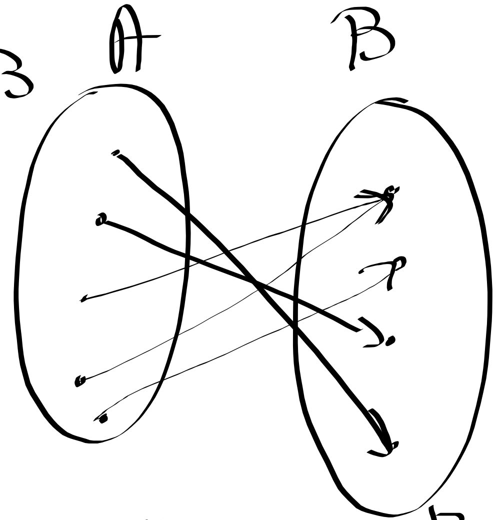
$\Rightarrow 3 | (a^2 - c^2)$ as $k + m \in \mathbb{Z}$

$\Rightarrow a \sim c$

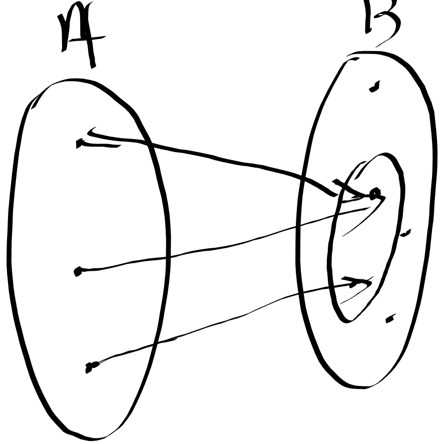
$\therefore R$ is transitive.

Onto Function

Let $f: A \to B$ be a function. f is onto if for every $y \in B$ there exists an $x \in A$, such that $f(x) = y$.



Theorem:- Let $f: A \to B$ be a function. Then f is onto iff $\text{Range}(f) = B$.



Q $f(x) = 4x - 2$.

Is f one-one? Is f onto?

$f: \mathbb{R} \to \mathbb{R}$

(1) Suppose $f(x_1) = f(x_2)$ $x_1, x_2 \in \mathbb{R}$

$\Rightarrow 4x_1 - \cancel{2} = 4x_2 - \cancel{2}$

$\Rightarrow \cancel{4}x_1 = \cancel{4}x_2$

$\Rightarrow x_1 = x_2$

Hence, f is one-one.

(2) Given $y \in \mathbb{R}$, does there exist $x \in \mathbb{R}$

such that $f(x) = 4x - 2 = y$

$4x = 2 + y$

$x = \dfrac{2+y}{4} \in \mathbb{R}$

$\therefore$ f is onto.

A function f is bijective if it is both one-one and onto.

Identity function

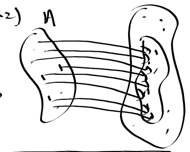
$f : A \to B$

$f(x) = x$ for all $x \in A$.

Suppose $f(x_1) = f(x_2)$

$x_1 = x_2$

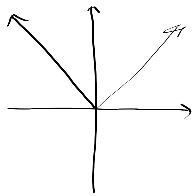
one-one



f is not onto unless $B = A$.

Absolute Value Function

$$|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$



18 $f : A \to B$

$f(x) = x^2 + 14x - 51$

(a) $A = \mathbb{N}$, $B = \{ b \in \mathbb{Z} \mid b \geq -100 \}$

Determine if f is one-one and if f is onto.

Suppose $f(x_1) = f(x_2)$

$x_1^2 + 14x_1 \cancel{-51} = x_2^2 + 14x_2 \cancel{-51}$

$(x_1^2 - x_2^2) + (14x_1 - 14x_2) = 0$

$(x_1 - x_2)(x_1 + x_2) + 14(x_1 - x_2) = 0$

$(x_1 - x_2)(x_1 + x_2 + 14) = 0$

$\Rightarrow x_1 = x_2$, $x_1 + x_2 + 14 \neq 0$ as $x_1, x_2 \in \mathbb{N}$

$\therefore$ f is one-one.

Is f onto?

Is range (f) $= B$?

Given $y \in B$, does there exist $x \in \mathbb{N}$

such that $f(x) = x^2 + 14x - 51 = y$?

let $y = -100 \Rightarrow x^2 + 14x - 51 = -100$

$x^2 + 14x = -49$

As $x \in \mathbb{N}$ so $x > 0$

$\Rightarrow x^2 + 14x > 0$

which cannot equal a negative number.

$\therefore$ f is not onto.

If $y = 1$

$x^2 + 14x - 51 = 1$

$x^2 + 14x - 52 = 0$

$$x = \frac{-14 \pm \sqrt{14^2 + 208}}{2}$$

Proposition :- A function f has an inverse function f^{-1} iff f is both one-one and onto.

$$f = \{(1, x), (2, y), (3, z)\}$$

Find f^{-1} if it exists.

f is one-one and onto. So f^{-1} exists.

$$f^{-1} = \{(x, 1), (y, 2), (z, 3)\}$$

$$f = \{(1, x), (2, y), (3, y)\}$$

f is not one-one, hence f^{-1} does not exist.

