3 Linear and Quadratic Functions

Outcome/Performance Criteria:

3. Understand and apply linear, quadratic, polynomial and rational functions.

   (a) Graph a line; determine the equation of a line.

   (b) Use a linear model to solve problems; create a linear model for a given situation.

   (c) Interpret the slope and intercept of a linear model.

   (d) Find the vertex and intercepts of a quadratic function; graph a quadratic function.

   (e) Solve a linear inequality; solve a quadratic inequality.

   (f) Solve a problem using a given quadratic model; create a quadratic model for a given situation.

   (g) Graph a parabola whose equation is given in $y = a(x-h)^2 + k$ form, relative to $y = x^2$.

   (h) Given the vertex of a parabola and one other point on the parabola, give the equation of the parabola.

   (i) Put the equation of a parabola in standard form $y = a(x-h)^2 + k$ by completing the square.

   (j) Find the average rate of change of a function over an interval, from either the equation of the function or the graph of the function. Include units when appropriate.

   (k) Find and simplify a difference quotient.

In Chapter 2 we introduced the idea of a function, and studied some general ideas about functions. We will now begin our study of particular kinds of functions with linear and quadratic functions, which you are probably somewhat familiar with from previous courses. Even though these two kinds of functions are amongst the simplest, they are very important in applications. For both kinds, our main focus will be on two things, how the equation and graph of the function are related and how these functions are used in applications.

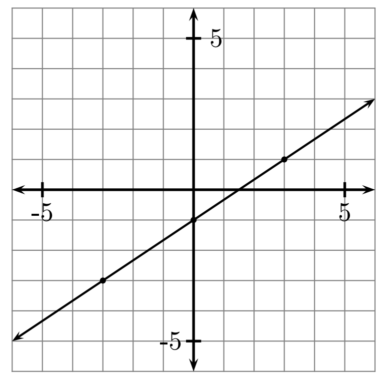
3.1 Linear Functions

Performance Criteria:

3. (a) Graph a line; determine the equation of a line.

Introduction

The simplest kind of functions are **linear functions**, called that because their graphs are lines. An example would be the function with equation $f(x) = \frac{2}{3}x - 1$, whose graph is shown to the right. We often use y instead of $f(x)$ when writing the equations of such functions, and we will see that their equations can always take the form $y = mx + b$, where m and b are constants (fixed numbers). The equation of the function whose graph is shown would then be written as $y = \frac{2}{3}x - 1$. We'll see that the graph is related to the $y = mx + b$ equation in a simple way. You are probably quite familiar with all this information; if you are, this section will provide a refresher.



Slopes of Lines

The slope of a line is essentially its "steepness," expressed as a number. Here are some facts about slopes of lines that you may recall:

- The slope of a horizontal line is zero, and the slope of a vertical line is undefined. (Sometimes, instead of "undefined," we say a vertical line has "no slope," which is not the same the same as a slope of zero!)

- Lines sloping "uphill" from left to right have positive slopes. Lines sloping downhill from left to right have negative slopes. (Note that lines with positive slopes are increasing functions, and lines with negative slopes are decreasing.)

- The steeper a line is, the larger the *absolute value* of its slope will be.

- To find the numerical value of the slope of a line we compute the"rise over the run." More precisely, this is the amount of vertical distance between two points on the line (*any* two points will do) divided by the horizontal distance between the same two points. If the two points are (x_1, y_1) and (x_2, y_2), the slope is given by the formula

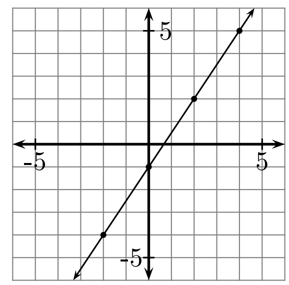
$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

where m represents the slope of the line. It is standard in mathematics to use the letter m for slopes of lines.

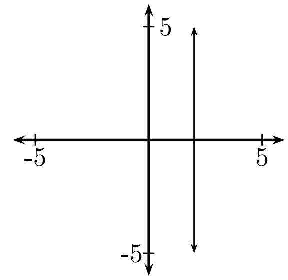
- Parallel lines have the same slope. Perpendicular lines have slopes that are negative reciprocals; for example, lines with slopes $m_1 = -\frac{2}{3}$ and $m_2 = \frac{3}{2}$ are perpendicular. (The subscripts here designate one slope from the other.)

◇ **Example 3.1(a):** Give the slope of the lines with the graphs shown below and to the right.

For Line 1 we first note that it slopes uphill from left to right, so its slope is positive. Next we need to determine the rise and run between two points; consider the points $(-2,-4)$ and $(2,2)$. Starting at $(-2,-4)$ and going straight up, we must go up six units to be level with the second point, so there is a rise of 6 units. We must then "run" four units to the right to get to $(2,2)$, so the run is 4. We already determined that the slope is positive, so for Line 1 we have $m = \frac{\text{rise}}{\text{run}} = \frac{6}{4} = \frac{3}{2}$.



Line 1



Line 2

Suppose we pick any two points on Line 2. The rise will be some number that will vary depending on which two points we choose. However, the run will be zero, regardless of which points we choose. Thus when we try to find the rise over the run we will get a fraction with zero in its denominator. Therefore the slope of Line 2 is undefined.

◇ **Example 3.1(b):** Give the slope of the line through $(-3,-4)$ and $(6,8)$.

Here we will let the first point given be (x_1, y_1), and the second point be (x_2, y_2). We then find the rise as $y_2 - y_1$ and the run as $x_2 - x_1$, and divide the rise by the run:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-(-4)}{6-(-3)} = \frac{12}{9} = \frac{4}{3}.$$

◇ **Example 3.1(c):** The slope of Line 1 is $-\frac{3}{5}$. Line 2 is perpendicular to Line 1 and Line 3 is parallel to Line 1. What are the slopes of Line 2 and Line 3?

Because Line 2 is perpendicular to Line 1, its slope must be the negative reciprocal of $-\frac{3}{5}$, or $\frac{5}{3}$. Line 3 is parallel to Line 1, so it must have the same slope, $-\frac{3}{5}$.

◇ **Example 3.1(d):** The slope of Line 1 is 0. If Line 2 is perpendicular to Line 1, what is its slope?

Because its slope is zero, Line 1 is horizontal. A perpendicular line is therefore vertical, so its slope is undefined.

Equations of Lines

There are three forms in which linear functions are usually given:

- **Standard form:** This is the form $Ax + By = C$, where A, B and C are all integers. This form is occasionally useful.

- **Slope-intercept form:** This is the form $y = mx + b$, obtained by solving an equation in standard form for y. The letter m represents the slope of the line, and b is the y-intercept. This form is useful because it is very easy to get the graph from the equation in this form, and vice-versa.

- **Point-slope form:** This is the form $y - y_0 = m(x - x_0)$, where m is the slope of the line and (x_0, y_0) is a distinct point on the line.

I will ask that you give all answers in standard form or slope-intercept form. Point-slope form is useful for determining the equation of a line, but that can be done using slope-intercept form as well. I will work almost exclusively with slope-intercept form here in these notes.

Recall a few basic things about equations and graphs of lines:

- The equation of a horizontal line through the point (a, b) is $y = b$. The equation of a vertical line through the same point is $x = a$.

- To graph a line, one really needs just two points, since lines are straight. Here are some suggestions for finding the two points to use:

  ▷ To graph a line whose equation is given in standard form, try letting $x = 0$ and solving for y to get one ordered pair (so one point) and then let $y = 0$ and solve for x to get another point. This method can be less than satisfying if fractions are obtained when solving!

