$$a_{\underline{n+1}} = 3a_{\underline{n}} - 4a_{\underline{n-1}}$$

$$a_n = 3a_{n-1} - 4a_{n-3}$$

$$\underbrace{a_n = r\,a_{n-1} + s\,a_{n-2} + f(n)}_{}$$

2^{nd} order linear recurrence relation.

If $f(n) = 0$, then the above recurrence relation is homogenous.

Non-homogenous relations (i.e. $f(n) \neq 0$)

$$y'' + 4y' + 3y = 0$$

$$r^2 + 4r + 3 = 0$$

$$(r+3)(r+1) = 0 \qquad y(x) = c_1 e^{-3x} + c_2 e^{-x}$$

$$r = -1, -3$$

$$y'' + 4y' + 3y = 2x + 1$$

$$y'' + 4y' + 3y = 0$$

Theorem :- Let p_n be a particular solution of the recurrence relation $a_n = r a_{n-1} + s a_{n-2} + f(n)$ ignoring initial conditions. Let q_n be the general solution of the corresponding homogeneous equation $a_n = r a_{n-1} + s a_{n-2}$ ignoring initial conditions. Then $\boxed{a_n = p_n + q_n}$ is the general solution of the recurrence relation

$$a_n = r a_{n-1} + s a_{n-2} + f(n).$$

The initial conditions are now used to find the coefficients of q_n.

13(b) $a_n = 5a_{n-1} - 6a_{n-2} + 3n$, $n \geq 2$

$a_0 = 2$
$a_1 = 14$

Homogeneous Relation

$$a_n = 5a_{n-1} - 6a_{n-2}$$

Characteristic Eqn

$$x^2 - 5x + 6 = 0$$
$$(x-3)(x-2) = 0$$
$$x = 2, 3$$
$$\therefore q_n = c_1 2^n + c_2 3^n$$

Particular Solution

Depends on the function $f(n)$

Choose p_n to be of the same "type" as $f(n)$.

$p_n = an + b$ as $f(n) = 3n$

$$a_n = 5a_{n-1} - 6a_{n-2} + 3n$$

As p_n is a solution, hence

$$p_n = 5p_{n-1} - 6p_{n-2} + 3n$$
$$an + b = 5(a(n-1)+b) - 6(a(n-2)+b) + 3n$$
$$an + b = 5an - 5a + 5b - 6an + 12a - 6b + 3n$$
$$an + b = -an + 7a - b + 3n$$
$$an + b = (7a - b) + n(3 - a)$$

Equating coefficients

$$3 - a = a \Rightarrow 2a = 3 \Rightarrow a = 3/2$$
$$7a - b = b \Rightarrow 7a = 2b \Rightarrow b = \frac{7a}{2} = \frac{21}{4}$$

$$\therefore p_n = \frac{3n}{2} + \frac{21}{4}$$

$$\therefore a_n = p_n + q_n$$
$$\therefore a_n = \frac{3n}{2} + \frac{21}{4} + c_1 2^n + c_2 3^n$$

Now use the initial conditions $a_0 = 2$ and $a_1 = 14$ to find c_1 and c_2.

$$a_0 = \frac{21}{4} + c_1 + c_2 = 2 \Rightarrow c_1 + c_2 = 2 - \frac{21}{4} = \frac{-13}{4} \quad - \times(-2)$$
$$a_1 = \frac{3}{2} + \frac{21}{4} + 2c_1 + 3c_2 = 14 \Rightarrow 2c_1 + 3c_2 = 14 - \frac{21}{4} - \frac{3}{2}$$
$$= 14 - \frac{27}{4}$$
$$2c_1 + 3c_2 = \frac{29}{4}$$
$$-2c_1 - 2c_2 = \frac{26}{4}$$

Adding $c_2 = \frac{55}{4}$

$$\Rightarrow c_1 = \frac{-13}{4} - c_2 = \frac{-13}{4} - \frac{55}{4} = -\frac{68}{4} = -17$$

$$\therefore \boxed{a_n = \frac{3}{2}n + \frac{21}{4} - 17 \cdot 2^n + \frac{55}{4} \cdot 3^n}$$

$$a_0 = \frac{21}{4} - 17 + \frac{55}{4} = \frac{76}{4} - 17 = 19 - 17 = 2 \checkmark$$

$$a_1 = \frac{3}{2} + \frac{21}{4} - (17 \cdot 2) + \left(\frac{55}{4}\right)3$$
$$= \frac{27}{4} - 34 + \frac{165}{4}$$
$$= \frac{192}{4} - 34$$
$$= 48 - 34 = 14 \checkmark$$

How to determine the correct particular solution p_n.