$f(x) = 3x + 4$ $\quad f: \mathbb{R} \to \mathbb{R}$

Find f^{-1} if it exists.

Is f one-one?

Let $f(x_1) = f(x_2)$ for $x_1, x_2 \in \mathbb{R}$

$$3x_1 + \cancel{4} = 3x_2 + \cancel{4}$$
$$3x_1 = 3x_2$$
$$x_1 = x_2$$

$\therefore$ f is one-one.

Is f onto?

Given $y \in \mathbb{R}$, does there exist $x \in \mathbb{R}$ such that

$$3x + 4 = y ?$$
$$3x = y - 4$$
$$x = \frac{y-4}{3} \in \mathbb{R}$$

$\therefore$ f is onto.

$$f(x) = 3x + 4$$
$$y = 3x + 4$$

Switch $\rightarrow$ $x = 3y + 4$

$$x - 4 = 3y$$

Solve $\longrightarrow$ $y = \frac{x-4}{3}$

$$\boxed{f^{-1}(x) = \frac{x-4}{3}}$$

1 $a \sim b$ iff a^2+b is even

$a, b \in \mathbb{N}$

① Reflexive - Let $a \in \mathbb{N}$

$$a^2+a = a(a+1)$$

Since a and $a+1$ are consecutive integers hence one of them is even, and so the product $a(a+1)$ is also even.

$\therefore a \sim a$. Hence $\sim$ is reflexive

② Symmetric - Let $a \sim b$

$\Rightarrow a^2+b$ is even.

$\Rightarrow a^2$ and b either both even or both odd.

Case 1: a^2 and b are both even.

a^2 even $\Rightarrow a$ is even.

b even $\Rightarrow b^2$ is even

$\Rightarrow b^2+a$ is even $\Rightarrow b \sim a$

Case 2: a^2 and b are both odd

a^2 odd $\Rightarrow a$ is odd

b odd $\Rightarrow b^2$ is odd

$\therefore b^2+a$ is even

$\Rightarrow b \sim a$

$\therefore \sim$ is symmetric

③ Transitive - Let $a \sim b$ and $b \sim c$, $a,b,c \in \mathbb{N}$.

$\Rightarrow a^2+b$ is even and b^2+c is even.

$\Rightarrow a^2$ and b are both even or both odd | and b^2 and c have the same parity

both even or both odd

Case 1: a^2 and b are both even.

$\Rightarrow b$ even $\Rightarrow b^2$ is even.

$\Rightarrow c$ is also even as b^2 and c have the same parity

$\Rightarrow a^2+c$ is also even

$\Rightarrow a \sim c$

Case 2 a^2 and b are both odd

b odd $\Rightarrow b^2$ is odd

$\therefore c$ is odd as b^2 and c have the same parity

$\Rightarrow a^2+c$ is even

$\Rightarrow a \sim c$

$\therefore \sim$ is transitive.

$$(f \circ g)(x) = f(g(x))$$

"of"

$$(f \circ g \circ h)(x)$$

$f(x) = \frac{1}{x-4}$ $\quad f(g(h(x)))$

$$h: A \to B$$

$$\forall \quad x \in A,\ h(x) \in \mathrm{Dom}(g),\ g(h(x)) \in \mathrm{Dom}(f)$$

$\forall$ "for every"

$\exists$ "there exists"

$$(f \circ f^{-1})(x) = x$$

$$f(f^{-1}(x)) = x$$

$$f(x) = \frac{2x-1}{5}$$

$$y = \frac{2x-1}{5}$$

$$x = \frac{2y-1}{5}$$

$$5x+1 = 2y$$

$$y = \frac{5x+1}{2}$$

$$f^{-1}(x) = \frac{5x+1}{2}$$

$$f(f^{-1}(x))$$

$$= f\left(\frac{5x+1}{2}\right)$$

$$= \frac{\cancel{2}\left(\frac{5x+1}{\cancel{2}}\right) - 1}{5}$$

$$= \frac{5x+\cancel{1}-\cancel{1}}{5} = \frac{5x}{5} = x$$

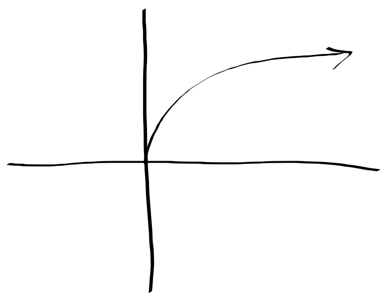
$$(f^{-1} \circ f)(x) = x$$

$$f^{-1}(f(x)) = f^{-1}\left(\frac{2x-1}{5}\right)$$

$$= \frac{\cancel{5}\left(\frac{2x-1}{\cancel{5}}\right)+1}{2}$$

$$= \frac{2x-\cancel{1}+\cancel{1}}{2}$$

$$= \frac{2x}{2} = x$$



$f: \underset{\sim}{y = \sqrt{x}}$

$f(x) = \sqrt{x}$

$y = \sqrt{x}$

$x = \sqrt{y}$

$f: [0,\infty) \to [0,\infty)$ $x^2 = y$ $f^{-1}: \underset{\sim}{\mathbb{R}} \to [0,\infty)$