  ▷ If you have the equation of a line in slope-intercept form (or you had it in standard form but solved for y to get it into slope-intercept form) use the following procedure to get the graph of the line:

    1. Plot the y-intercept b. (Note that you are really getting an ordered pair by letting $x = 0$.)
    2. If the slope is not given as a fraction $\frac{p}{q}$, make it one by putting it over one. *Starting at the y-intercept*, go up or down p units. Then go left or right by q units - choose whichever direction will give the correct sign (positive or negative) of the slope when connected with the y-intercept. Plot a point where you end up.
    3. Draw a line through your two points, extending to the edges of your coordinate grid. Put arrowheads at the ends of the line to indicate that it keeps going.

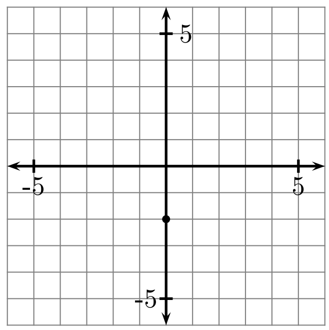
All of the sorts of things I will expect you to be able to do relating to equations and graphs of lines are illustrated in the following examples.

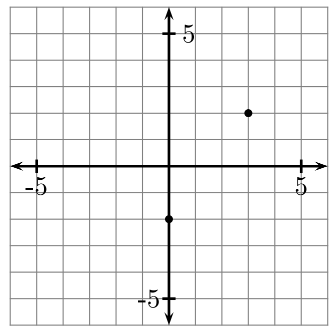
◇ **Example 3.1(e):** Graph the line with equation $4x - 3y = 6$.

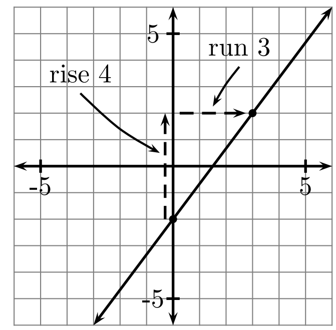
The equation will be easiest to graph if we first put it into $y = mx + b$ form:

$$\begin{aligned} 4x - 3y &= 6 \\ -3y &= -4x + 6 \\ y &= \frac{-4x + 6}{-3} \\ y &= \tfrac{4}{3}x - 2 \end{aligned}$$

We then can let $x = 0$ and find the corresponding value of y, -2. We then plot the point $(0, -2)$, as done in the first graph below. Next we let x take the next easiest value to work with, 3. This gives us $y = 2$, so we then plot the ordered pair $(3, 2)$, as shown in the second picture below. Finally we connect those two points and extend our line to the edges of the grid, putting arrowheads on the ends to indicate that the graph continues. This is shown on the third graph below.



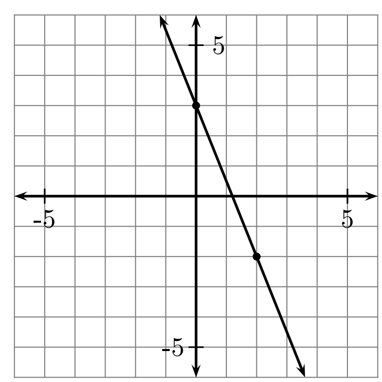




Another way we can think of getting our two points to graph from the equation $y = \frac{4}{3}x - 2$ is that $b = -2$ is the y-intercept, so we can first plot that point. Next we look at the slope $m = \frac{4}{3}$. We rise four units from the y-intercept, then "run" three units in the direction that will give us a positive slope, to the right. There we plot a point, which is the point $(3, 2)$. This is illustrated by the dashed lines in the third diagram above. Finally, as before, we draw in the line.

◇ **Example 3.1(f):** Give the equation of the line graphed below and to the right.

Here we begin by noting that the equation of any line can be written in the form $y = mx + b$, and the task essentially boils down to finding m and b. In our case b is the y-intercept 3. Next we use any two points on the line to find the slope of the line, noting that the slope is negative since the line slopes "downhill" from left to right. Using the y-intercept point and the point $(2, -2)$, we see that the vertical change is five units and the horizontal change is two units. Therefore the slope is $m = -\frac{5}{2}$ and the equation of the line is $y = -\frac{5}{2}x + 3$.



Students sometimes give their result from an exercise like the last example as $y = -\frac{5}{2} + 3$. This is the same as $y = \frac{4}{3}$, which *IS* the equation of a line (the horizontal line with y-intercept $\frac{4}{3}$), but it is not the equation of the line graphed above. Be sure to include the x from $y = mx + b$ in your answer to this type of question, or the two that follow.

◇ **Example 3.1(g):** Find the equation of the line through $(-4, 5)$ and $(2, 1)$ **algebraically**.

Again we want to use the "template" $y = mx + b$ for our line. In this case we first find the slope of the line:

$$m = \frac{5 - 1}{-4 - 2} = \frac{4}{-6} = -\frac{2}{3}.$$

At this point we know the equation looks like $y = -\frac{3}{2}x + b$, and we now need to find b. To do this we substitute the values of x and y from *either* ordered pair that we are given into $y = -\frac{2}{3}x + b$ (being careful to get each into the correct place!) and solve for b:

$$\begin{aligned} 1 &= -\tfrac{2}{3}(2) + b \\ \tfrac{3}{3} &= -\tfrac{4}{3} + b \\ \tfrac{7}{3} &= b \end{aligned}$$

We now have the full equation of the line: $y = -\frac{2}{3}x + \frac{7}{3}$.

◇ **Example 3.1(h):** Find the equation of the line containing $(1, -2)$ and perpendicular to the line with equation $5x + 4y = 12$.

In this case we wish to again find the m and b in the equation $y = mx + b$ of our line, but we only have one point on the line. We'll need to use the other given information, about a second line. First we solve its equation for y to get $y = -\frac{5}{4}x + 3$. Because our line is perpendicular to that line, the slope of our line is $m = \frac{4}{5}$ and its equation is $y = \frac{4}{5}x + b$. As in the previous situation, we then substitute in the given point on our line to find b:

$$\begin{aligned} -2 &= \tfrac{4}{5}(1) + b \\ -\tfrac{10}{5} &= \tfrac{4}{5} + b \\ -\tfrac{14}{5} &= b \end{aligned}$$

The equation of the line is then $y = \frac{4}{5}x - \frac{14}{5}$.

◇ **Example 3.1(i):** Find the equation of the line through $(2, -3)$ and $(2, 1)$.

This appears to be the same situation as Example 3.1(g), so let's proceed the same way by first finding the slope of our line:

$$m = \frac{-3 - 1}{2 - 2} = \frac{-4}{0},$$

which is undefined. Hmmm... what to do? Well, this means that our line is vertical. Every point on the line has a x-coordinate of 2, regardless of its y-coordinate, so the equation of the line is $x = 2$.

This last example points out an idea that we will use a fair amount later:

Equations of Vertical and Horizontal Lines

The equation of a horizontal line through the point (a, b) is $y = b$, and the equation of a horizontal line through the same point is $x = a$.

Section 3.1 Exercises

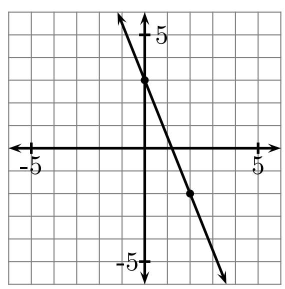
1. Find the slopes of the lines through the following pairs of points.

