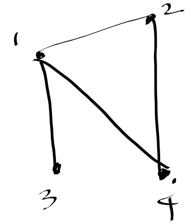
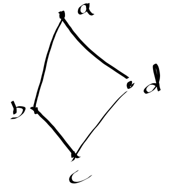
$G_1 = (V_1, E_1)$



$G_2 = (V_2, E_2)$

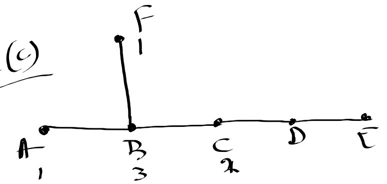


2, 2, 2, 2

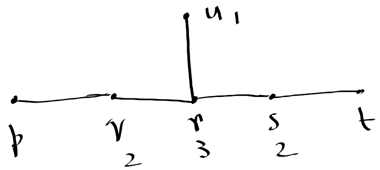
$\varphi(v) = u$

3, 2, 2, 1

4(c)



3, 2, 2, 1, 1, 1



3, 2, 2, 1, 1, 1

$\varphi(B) = r$

Since B and r have the same degree hence if φ is an isomorphism, then $\varphi(B) = r$.
The adjacent vertices of B have degrees 2, 1, 1, whereas the adjacent vertices of r have degrees 2, 2, 1. This implies that such a bijection φ does not exist.
Hence, the graphs are not isomorphic.

10.1 Definitions

Walk - A walk is defined as a sequence of vertices, where consecutive vertices in the sequence are adjacent vertices in the graph.

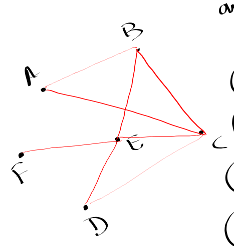
A closed walk is a walk in which the first and last vertices are the same.

Trail - A trail is a walk where no edges are repeated (each edge is used at most once)

A circuit is a closed trail. (A trail in which the first and last vertices are the same)

Path - A path is a walk, where no vertices are repeated (i.e. each vertex is used at most once)

A cycle is a closed path (A path in which the first and last vertices are the same)



1. ABECBE is a walk
2. ABECB is a trail.
3. ABECA is a circuit
4. ABEDC is a path
5. ABEDCA is a cycle

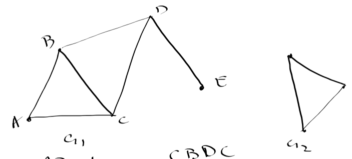
Euler Trail (or Euler Path) - An Euler trail is a walk that uses every edge of a graph exactly once.

Euler Circuit (or Euler Cycle) - An Euler Circuit is an Euler Trail that uses every edge of a graph exactly once.

Note
1. An Euler Trail starts and ends at different vertices
2. An Euler Circuit starts and ends at the same vertex.

A graph is called Eulerian if it contains an Euler circuit.

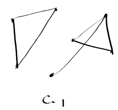
Connected Graph - A graph is connected iff there exists a walk between any two vertices of the graph.



ABCA CBDC

3-cycle 3-cycle

$\varphi(A) = V$
$\varphi(B) = U$



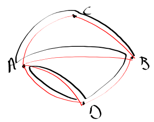
Four-Color Theorem

Question - Is it possible to determine whether a graph has an Euler Trail or Euler Circuit, without necessarily having to find one explicitly?

Euler Circuit -

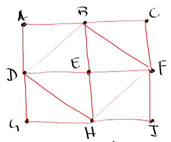
Theorem :- A graph has an Euler Circuit (or is Eulerian) iff it is connected and every vertex is even.

Euler Trail - A graph has an Euler Trail (or Euler Path) iff there are exactly two odd vertices.



BCABDAD

(e)

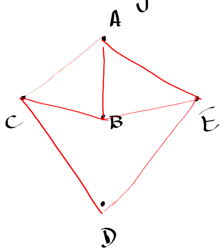


B has odd degree and hence the graph is not Eulerian.

