$$\overline{a} = \{ x \in \mathbb{R} \mid x \sim a \}$$

$0 \leq a < 1$

$$\overline{3.2} = \{ x \in \mathbb{R} \mid x \sim 3.2 \}$$

x

5.2 7.2

$\overline{4.1}$

$x = n + a$

0.2

3.3 One-one correspondence and the cardinality of a set

$\boxed{\text{Set Theory}}$

$A = \{1, 2, 3\}$

$|A| = 3$

$A = (0,1)$ $B = (2, 5)$

$|A| = ?$ $|B| = ?$

$|A| = |B|$

Definition :- The cardinality of a set A is the number of elements in the set A and is denoted by $|A|$.

---

one-one correspondence $\Leftrightarrow$ A function f that is bijective.
i.e. both one-one and onto.

Definition :- Sets A and B have the same cardinality iff A and B has a one-one correspondence between them.

i.e. we can find a bijective function $f: A \to B$.

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$2\mathbb{Z} = \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}$

$\mathbb{Z}$ and $2\mathbb{Z}$ have the same cardinality.

$f: \mathbb{Z} \to 2\mathbb{Z}$

$f(n) = 2n$

<u>One-one</u>

let $f(n_1) = f(n_2)$, $n_1, n_2 \in \mathbb{Z}$

$2n_1 = 2n_2$

$n_1 = n_2$

$\therefore f$ is one-one.

<u>Onto</u>

let $y \in 2\mathbb{Z}$

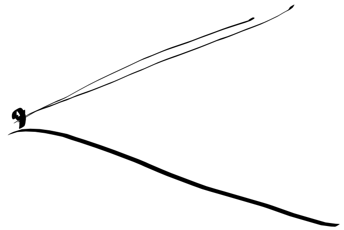
$\Rightarrow y = 2n$ for some $n \in \mathbb{Z}$

$\Rightarrow n = \frac{y}{2} \in \mathbb{Z}$

$\therefore f\left(\frac{y}{2}\right) = y$

Hence f is onto.

$\therefore f$ is bijective and hence $\mathbb{Z}$ and $2\mathbb{Z}$ have the same cardinality.



Check for well-defined function.

Let $x_1 = x_2$

Find out if $f(x_1) = f(x_2)$

Let $n_1 = n_2$

$f(n_1) = 2n_1$ $\qquad$ $f(n_2) = 2n_2$

$\quad = 2n_2 = f(n_2)$

$\therefore$ f is well-defined.

Proposition :- Any two open intervals in $\mathbb{R}$ have the same cardinality.

i.e. (a,b) and (c,d) have the same cardinality.

$$f : (0,1) \to (4,8)$$

$f(0) = 4$
$f(1) = 8$

$f(x) = kx + l$ $\quad$ k, l have to be found out.

$f(0) = k \cdot 0 + l = 4$
$\Rightarrow l = 4$

$f(1) = k \cdot 1 + l$
$\Rightarrow k + 4 = 8$
$\Rightarrow k = 4$

$\therefore f(x) = 4x + 4$

$f : (0,1) \to (4,8)$
$f(x) = 4x + 4$

One-one $\quad$ Let $f(x_1) = f(x_2)$, $x_1, x_2 \in (0,1)$
$4x_1 + \cancel{4} = 4x_2 + \cancel{4}$
$x_1 = x_2$

$\therefore f$ is one-one

Onto $\quad$ Let $y \in (4,8)$
$4x + 4 = y$
$x = \frac{y-4}{4} \in (0,1)$

$4 \leq y \leq 8$
$0 \leq y - 4 \leq 4$
$0 \leq \frac{y-4}{4} \leq 1$
$0 \leq x \leq 1$

$\therefore f\left(\frac{y-4}{4}\right) = y$.

$\therefore f$ is onto.

Hence, $(0,1)$ and $(4,8)$ have the same cardinality.

Q Show that $(0,1)$ and (a,∞) has the same cardinality for any $a \in \mathbb{R}$.

A.

$0 \to a$

$1 \to \infty$

$$f(x) = \frac{1}{1-x} - 1 + a$$

$$g(x) = \frac{a}{1-x}$$

<u>One-one</u> Let $f(x_1) = f(x_2)$

$$\frac{1}{1-x_1} \cancel{-1} + \cancel{a} = \frac{1}{1-x_2} \cancel{-1+a}$$

$$\cancel{1} - x_2 = \cancel{1} - x_1$$

$$x_1 = x_2$$

$\therefore f$ is one-one.

<u>One-one</u> Let $g(x_1) = g(x_2)$

$$\frac{a}{1-x_1} = \frac{a}{1-x_2}$$

$a \neq 0$ then g is one-one.

<u>Onto</u>: Let $y \in (a,\infty)$

$$\frac{1}{1-x} - 1 + a = y$$

$$\frac{1}{1-x} = y + 1 - a$$

$$1 - x = \frac{1}{y+1-a}$$

$$x = 1 - \frac{1}{y+1-a} \in (0,1)$$

$a \leq y \leq \infty$

$1 \leq y + 1 - a \leq \infty$

$\frac{1}{1} \geq \frac{1}{y+1-a} \geq 0$

$-1 \leq -\frac{1}{y+1-a} \leq 0$

$0 \leq 1 - \frac{1}{y+1-a} \leq 1$

$1 \leq 3 \leq 5$

$1 \geq \frac{1}{3} \geq \frac{1}{5}$

Hence $f\left(1 - \frac{1}{y+1-a}\right) = y$

$\therefore f$ is onto.

Definition :- A set A is countably infinite or countable iff $|A| = |\mathbb{N}|$. A set that is not countable is uncountable.

Standard Notation

$$|\mathbb{N}| = \aleph_0 \text{ (aleph naught)}$$

$\mathbb{Z}$ is countable

$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

$$f : \mathbb{N} \to \mathbb{Z}$$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ even} \\ -\left(\frac{n-1}{2}\right) & \text{if } n \text{ odd.} \end{cases}$$

$\mathbb{Z}: 0, 1, -1, 2, -2, 3, -3, \ldots$

$\mathbb{N}: 1\ 2\ 3\ 4\ 5\ 6\ 7\ \ldots$

$\boxed{\mathbb{Q} \text{ are countable}}$

$f : \mathbb{N} \to \mathbb{Q}$ Quite hard.

a \ b	1	2	3	4	5
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
2	$\frac{2}{1}$	$\cancel{\frac{2}{2}}$	$\frac{2}{3}$	$\cancel{\frac{2}{4}}$	$\frac{2}{5}$
3	$\frac{3}{1}$	$\frac{3}{2}$	$\cancel{\frac{3}{3}}$	$\frac{3}{4}$	$\frac{3}{5}$
4	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$

$$|\mathbb{N} \times \mathbb{N}| = \aleph_0$$

$$|\mathbb{N} \times \mathbb{N}| = |\mathbb{Q}|$$

$\mathbb{Q}$

$\frac{a}{b}$