   (a) $(-4,5)$ and $(2,1)$ (b) $(-3,7)$ and $(7,11)$ (c) $(3,2)$ and $(-5,2)$

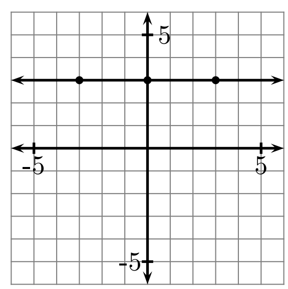
   (d) $(1,7)$ and $(1,-1)$ (e) $(-4,-5)$ and $(-1,4)$ (f) $(3,-1)$ and $(-3,2)$

2. Find the slope of each line.

(a)



(b)



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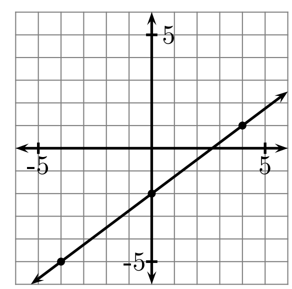
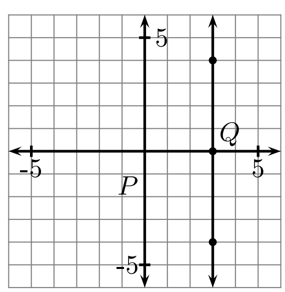


(d)



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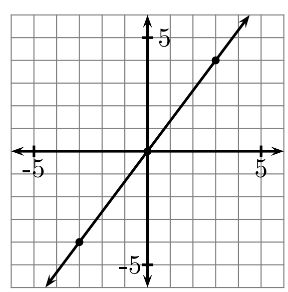
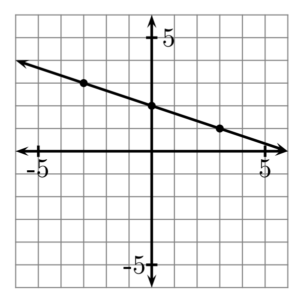


(f)



3. Give the equation of each line in the previous exercise.

4. Line 1 has slope $-\frac{1}{3}$, and Line 2 is perpendicular to Line 1. What is the slope of Line 2?

5. Line 1 has slope $\frac{1}{2}$. Line 2 is perpendicular to Line 1, and Line 3 is perpendicular to Line 2. What is the slope of Line 3?

6. Sketch the graph of each line.

   (a) $y = -\frac{1}{2}x + 3$ (b) $y = \frac{4}{5}x - 1$ (c) $3x - y = 4$

   (d) $x + y = 2$ (e) $3x + 5y = 10$

7. (a) Sketch the graph of $x = -2$. (b) Sketch the graph of $y = 4$.

8. Find the equation of the line through the two points **algebraically**.

(a) $(3, 6),\ (12, 18)$	(b) $(-3, 9),\ (6, 3)$	(c) $(3, 2),\ (-5, 2)$
(d) $(-8, -7),\ (-4, -6)$	(e) $(2, -1),\ (6, -4)$	(f) $(-2, 4),\ (1, 4)$
(g) $(-4, -2),\ (-2, 4)$	(h) $(1, 7),\ (1, -1)$	(i) $(-1, -5),\ (1, 3)$
(j) $(1, 7), (3, 11)$	(k) $(-6, -2),\ (5, -3)$	(l) $(-2, -5),\ (2, 5)$

9. Are the two lines $3x - 4y = 7$ and $8x - 6y = 1$ perpendicular, parallel, or neither?

10. Find the equation of the line containing the point $P(2, 1)$ and perpendicular to the line with equation $2x + 3y = 9$. Give your answer in slope-intercept form ($y = mx + b$).

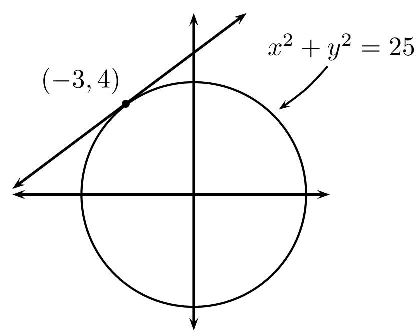
11. Find the equation of a line with x-intercept -4 and y-intercept 3.

12. Find the equation of a horizontal line through the point $(2, -5)$.

13. Find the equation of the line through $(2, -3)$ and parallel to the line with equation $4x + 5y = 15$.

14. Determine, *without graphing*, whether $A(-2, 0)$, $B(2, 3)$ and $C(-6, -3)$ lie on the same line. Show work supporting your answer. (**Hint:** Find the slope of the line through points A and B, then through B and C. How should they compare if the points are all on the same line?)

15. A line that touches a circle in only one point is said to be **tangent** to the circle. The picture to the right shows the graph of the circle $x^2 + y^2 = 25$ and the line that is tangent to the circle at $(-3, 4)$. Find the equation of the tangent line. (**Hint:** The tangent line is perpendicular to the segment that goes from the center of the circle to the point $(-3, 4)$.)



3.2 Applications of Linear Functions

Outcome/Criteria:

3. (b) Use a linear model to solve problems; create a linear model for a given situation.

   (c) Interpret the slope and intercept of a linear model.

Using a Linear Model to Solve Problems

An insurance company collects data on amounts of damage (in dollars) sustained by houses that have caught on fire in a small rural community. Based on their data they determine that the expected amount D of fire damage (in dollars) is related to the distance d (in miles) of the house from the fire station. (Note here the importance of distinguishing between upper case variables and lower case variables!) The equation that seems to model the situation well is

$$D = 28000 + 9000d$$

This tells us that the damage D is a function of the distance d of the house from the fire station. Given a value for either of these variables, we can find the value of the other.

◇ **Example 3.2(a):** Determine the expected amount of damage from a house fire that is 3.2 miles from the fire station, 4.2 miles from the fire station and 5.2 miles from the fire station.

For convenience, let's rewrite the equation using function notation, and in the slope-intercept form: $D(d) = 9000d + 28000$. Using this we have

$$D(3.2) = 9000(3.2) + 28000 = 56800, \qquad D(4.2) = 65800, \qquad D(5.2) = 74800$$

The damages for distances of 3.2, 4.2 and 5.2 miles from the fire station are $\$56,800$, $\$65,800$ and $\$74,800$.

Note that in the above example, for each additional mile away from the fire station, the amount of damage increased by $9000.

◇ **Example 3.2(b):** If a house fire caused $47,000 damage, how far would you expect that the fire might have been from the fire station? *Round to the nearest tenth of a mile.*

Here we are given a value for D and asked to find a value of d. We do this by substituting the given value of D into the equation and solving for d:

$$\begin{aligned} 47000 &= 9000d + 28000 \\ 19000 &= 9000d \\ 2.1 &= d \end{aligned}$$

We would expect the house that caught fire to be about 2.1 miles from the fire station.

◇ **Example 3.2(c):** How much damage might you expect if your house was right next door to the fire station?

A house that is right next door to the fire station is essentially a distance of zero miles away. We would then expect the damage to be

$$D(0) = 9000(0) + 28000 = 28000.$$

The damage to the house would be $\$28,000$.

Interpreting the Slope and Intercept of a Linear Model

There are a few important things we want to glean from the above examples.

- When we are given a mathematical relationship between two variables, if we know one we can find the other.
- Recall that slope is rise over run. In this case rise is damage, measured in units of dollars, and the run is distance, measured in miles. Therefore the slope is measured in $\frac{\text{dollars}}{\text{miles}}$, or dollars per mile. The slope of 9000 dollars per mile tells us that for each additional mile farther from the fire station that a house fire is, the amount of damage is expected to increase by \$9000.
- The amount of damage expected for a house fire that is essentially right at the station is \$28,000, which is the D-intercept for the equation.

In general we have the following.