If $f(n) = 2^n$ then $p_n = a \cdot 2^n$

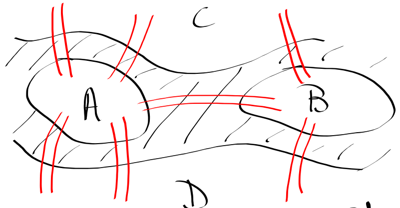
If $f(n) = n^2 \cdot 2^n$ then $p_n = (an^2 + bn + c) \cdot 2^n$

If $f(n) = n^3$ then $p_n = an^3 + bn^2 + cn + d$

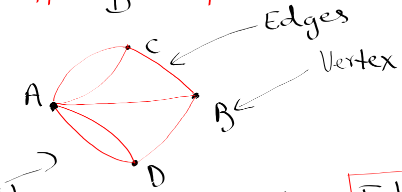
If $f(n) = 5$ then $p_n = a$

Graphs

The bridges of Königsberg



Euler

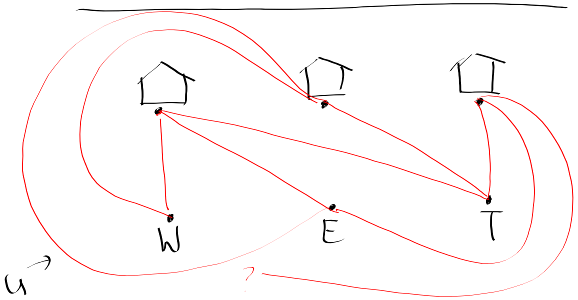


Graph

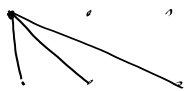
Does there exist an Eulerian ~~path.~~ cycle?

No

Three Houses and Three Utility Problem



No

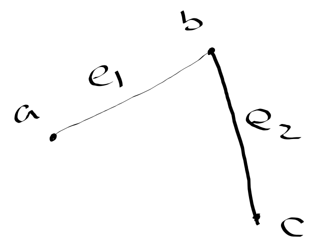


Is G a planar graph?

Definition :-

A graph is a pair (V, E) of sets V (non-empty), and E a set of two distinct elements of V. The elements of V are called vertices, and the elements of E are called edges.

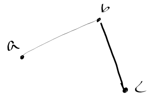
Ex:- $V = \{a, b, c\}$ $E = \{ab, bc\}$



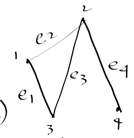
Two vertices are adjacent if they are the end vertices of an edge.

In the previous example vertices a and c are not adjace... ...er, vertices a and b are adjacent. So is b ... c.

Two edges are adjacent if they have a vertex in common.

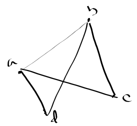


ab and ac are adjacent edges (b is the common vertex)



e_1 and e_4 are not adjacent edges.

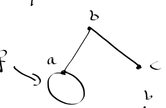
<u>Definition</u> :- The number of edges incident with a vertex v is the degree of the vertex v.



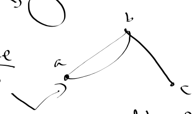
$\deg(a) = 3$
$\deg(b) = 3$
$\deg(c) = 2$
$\deg(d) = 2$

A pseudograph is like a graph, but it may contain loops and/or multiple edges

<u>Loop</u>



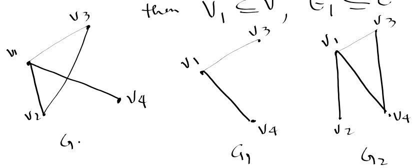
<u>Multiple Edges</u>



<u>Definition</u>:- A graph G_1 is a subgraph of another graph G (parent graph) iff the vertex and edge sets of G_1 are respectively subsets of the vertex and edge sets of G.

