MATH 327
DISCRETE MATHEMATICS
EXAMINATION 1

April 22, 2019

Name (Last, First) Key

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Directions

- There are 6 problems on the test.
- The total points for the test is 40.
- You have a maximum of 50 minutes for this exam.
- You will only get partial credit on problems depending on your solution.
- You must show your work in every problem.

TOTAL POINTS SCORED________________

1. (*6 points*) Let $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 4, 5\}$, and $C = \{3, 4, 5, 6\}$.

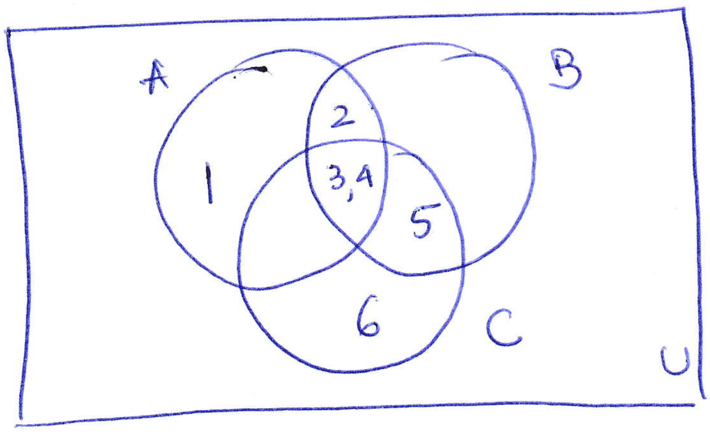
(a) Draw a Venn diagram of these sets with the elements in their proper places.

(b) Find the sets

(i) $A \oplus (B \cap C)$

(ii) $(A^c \cup B)^c$

(a)



(b) $B \cap C = \{3, 4, 5\}$

(i) $\boxed{A \oplus (B \cap C) = \{1, 2, 5\}}$

(ii) $A^c = \{5, 6\}$

$A^c \cup B = \{2, 3, 4, 5, 6\}$

$\boxed{(A^c \cup B)^c = \{1\}}$

2. (*7 points*) Show that for all sets A, B, and C,

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

Show your work.

<u>Case 1</u> :- WTS $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

Let $(x, y) \in A \times (B \cup C)$

$\Rightarrow x \in A$ and $y \in B \cup C$

$\Rightarrow x \in A$ and $(y \in B$ or $y \in C)$

$\Rightarrow (x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$

$\Rightarrow x \in A \times B$ or $x \in A \times C$

$\Rightarrow x \in (A \times B) \cup (A \times C)$

$\therefore A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

<u>Case 2</u> :- WTS $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$

Let $(x, y) \in (A \times B) \cup (A \times C)$

$\Rightarrow (x, y) \in A \times B$ or $(x, y) \in A \times C$

$\Rightarrow (x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$

$\Rightarrow (x \in A$ and $y \in B \cup C)$ or $(x \in A$ and $y \in B \cup C)$

$\Rightarrow (x, y) \in A \times (B \cup C)$ in both cases

$\therefore (A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$

$\therefore A \times (B \cup C) = (A \times B) \cup (A \times C)$

3. (*7 points*) Determine whether the given relation R is reflexive, symmetric, transitive, or none of these. Justify your answers. Show your work.

$$\text{For all } x, y \in \mathbb{R}, \quad x\,R\,y \quad \Leftrightarrow \quad x^2 + y^2 = 1$$

(1) Reflexive - If $x = 5 \in \mathbb{R}$

then $5^2 + 5^2 \neq 1$ or $5 \not{R} 5$.

$\therefore$ $\boxed{R \text{ is not reflexive}}$

(2) Symmetric :- Let $xRy \Rightarrow x^2 + y^2 = 1$

$\Rightarrow y^2 + x^2 = 1$

$\Rightarrow yRx$.

$\therefore$ $\boxed{R \text{ is symmetric.}}$

(3) Transitive :- $\frac{\sqrt{3}}{2} R \frac{1}{2}$ as $\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1$

also, $\frac{1}{2} R \frac{\sqrt{3}}{2}$ as $\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$

but $\frac{\sqrt{3}}{2} \not{R} \frac{\sqrt{3}}{2}$ as $\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 \neq 1$.

$\therefore$ $\boxed{R \text{ is not transitive}}$

4. (*7 points*) For $a, b \in \mathbb{R}$, define $a \sim b$ if and only if $a - b \in \mathbb{Z}$.