Interpreting Slopes and Intercepts of Lines

When an "output" variable depends linearly on another "input" variable,

- the slope has units of the output variable units over the input variable units, and it represents the amount of increase (or decrease, if it is negative) in the output variable for each one unit increase in the input variable,
- the output variable intercept ("y"-intercept) is the value of the output variable when the value of the input variable is zero, and its units are the units of the output variable. *The intercept is not always meaningful.*

The first of these two points illustrates what was pointed out after 3.2(a). As the distance increased by one mile from 3.2 miles to 4.2 miles the damage increased by $65800 - 56800 = 9000$ dollars, and when the distance increased again by a mile from 4.2 miles to 5.2 miles the damage again increased by $\$9000$.

When dealing with functions we often call the "input" variable (which is *ALWAYS* graphed on on the horizontal axis) the **independent variable**, and the "output" variable (which is always graphed on the vertical axis) is the **dependent variable**. Using this language we can reword the items in the previous box as follows.

Slope and Intercept in Applications

For a linear model $y = mx + b$,

- the slope m tells us the amount of increase in the dependent variable for every one unit increase in the independent variable
- the vertical axis intercept tells us the value of the dependent variable when the independent variable is zero

Section 3.2 Exercises

1. The amount of fuel consumed by a car is a function of how fast the car is traveling, amongst other things. For a particular model of car this relationship can be modeled mathematically by the equation $M = 18.3 - 0.02s$, where M is the mileage (in miles per gallon) and s is the speed (in miles per hour). *This equation is only valid when the car is traveling on the "open road" (no stopping and starting) and at speeds between 30 mph and 100 mph.*

   (a) Find the mileage at a speed of 45 miles per hour and at 50 miles per hour. Did the mileage increase, or did it decrease, as the speed was increased from 45 mph to 50 mph? By how much?

   (b) Repeat (a) for 90 and 95 mph.

   (c) Give the slope of the line. Multiply the slope by 5. What do you notice?

   (d) Find the speed of the car when the mileage is 17.5 mpg.

   (e) Give the slope of the line, with units, and interpret its meaning in the context of the situation.

   (f) Although we can find the m-intercept and nothing about it seems unusual, why should we not interpret its value?

2. The weight w (in grams) of a certain kind of lizard is related to the length l (in centimeters) of the lizard by the equation $w = 22l - 84$. This equation is based on statistical analysis of a bunch of lizards between 12 and 30 cm long.

   (a) Find the weight of a lizard that is 3 cm long. Why is this not reasonable? What is the problem here?

   (b) What is the w-intercept, and why does it have no meaning here?

   (c) What is the slope, with units, and what does it represent?

3. A salesperson earns \$800 per month, plus a 3% commission on all sales. Let P represent the salesperson's gross pay for a month, and let S be the amount of sales they make in a month. (Both are of course in dollars. Remember that to compute 3% of a quantity we multiply by the quantity by 0.03, the decimal equivalent of 3%.)

(a) Find the pay for the salesperson when they have sales of \$50,000, and when they have sales of \$100,000.

(b) Find the equation for pay as a function of sales, given that this is a linear relationship.

(c) What is the slope of the line, and what does it represent?

(d) What is the P-intercept of the line, and what does it represent?

4. The equation $F = \frac{9}{5}C + 32$ gives the Fahrenheit temperature F corresponding to a given Celsius temperature C. This equation describes a line, with C playing the role of x and F playing the role of y.

   (a) What is the F-intercept of the line, and what does it tell us?

   (b) What is the slope of the line, and what does it tell us?

5. We again consider the manufacture of Widgets by the Acme Company. The costs for one week of producing Widgets is given by the equation $C = 7x + 5000$, where C is the costs, in dollars, and x is the number of Widgets produced in a week. This equation is clearly linear.

   (a) What is the C-intercept of the line, and what does it represent?

   (b) What is the slope of the line, and what does it represent?

   (c) If they make 1,491 Widgets in one week, what is their total cost? What is the cost for each individual Widget made that week? (The answer to this second question should *NOT* be the same as your answer to (b).)

6. The cost y (in dollars) of renting a car for one day and driving x miles is given by the equation $y = 0.24x + 30$. Of course this is the equation of a line. Explain what the slope and y-intercept of the line represent, *in terms of renting the car.*

7. A baby weighs 8 pounds at birth, and four years later the child's weight is 32 pounds. Assume that the childhood weight W (in pounds) is linearly related to age t (in years).

   (a) Give an equation for the weight W in terms of t. *Test it for the two ages that you know the child's weight, to be sure it is correct!*

   (b) What is the slope of the line, with units, and what does it represent?

   (c) What is the W-intercept of the line, with units, and what does it represent?

   (d) Approximately how much did the child weigh at age 3?

3.3 Quadratic Functions

Performance Criteria:

3. (d) Find the vertex and intercepts of a quadratic function; graph a quadratic function.

Introduction

Quadratic functions, like linear functions, show up naturally in a number of applications. You have seen some of them already: height of an object that is projected up in the air and the dependence of revenue on the number of items sold, for example. One of the most important things about quadratic functions is that their graphs are parabolas that open either up or down, so every quadratic function has either an absolute maximum value or an absolute minimum value.

We will examine this idea in some detail in this section and the next one. We'll see how to quickly find the maximum or minimum value of a quadratic function, and the input value that gives that maximum or minimum value. We'll see that there is a standard form for the equation of a parabola, sort of the quadratic function version of $y = mx + b$, that makes it very easy to construct the graph of a quadratic function.

Quadratic functions are functions of the form

$$f(x) = ax^2 + bx + c\ ,$$

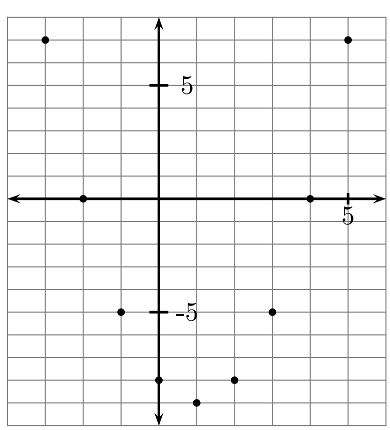
where a, b, and c are real numbers and $a \neq 0$. You have seen a number of these functions already, of course, including several in "real world" situations. We will refer to the value a, that x^2 is multiplied by, as the **lead coefficient**, and c will be called the **constant term**.

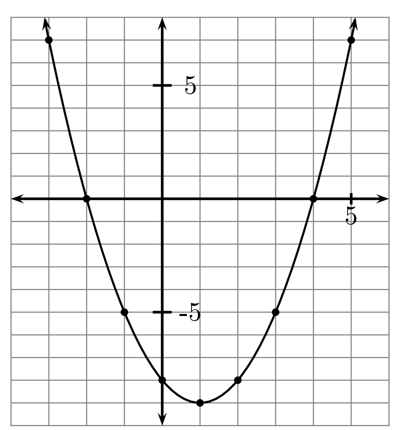
Our main goal in this section will be to learn how to graph such a function quickly, but let's do it once by just plotting a bunch of points before we go on.

◇ **Example 3.3(a)**: Graph the quadratic function $f(x) = x^2 - 2x - 8$.

To the left below are some pairs, and the corresponding points are plotted on the first grid below. In this case the points appear to lie on the parabola graphed on the second grid.

x	$f(x)$
-3	7
-2	0
-1	-5
0	-8
1	-9
2	-8
3	-5
4	0
5	7



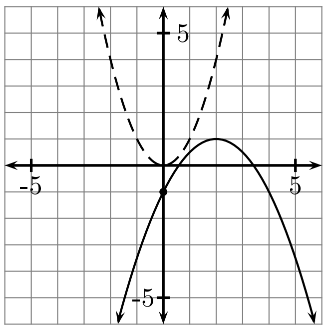


Whew - that was a lot of work (well,for me anyway...)! Let's see if we can figure out an easier way.