$G = (V, E)$ $\quad G_1 = (V_1, E_1)$

then $V_1 \subseteq V$, $E_1 \subseteq E$



G_1 is a subgraph of G.
G_2 is not a subgraph of G. (v_3v_4 is not an edge in G).

$\underbrace{3n^2+3n}$ $+ 6k_1$

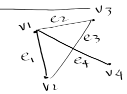
$3(n^2+n) = 3\underbrace{n(n+1)}_{2k}$.

If n is even n^2 is even
n^2+n is even so $n^2+n = 2k$.

If n is odd then n^2 is odd
n^2+n is even so n^2+n is $2k$

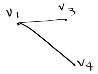
$\therefore 3(n^2+n) = 6k$.

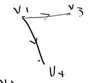
G

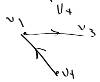


$G \setminus \{v_2\}$

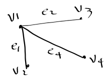
complement



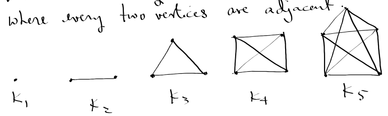




$G \setminus \{e_3\}$



Definition :- A complete graph of n vertices denoted by K_n is that graph where every two vertices are adjacent.



# of edges $K_1 = 0$
" $K_2 = 1 = 1$
" $K_3 = 3 = 1+2$
" $K_4 = 6 = 1+2+3$
" $K_5 = 10 = 1+2+3+4$
"
:
$K_n = \frac{(n-1)n}{2} = 1+2+3+\cdots+(n-1)$

$\frac{n(n+1)}{2}$

$\frac{(n-1)(n)}{2}$

$a_{k+1} = k a_k$ $a_1 = 1$

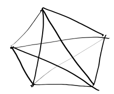
$a_2 = 2a_1 = 2\cdot 1$

$a_3 = 3\cdot a_2 = 3\cdot 2\cdot 1 = 3!$

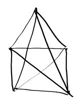
$a_4 = 4\cdot a_3 = 4\cdot 3\cdot 2\cdot 1 = 4!$

:

$a_n = n!$



edges K_5 # edges in $K_4 + 4$



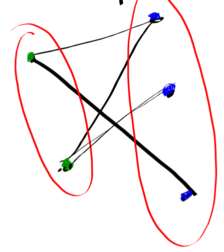
edges K_n = $\underbrace{\text{\# edges in } K_{n-1}}$ $+(n-1)$

$\frac{(n-2)(n-1)}{2} + (n-1)$

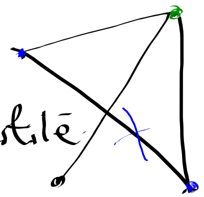
$(n-1)\left[\frac{n-2}{2}+1\right]$

$(n-1)\left[\frac{n}{2}\right] = \frac{(n-1)(n)}{2}$

<u>Definition</u> :- A bipartite graph is a graph whose vertices can be partitioned into two disjoint sets V_1 and V_2 in which there are only edges from vertices between V_1 and V_2.



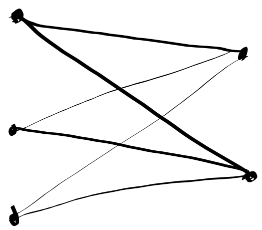
Algorithm to check whether a graph is bipartite.



A complete bipartite graph is a bipartite graph in which every vertex in V_1 is connected to every vertex in V_2 via an edge.

If V_1 has m vertices, and V_2 has n vertices then $K_{m,n}$ is the notation of the complete bipartite graph in this case

$K_{3,2}$



\# edges of $K_{m,n} = mn$

Algorithm

1. Pick two colors. Color vertex v_1 with red and v_2 with blue if there is an edge between the vertices v_1 and v_2. Continue doing this for every edge making sure that an edge has different colored end vertices.

2. If this process terminates with all vertices getting colored, then the graph is bipartite.