(a) Prove that $\sim$ is an equivalence relation.

(b) What is the equivalence class of $6\frac{1}{2}$?

(c) What are all the equivalence classes defined by this relation?

(a) (1) Reflexive $\rightarrow$ $a - a = 0 \in \mathbb{Z}$ $\therefore a \sim a$.

So $\sim$ is reflexive.

(2) Symmetric $\rightarrow$ let $a \sim b$

$\Rightarrow a - b \in \mathbb{Z}$

$\Rightarrow a - b = n \in \mathbb{Z}$.

$\Rightarrow b - a = -n \in \mathbb{Z}$.

$\therefore b \sim a$. So $\sim$ is symmetric.

(3) Transitive $\rightarrow$ let $a \sim b$ and $b \sim c$

$\therefore a - b \in \mathbb{Z}$ and $b - c \in \mathbb{Z}$.

$\Rightarrow a - b = n$ and $b - c = m$ where $n, m \in \mathbb{Z}$.

$\Rightarrow (a - b) + (b - c) = n + m$

$\Rightarrow a - c = n + m \in \mathbb{Z}$.

$\Rightarrow a \sim c$

$\therefore \sim$ is transitive. Hence $\sim$ is an equivalence relation.

(b) $\overline{6\frac{1}{2}} = \{x \in \mathbb{R} \mid x \sim 6\frac{1}{2}\}$

$\Rightarrow x - 6\frac{1}{2} \in \mathbb{Z}$.

$\Rightarrow \left(x - \frac{1}{2}\right) - 6 \in \mathbb{Z}$.

$\Rightarrow x - \frac{1}{2} \in \mathbb{Z}$. $\Rightarrow x = \{n + \frac{1}{2} \mid n \in \mathbb{Z}\}$.

(c) Any $x \in \mathbb{R}$ can be written as $n + a$ for some $n \in \mathbb{Z}$ and $a \in \mathbb{R}$, $0 \leq a < 1$

$\therefore \overline{x} = \{n + a \mid n \in \mathbb{Z}, 0 \leq a < 1, a \in \mathbb{R}\}$.

5. (*7 points*) Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = x^3 - x + 1$. Determine (with reasons) whether f is one-to-one and/or onto. Show your work.

Check for one-one

let $x_1, x_2 \in \mathbb{Z}$ such that $f(x_1) = f(x_2)$

$\Rightarrow x_1^3 - x_1 + \cancel{1} = x_2^3 - x_2 + \cancel{1}$

$\Rightarrow (x_1^3 - x_2^3) - (x_1 - x_2) = 0$

$\Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) - (x_1 - x_2) = 0$

$\Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2 - 1) = 0$.

$x_1^2 + x_1x_2 + x_2^2 - 1 = 0$

is true if $x_1 = 1$ and $x_2 = -1$ $\quad \therefore f(1) = f(-1)$

but $1 \neq -1$

$\therefore$ $\boxed{f \text{ is not one-one.}}$

Check for onto

let $y \in \mathbb{Z}$. Does $x^3 - x + 1 = y$ have a solution in $x \in \mathbb{Z}$ for all $y \in \mathbb{Z}$?

let $y = 8$

then $x^3 - x + 1 = 8$

$x^3 - x = 7$

$x(x-1)(x+1) = 7$

7 is a prime number, and has no factors other than 1 and 7. So the product of three distinct integers cannot be 7. So the $x^3 - x + 1 = 8$ has not solutions in $\mathbb{Z}$.

Thus

$\therefore$ $\boxed{f \text{ is not onto}}$

6. (*6 points*) Define $f : \mathbb{R}2 \setminus \{-5\} \to \mathbb{R}$ by

$$f(x) = \frac{2x-1}{x+5}$$

It is known that f is a bijective function. Find f^{-1}. Show your work.

$$y = \frac{2x-1}{x+5}$$

<u>Switch</u> $\quad x = \dfrac{2y-1}{y+5}$

$$\Rightarrow x(y+5) = 2y-1$$

$$\Rightarrow xy + 5x = 2y - 1$$

$$\Rightarrow 5x + 1 = 2y - xy.$$

$$\Rightarrow 5x+1 = y(2-x)$$

$$\Rightarrow y = \frac{5x+1}{2-x}$$

$$\Rightarrow \boxed{f^{-1}(x) = \frac{5x+1}{2-x}}$$

$f^{-1} : \mathbb{R}\setminus\{2\} \to \mathbb{R}.$