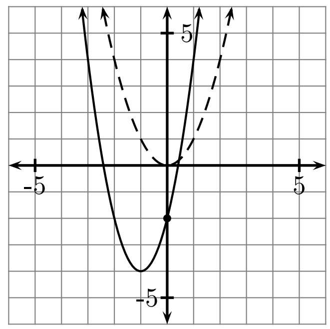
Graphs of Quadratic Functions

On each graph below, the solid curve is the graph of the given quadratic function, and the dashed curve is the graph of $y = x^2$.

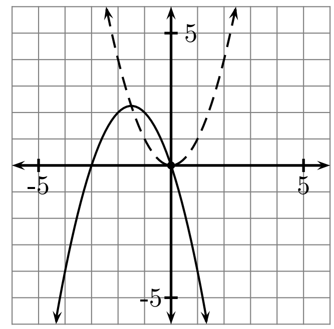
$f(x) = -\frac{1}{2}x^2 + 2x - 1$



$g(x) = 2x^2 + 4x - 2$



$h(x) = -x^2 - 3x$



As you know, the shapes of the curves in these graphs are called **parabolas**. Our first goal is to get some idea, without finding any solution pairs, how the graph of a parabola relates to its equation. There are three things we can see from the above examples, two of which are obvious and the third of which is less obvious.

Note that the graphs of the first and third are parabolas that "open downward", and the graph of the second "opens upward." What is it that the first and third have in common, that is different for the second? Well, it is the sign of the lead coefficient. It is positive for the functions whose graph that opens upward, and negative for the one that opens downward. What we are seeing is that if $a > 0$ the graph of the quadratic function $f(x) = ax^2 + bx + c$ will open upward, and if $a < 0$, the graph of the quadratic function will open downward. (If a were zero, we would no longer have a quadratic function, but a linear function. This is why I stated earlier that $a \neq 0$.)

NOTE: It is possible to have equations whose graphs are parabolas that open left or right instead of up or down, as you saw in Chapter 1, but those equations do not describe functions because for some x-values there are two corresponding y-values. In this chapter we will be working with quadratic *functions*, so the parabolas will always open either upward or downward.

The two tails of the graph of a quadratic function climb steeply upward or descend steeply downward. Either way, they are also always spreading apart, so every quadratic function will always have a y-intercept, and it should be clear that the y-intercept is at the value c from $f(x) = ax^2 + bx + c$.

Finally, if we look carefully at the graphs of f and g, we can see that the parabola for f is "broader" than the one for $y = x^2$, and the parabola for g is "narrower" than the one for $y = x^2$. The parabolas for h and $y = x^2$ have the same shape. It turns out that if $|a| < 1$ the graph of $f(x) = ax^2 + bx + c$ will be broader than the graph of $y = x^2$, and if $|a| > 1$ it will be narrower. If $|a| = 1$ the parabola will have the same width as $y = x^2$.

Let's now summarize all that we have seen about the graph of $f(x) = ax^2 + bx + c$:

- The graph is a parabola.
- If $a > 0$, the parabola opens upward, and if $a < 0$, it opens downward.

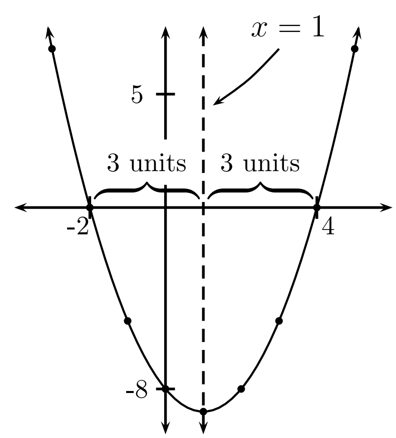
- If $|a| < 1$ the graph of $f(x) = ax^2 + bx + c$ will be broader than the graph of $y = x^2$, and if $|a| > 1$ it will be narrower. If $|a| = 1$ the parabola will have the same width as $y = x^2$.
- The y-intercept of the graph is c.

The Vertex of a Parabola

These things give us some idea of what the parabola for a quadratic function will look like, but how do we actually get the graph of the parabola without so much effort as in Example 3.3(a)? The only point we get from the above is the y-intercept; the rest of the information is qualitative.

Well, graphing always boils down to finding some solution pairs and plotting them. However, when we knew an equation was that of a line we then only had to plot two points to get the graph of the line. With a parabola we need to plot more than that, but we'll see that we can get a very good graph by finding two or three solution pairs and then plotting three or five points. (How we get three or five points from two or three solutions will be clear soon.) There are two keys to getting a graph quickly:

- The most important point on any parabola is the high or low point, called the **vertex** of the parabola. That is the first point we will want to find and graph. The plural of vertex is *vertices*, and the vertices of the parabolas whose graphs are at the top of the previous page are $(2, 1)$, $(-1, -4)$ and $(-1\frac{1}{2}, 2\frac{1}{4})$.
- There is a vertical line down the center of any parabola, called the **line of symmetry**, and for each point on the parabola other than the vertex, there is another point an equal distance away on the other side of the line of symmetry.



To the right is the graph of $f(x) = x^2 - 2x - 8$, the function from Example 3.3(a). Note first off that the line of symmetry is the dashed vertical line $x = 1$ and, for every point on one side of the parabola there is another point the same distance from the dashed line but on the opposite side of it horizontally. This illustrates the second point above.

Now we'll use the same function to see how to find the vertex of a parabola. We can see that the x-intercepts are -2 and 4, but let's find them algebraically. Recall that to get the x-intercepts we set $y = 0$, resulting in the equation $0 = x^2 - 2x - 8$. We then solve for x. This is quite easy to do by factoring, but in this case let's use the quadratic formula:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-8)}}{2(1)} = \frac{2 \pm \sqrt{36}}{2} = \frac{2}{2} \pm \frac{6}{2} = 1 \pm 3.$$

How does this relate to the graph and the vertex of the parabola? Well, we can see that the x-coordinate of the vertex is $x = 1$, and we go three units each way from there on the x-axis to get the x-intercepts. This shows us that the x-coordinate of the vertex of a parabola is given by the part $x = \frac{-b}{2a} = -\frac{b}{2a}$ of the quadratic formula! To get the y-coordinate of the vertex, we simply evaluate the function for the x-coordinate of the vertex.

◇ **Example 3.3(b):** Determine the coordinates of the vertex of the parabola with equation $f(x) = -\frac{1}{2}x^2 + 2x - 1$

Using the formula $x = -\dfrac{b}{2a}$ we find that the x-coordinate of the vertex is

$$x = -\frac{2}{2(-\frac{1}{2})} = -\frac{2}{-1} = 2.$$

The y-coordinate of the vertex is found by substituting the value 2 that we just found for x into the original equation:

$$f(2) = -\frac{1}{2}(2)^2 + 2(2) - 1 = -2 + 4 - 1 = 1$$

Therefore the vertex of the parabola is $(2, 1)$.