$f^{-1}(x) = x^2$

$f \circ f^{-1}(x)$ ✓ $(f^{-1} \circ f)(x)$

$f: \underset{\sim}{[0,\infty)} \to \underset{\sim}{\mathbb{Z}}$ $f(x) = 1$

$g: \underset{\sim}{\mathbb{Z}} \to \mathbb{R}$ $g(x) = x-2$

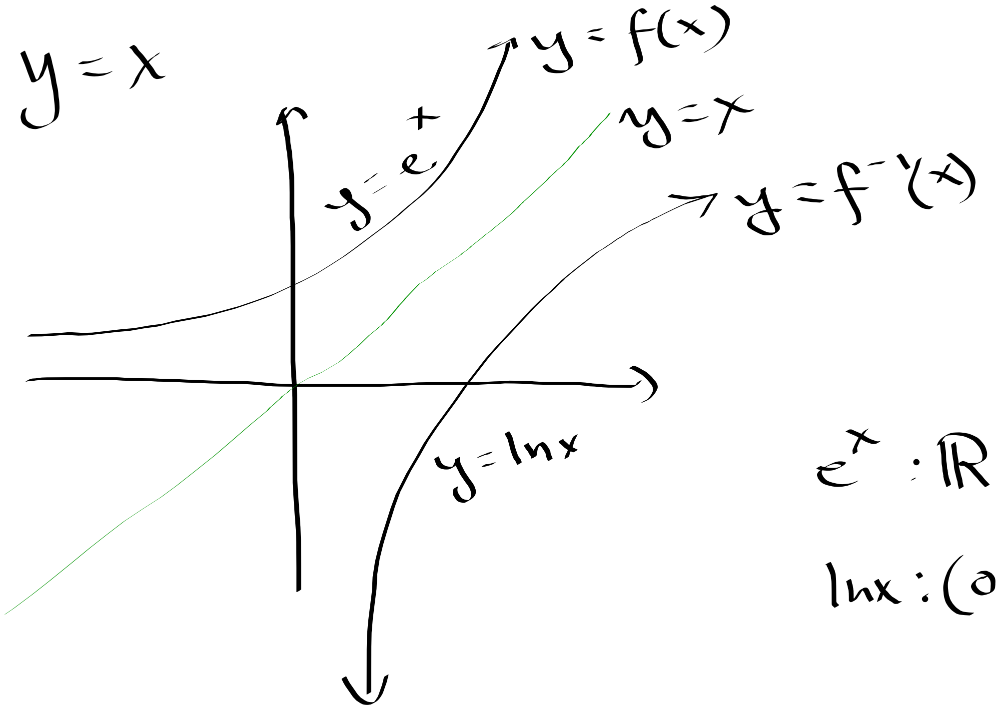
$(g \circ f)(x)$ ✓ $(f \circ \underset{\sim}{g})(\mathbb{Z}x)$ does not need to be defined.

$f(g(-3))$

$f(-5)$ is not defined

What is the relationship between the graphs of $y=f(x)$ and $y=f^{-1}(x)$?

The graphs are symmetric about the line $y=x$



$e^x : \mathbb{R} \to (0,\infty)$

$\ln x : (0,\infty) \to \mathbb{R}$

Show

$(A-B)\cup(C-B) = (A\cup C)-B$

Case 1 WTS $(A-B)\cup(C-B) \subseteq (A\cup C)-B$

let $x \in (A-B)\cup(C-B)$

$\Rightarrow x \in A-B$ or $x \in C-B$

$\Rightarrow (x\in A \text{ and } x \notin B)$ or $(x\in C \text{ and } x\notin B)$

Case 1(a) $x\in A$ and $x\notin B$

$\Rightarrow x\in A\cup C$ and $x\notin B$

$\Rightarrow x \in (A\cup C)-B$

Case 1(b) $x\in C$ and $x\notin B$

$\Rightarrow x\in A\cup C$ and $x\notin B$

$\Rightarrow x\in(A\cup C)-B$

$x\in A$

$x\in A\cup \cancel{X}$

$\therefore (A-B)\cup(C-B) \subseteq (A\cup C)-B$

Case 2 WTS $(A\cup C)-B \subseteq (A-B)\cup(C-B)$

let $x\in(A\cup C)-B$

$\Rightarrow x\in(A\cup C)$ and $x\notin B$

$\Rightarrow (x\in A$ or $x\in C)$ and $x\notin B$

Case 2(a) $x\in A$ and $x\notin B$

$\Rightarrow x\in A-B$

$\Rightarrow x\in(A-B)\cup(C-B)$

Case 2(b) $x\in C$ and $x\notin B$

$\Rightarrow x\in C-B$

$\Rightarrow x\in(A-B)\cup(C-B)$

$\therefore (A\cup C)-B \subseteq (A-B)\cup(C-B)$

Hence, the sets are equal.