Hamiltonian Cycles

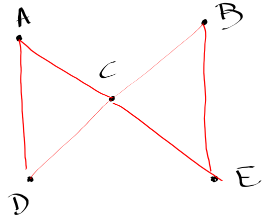
Definition:- A Hamiltonian Cycle in a graph is a cycle that contains every vertex of the graph exactly once. A graph is Hamiltonian if it contains a Hamiltonian cycle.

Note that a Hamiltonian cycle and a Hamiltonian circuit are synonamous.



G_1

DCBAED is a Hamiltonian cycle



G_2

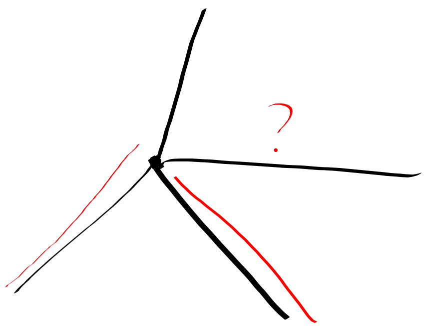
G_2 is not Hamiltonian.

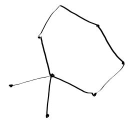
There is no known algorithm which helps in determining if a graph is Hamiltonian unlike Eulerian graphs.

More edges → More likely to be Hamiltonian.

Properties of cycles in graphs.

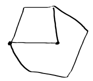
(1) Let C be a cycle in a graph G.
For every vertex v of C, precisely two edges incident with v are in C, hence if C is a Hamiltonian cycle of G and a vertex has degree 2, then both edges incident with v must be part of C.



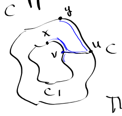


Continued from 5/30/2019

(2) Let C be a cycle in a graph G.
Then the only cycle contained in C is C itself.



Quick Proof: On the contrary, suppose C_1 is a cycle within C

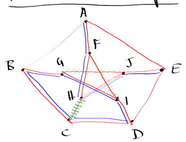


let $y \in C$ but not in C_1

let $x \in C_1$

There exists a path from the vertex y to x.

Peterson's Graph



The Peterson's Graph is not Hamiltonian.

Theorem (Dirac) – If a graph G has $n \geq 3$ vertices and every vertex has degree at least $n/2$, then G is Hamiltonian.

K_n is Hamiltonian.

$\deg(v) = n-1 \geq \frac{n}{2}$

$n = 3$

$P(3)$: LHS: $3-1=2$ $\quad 2 \geq 1.5$ ✓

RHS: $\frac{3}{2} = 1.5$

$P(3)$ is true.

let $P(k)$ be true for $n=k$

i.e $k-1 \geq \frac{k}{2}$

$\boxed{k \geq \frac{k+1}{2}}$

$P(k+1)$ $\quad k \geq \frac{k}{2}+1$

$k \geq \frac{k+2}{2}$

$k \geq \frac{k+1}{2} + \frac{1}{2}$

$\Rightarrow k \geq \frac{k+1}{2}$

$\Rightarrow (k-1)+1 \geq \frac{k+1}{2}$

$\therefore P(k+1)$ is true.

Hence the statement is true for all $n \geq 3$.

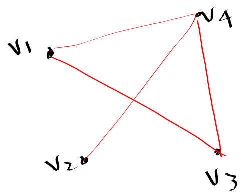
10.3 Adjacency Matrices

Definition :- Let G be a graph with n vertices labeled $v_1, v_2, \ldots, v_n$. For each i and j, with $1 \leq i, j \leq n$, define

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \text{ is an edge} \\ 0 & \text{if } v_i v_j \text{ is not an edge.} \end{cases}$$

The adjacency matrix of G is the $n \times n$ matrix $A = [a_{ij}]$ where the (i,j)th entry is a_{ij}.

Ex :-



$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

a_{ij} Symmetric Matrix

$A^T = A$

① The diagonal entries are all zeros (except when G is a pseudograph)

② If G is an undirected graph, then A is symmetric, i.e. $A^T = A$.