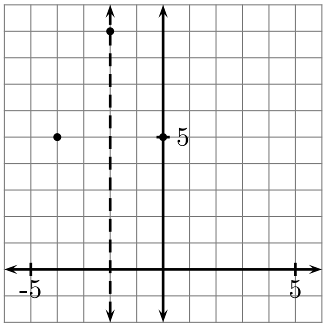
We can now describe an efficient procedure for getting the graph of a quadratic function:

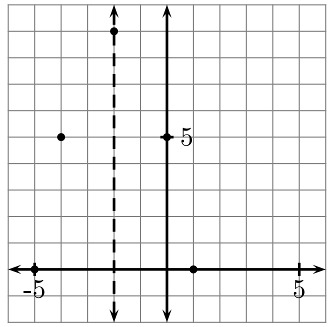
Graphing $f(x) = ax^2 + bx + c$

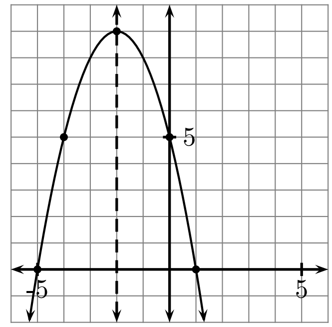
1) Use $x = -\dfrac{b}{2a}$ to find the x-coordinate of the vertex, then evaluate the function to find the y-coordinate of the vertex. Plot that point.
2) Find and plot the y-intercept and the point opposite it. Sketch in the line of symmetry if you need to in order to do this.
3) Pick an x value other than ones used for the three points that you have plotted so far and evaluate the function. Plot the resulting point and the point opposite it.
4) Draw the graph of the function. As always, include arrowheads to show where the the graph continues beyond the edges of the graph.

◇ **Example 3.3(c):** Graph the function $f(x) = -x^2 - 4x + 5$. Indicate clearly and accurately five points on the parabola.

First we note that the graph will be a parabola opening downward, with y-intercept 5. The x-coordinate of the vertex is $x = -\dfrac{-4}{2(-1)} = -2$. Evaluating the function for this value of x gives us $f(-2) = -(-2)^2 - 4(-2) + 5 = -4 + 8 + 5 = 9$, so the vertex is $(-2, 9)$. On the first graph at the top of the next page we plot the vertex, y-intercept, and the axis of symmetry, and the point opposite the y-intercept, $(-4, 5)$. To get two more points we can evaluate the function for $x = 1$ to get $f(1) = -1^2 - 4(1) + 5 = 0$. We then plot the resulting point $(1, 0)$ and the point opposite it, as done on the second graph. The third graph then shows the graph of the function with our five points clearly and accurately plotted.







Section 3.3 Exercises

1. Graph $g(x) = x^2 + 2x + 3$, indicating five points on the graph clearly and accurately.

2. Graph the function $h(x) = x^2 + 4x - 5$. Indicate clearly and accurately five points on the parabola. How does the result compare with that from Example 3.3(c)?

3. Discuss the x-intercept situation for quadratic functions. (Do they always have at least one x-intercept? Can they have more than one? Sketch some parabolas that are arranged in different ways (but always opening up or down) relative to the coordinate axes to help you with this.

4. Here you will find that the method used before Example 3.3(b) for finding the x-coordinate of the vertex of a parabola will not always work, but that you can always find the x-coordinate of the vertex using the formula $x = -\frac{b}{2a}$.

   (a) Find the x-intercepts of $g(x) = x^2 - 6x + 13$. What goes wrong here?

   (b) Which way does the parabola open?

   (c) Find the vertex or the parabola.

   (d) Tell how your answers to (b) and (c) explain what happened in (a).

   (e) Give the coordinates of four other points on the parabola.

5. Discuss the existence of relative and absolute maxima and/or minima for quadratic functions.

6. Consider the function $f(x) = x^2 - 6x - 7$. Here you will graph the function in a slightly different manner than used in Example 3.3(c).

   (a) You know the graph will be a parabola - which way does it open?

   (b) Find the y-intercept and plot it.

   (c) Find the x-intercepts, and plot them.

   (d) Sketch the line of symmetry on your graph - you should be able to tell where it is, based on the x-intercepts. What is the x-coordinate of the vertex?

(e) Find the y-coordinate of the vertex of the parabola by substituting the x-coordinate into the equation.

(f) Plot the vertex and the point opposite the y-intercept and graph the function.

(g) Where (at what x value) does the minimum or maximum value of the function occur? Is there a minimum, or a maximum, there? (This last part should not be hard to answer, but you should always state whether you are dealing with a minimum, or a maximum.) What is the minimum or maximum value of the function (y value)?

7. A parabola has x-intercepts 3 and 10. What is the x-coordinate of the vertex?

8. Consider the function $g(x) = 10 + 3x - x^2$.

   (a) What are the values of a, b and c for this parabola?

   (b) Use the procedure of Example 3.3(c) to sketch the graph of g.

9. For each of the quadratic functions below,

   - give the general form of the graph (parabola opening upward, parabola opening downward),
   - give the coordinates of the vertex,
   - give the y-intercept of the graph and the coordinates of one other point on the graph,
   - sketch the graph of the function,
   - tell what the maximum or minimum value of the function is (stating whether it is a maximum or a minimum) and where it occurs.

   (a) $f(x) = -x^2 + 2x + 15$

   (b) $g(x) = x^2 - 9x + 14$

   (c) $h(x) = -\frac{1}{4}x^2 + \frac{5}{2}x - \frac{9}{4}$

   (d) $s(t) = \frac{1}{2}t^2 + 3t + \frac{9}{2}$

10. Consider the function $f(x) = (x - 3)^2 - 4$.

   (a) Get the function in the form $f(x) = ax^2 + bx + c$.

   (b) Find the coordinates of the vertex of the parabola. Do you see any way that they might relate to the original form in which the function was given?

   (c) Find the x and y-intercepts of the function and sketch its graph.

   (d) On the same coordinate grid, sketch the graph of $y = x^2$ *accurately.* (Find some ordered pairs that satisfy the equation and plot them to help with this.) Compare the two graphs. Check your graphs with your calculator.

11. Consider the function $y = -\frac{1}{2}(x-2)(x+3)$. Do all of the following without multiplying this out!

(a) If you were to multiply it out, what would the value of a be? Which way does the parabola open?

(b) Find the y-intercept of the function.

(c) Find the x-intercepts of the function. In this case it should be easier to find the x-intercepts than the y-intercept!

(d) Find the coordinates of the vertex.

(e) Graph the function, labeling the vertex with its coordinates.

3.4 Linear and Quadratic Inequalities

Outcome/Criteria:

3. (e) Solve a linear inequality; solve a quadratic inequality.

Let's recall the notation $a > b$, which says that the number a is greater than the number b. Of course that is equivalent to $b < a$, b is less than a. So, for example, we could write $5 > 3$ or $3 < 5$, both mean the same thing. Note that it is not correct to say $5 > 5$, since five is not greater than itself! On the other hand, when we write $a \geq b$ it means a is greater than *or equal to* b, so we could write $5 \geq 5$ and it would be a true statement. Of course $a \geq b$ is equivalent to $b \leq a$.

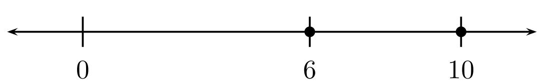
In this section we will consider algebraic inequalities; that is, we will be looking at inequalities that contain an unknown value. An example would be

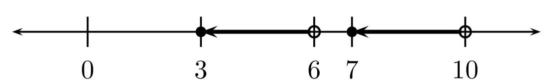
$$3x + 5 \leq 17,$$

and our goal is to *solve* the inequality. This means to find all values of x that make this true. This particular inequality is a **linear inequality**; before solving it we'll take a quick look at how linear inequalities behave.

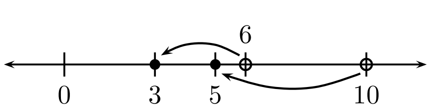
Solving Linear Inequalities

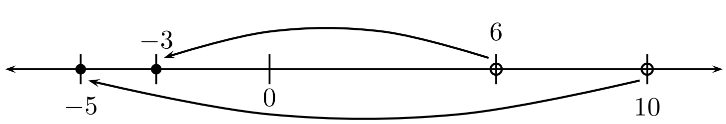
Consider the *true* inequality $6 < 10$; if we put dots at each of these values on the number line the larger number, ten, is farther to the right, as shown to the left below. If we subtract, say, three from both sides of the inequality we get $3 < 7$. The effect of this is to shift each number to the left by five units, as seen in the diagram below and to the right, so their relative positions don't change.





Suppose now that we divide both sides of the same inequality $6 < 10$ by two. We then get $3 < 5$, which is clearly true, and shown on the diagram below and to the left. If, on the other hand, we divide both sides by -2 we get $-3 < -5$. This is not true! The diagram below and to the right shows what is going on here; when we divide both sides by a negative we reverse the positions of the two values on the number line, so to speak.





The same can be seen to be true if we multiply both sides by a negative. This leads us to the following principle.

> **Solving Linear Inequalities**
>
> To solve a linear inequality we use the same procedure as for solving a linear equation, *except that we reverse the direction of the inequality whenever we multiply or divide BY a negative.*

◇ **Example 3.4(a):** Solve the inequality $3x + 5 \leq 17$.

We'll solve this just like we would solve $3x + 5 = 17$. At some point we will divide by 3, but since we are not dividing by a negative, we won't reverse the inequality.

$$\begin{aligned} 3x + 5 &\leq 17 \\ 3x &\leq 12 \\ x &\leq 4 \end{aligned}$$

This says that the original inequality is true for any value of x that is less than or equal to 4. Note that zero is less than 4 and makes the original inequality true; this is a check to make sure at least that the inequality in our answer goes in the right direction.

• **Example 3.4(b):** Solve $-3x + 7 > 19$.

Here we can see that at some point we will divide by -3, and *at that point* we'll reverse the inequality.

$$\begin{aligned} -3x + 7 &> 19 \\ -3x &> 12 \\ x &< -4 \end{aligned}$$

This says that the original inequality is true for any value of x that is less than -4. Note that -5 is less than -4 and makes the original inequality true, so the inequality in our answer goes in the right direction.

Solving Quadratic Inequalities

Suppose that we wish to know where the function $f(x) = -x^2 + x + 6$ is positive. Remember that what this means is we are looking for the x values for which the corresponding values of $f(x)$ are greater than zero; that is, we want $f(x) > 0$. But $f(x)$ is $-x^2+x+6$, so we want to solve the inequality

$$-x^2 + x + 6 > 0.$$

Let's do this by first multiplying both sides by -1 (remembering that this causes the direction of the inequality to change!) and then factoring the left side:

$$\begin{aligned} -x^2 + x + 6 &> 0 \\ x^2 - x - 6 &< 0 \\ (x-3)(x+2) &< 0 \end{aligned}$$

In order for this last inequality to be true, one of the factors $x-3$ and $x+2$ must be negative and the other must be positive, so that when we multiply them we get a negative. There are two ways that this can happen:

$$\begin{array}{rcl} x-3<0 & \text{and} & x+2>0 \\ x<3 & \text{and} & x>-2 \end{array} \qquad OR \qquad \begin{array}{rcl} x-3>0 & \text{and} & x+2<0 \\ x>3 & \text{and} & x<-2 \end{array}$$

The second situation, $x>3$ and (meaning at the same time) $x<-2$ is impossible, so it is the first situation we are looking for, $x<3$ and $x>-2$. These can be combined into one inequality, $-2<x<3$, or can be written using interval notation $(-2,3)$. note that zero is in this interval, and we can see that $x=0$ makes the original inequality true, so that gives us some confidence in our result.

Sign Charts

You may have found the logic in this last example to be difficult to follow. Unless one is used to working with such things, it can be somewhat confusing. Now let's look at a different method for solving the same kind of problem. It will involve creating something called a **sign chart**. To solve an equality like the one

$$3x^2+13x-10\geq 0$$

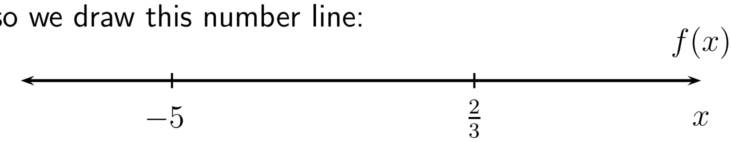
we first define a function $f(x)=3x^2+13x-10$. Then we construct the sign chart for the function as follows:

- Draw a number line, with all the x-values for which the function has value zero marked and labeled. Write x below the right end of the number line.
- Write zero *above* every x-value where $f(x)=0$, and write $f(x)$ above the right end of the line to indicate that the information above the line is about $f(x)$ rather than x.
- The places where the function is zero divide the number line into a number of intervals (in this case, three intervals). Pick a value of x in each interval and find the *sign* of $f(x)$. (Note that you do not need to find the actual value!) This is easily done from the factored form by determining the sign of each factor, then determining what the product will be. Once you have determined the sign of the function in an interval, put that sign over the interval, above the number line and between two zeros.

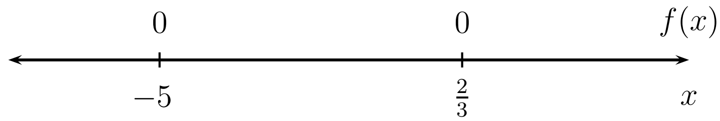
Let's see how this process works:

⋄ **Example 3.4(c):** Create the sign chart for $f(x)=3x^2+13x-10$ and use it to solve the inequality $3x^2+13x-10\geq 0$.

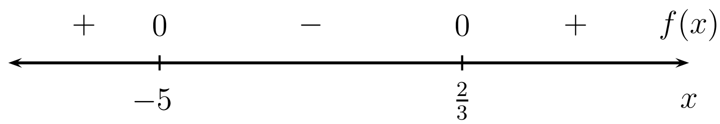
From the factored form $f(x)=(3x-2)(x+5)$ we know that $f(x)=0$ when $x=\frac{2}{3}$ and $x=-5$ so we draw this number line:



We show that $f(x)=0$ at $x=-5$ and $x=\frac{2}{3}$ like this:

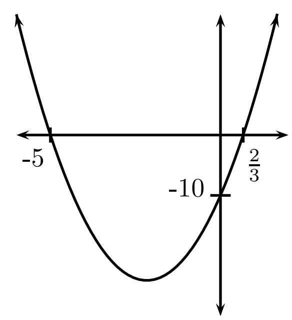


When $x=-6$, $3x-2<0$ and $x+5<0$, so the sign of $f(x)$ is $(-)(-)=+$. When $x=0$, the sign of $f(x)$ is $(-)(+)=-$ and when $x=1$, the sign of $f(x)$ is $(+)(+)=+$ We show these like this:



What the sign chart shows us is that $f(x)>0$ for $x<-5$ and $x>\frac{2}{3}$, $f(x)<0$ for $-5<x<\frac{2}{3}$, and $f(x)=0$ for $x=-5$ and $x=\frac{2}{3}$. From this it is easy to see that the inequality $3x^2+13x-10\geq 0$ holds for $x\leq -5$ and $x\geq \frac{2}{3}$.

To the right is the graph of $f(x)=3x^2+13x-10$. Compare the sign chart from this last example with the graph of the function to see how the sign chart relates to the graph. The sign chart gives us the intervals where the function is positive or negative - remember when looking at the graph that "the function" means the y values.



We can also use our knowledge of what the graph of a quadratic function looks like to get a sign chart, as demonstrated in the following example.

⋄ **Example 3.4(d):** Solve the quadratic inequality $5>x^2+4x$.

The given inequality is equivalent to $0>x^2+4x-5$. Setting $y=x^2+4x-5$ and factoring gives us $y=(x+5)(x-1)$. Therefore the graph of y is a parabola, opening upward and with x-intercepts -5 and 1. y is then less than zero between -5 and 1, not including either. The solution to the inequality is then $-5<x<1$ or $(-5,1)$.

Section 3.4 Exercises

1. Solve each of the following linear inequalities.

(a) $5x + 4 \geq 18$

(b) $6x - 3 < 63$

(c) $8 - 5x \leq -2$

(d) $4x + 2 > x - 7$

(e) $5x - 2(x - 4) > 35$

(f) $5y - 2 \leq 9y + 2$

(g) $3(x - 2) + 7 < 2(x + 5)$

(h) $7 - 4(3x + 1) \geq 2x - 5$

(i) $4(x + 3) \geq x - 3(x - 2)$

(j) $-4x + 3 < -2x - 9$

(k) $8 - 5(x + 1) \leq 4$

(l) $7 - 2x \leq 13$

2. Solve each of the following inequalities. (**Hint:** Get zero on one side first, if it isn't already.) Give your answers as inequalities, OR using interval notation.

(a) $x^2 + 8 < 9x$

(b) $2x^2 \leq x + 10$

(c) $21 + 4x \geq x^2$

(d) $8x^2 < 16x$

(e) $2x + x^2 < 15$

(f) $x^2 + 7x + 6 \geq 0$

(g) $\frac{2}{3}x^2 + \frac{7}{3}x \leq 5$

(h) $y^2 + 3y \geq 18$

(i) $5x^2 + x > 0$

(j) $\frac{1}{15}x^2 \leq \frac{1}{6}x - \frac{1}{10}$

3. For each of the following, factor and use a sign chart to solve the inequality.

(a) $x^3 + 3x^2 + 2x \geq 0$

(b) $10x^3 < 29x^2 + 21x$

4. Try solving each of the following inequalities.

(a) $(x - 3)(x + 1)(x + 4) < 0$

(b) $(x - 5)(x + 2)(x - 2) \geq 0$

(c) $(x - 3)^2(x + 2) > 0$

(d) $(x - 2)^3 \leq 0$

3.5 Applications of Quadratic Functions

Outcome/Criteria:

3. (f) Solve a problem using a given quadratic model; create a quadratic model for a given situation.

Quadratic functions arise naturally in a number of applications. As usual, we often wish to know the output for a given input, or what inputs result in a desired output. However, there are other things we may want to know as well, like the maximum or minimum value of a function, and we are now prepared to do that for quadratic functions. We can also determine when a quadratic function is positive or negative.

For this section we will dispense with examples; you should be able to do everything necessary with the skills developed in the last two sections, and in Chapter 2.

Section 3.5 Exercises

1. The height h (in feet) of an object that is thrown vertically upward with a starting velocity of 48 feet per second is a function of time t (in seconds) after the moment that it was thrown. The function is given by the equation $h(t) = 48t - 16t^2$.

   (a) How high will the rock be after 1.2 seconds? Round your answer to the nearest tenth of a foot.

   (b) When will the rock be at a height of 32 feet? You should be able to do this by factoring, and you will get two answers; explain why this is, physically.

   (c) Find when the rock will be at a height of 24 feet. Give your answer rounded to the nearest tenth of a second.

   (d) When does the rock hit the ground?

   (e) What is the mathematical domain of the function? What is the feasible domain of the function?

   (f) What is the maximum height that the rock reaches, and when does it reach that height?

   (g) For what time periods is the height less than 15 feet? Round to the nearest hundredth of a second.

2. In statistics, there is a situation where the expression $x(1 - x)$ is of interest, for $0 \leq x \leq 1$. What is the maximum value of the function $f(x) = x(1 - x)$ on the interval $[0, 1]$? Explain how you know that the value you obtained is in fact a maximum value, not a minimum.

3. Use the quadratic formula to solve $s = s_0 + v_0t - \frac{1}{2}gt^2$ for t. (**Hint:** Move everything to the left side and apply the quadratic formula. Keep in mind that t is the variable, and all other letters are constants.)

4. The Acme Company also makes and sells Geegaws, in addition to Widgets and Gizmos. Their weekly profit P (in dollars) is given by the equation $P = -800 + 27x - 0.1x^2$, where x is the number of Geegaws sold that week. Note that it will be possible for P to be negative; of course negative profit is really loss!

   (a) What is the P-intercept of the function? What does it mean, "in reality"?

   (b) Find the numbers of Geegaws that Acme can make and sell in order to make a true profit; that is, we want P to be positive. Round to the nearest whole Geegaw.

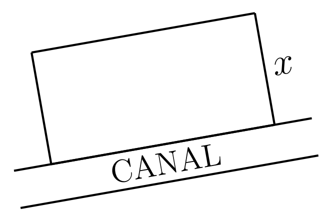
   (c) Find the maximum profit they can get. How do we know that it is a maximum, and not a minimum?

5. In a previous exercise you found the revenue equation $R = 20000p - 100p^2$ for sales of Widgets, where p is the price of a widget and R is the revenue obtained at that price.

   (a) Find the price that gives the maximum revenue, and determine the revenue that will be obtained at that price.

   (b) For what prices will the revenue be at least \$500,000?

6. Another farmer is going to create a rectangular field (without compartments) against a straight canal, by putting fence along the three straight sides of the field that are away from the canal. (See picture to the right.) He has 1000 feet of fence with which to do this.



   (a) Write an equation (in simplified form) for the area A of the field, in terms of x.

   (b) Give the feasible domain of the function.

   (c) Determine the maximum area of the field.

   (d) Determine all possible values of x for which the area is at least 90,000 square feet.

3.6 Quadratic Functions in Standard Form

Outcome/Criteria:

3. (g) Graph a parabola with equation given in $y = a(x-h)^2+k$ form, relative to $y = x^2$.

   (h) Given the vertex of a parabola and one other point on the parabola, give the equation of the parabola.

   (i) Put the equation of a parabola in the form $y = a(x-h)^2+k$ by completing the square.

Graphing $y = a(x-h)^2 + k$

In this section we will see that when the equations of quadratic functions are in a certain standard for it is quite easy to determine the appearance of the graph of the function.

◇ **Example 3.6(a):** Put the function $f(x) = -\frac{1}{2}(x+1)^2 - 2$ into $f(x) = ax^2 + bx + c$ form, then find the coordinates of the vertex.

We see that $f(x) = -\frac{1}{2}(x+1)^2 - 2 = -\frac{1}{2}(x^2+2x+1) - 2 = -\frac{1}{2}x^2 - x - \frac{5}{2}$. From this we find that the x-coordinate of the vertex is $x = -\dfrac{-1}{2(-\frac{1}{2})} = -\dfrac{-1}{-1} = -1$. Substituting this into the function gives us $f(1) = -\frac{1}{2}(-1+1)^2 - 2 = -2$, which is then the y-coordinate of the vertex. The vertex of the parabola is $(-1, -2)$.

As seen, the equation $f(x) = -\frac{1}{2}(x+1)^2 - 2$ is the equation of a parabola. We should recognize that the value of x that makes $x+1$ equal to zero, $x = -1$, should be important for some reason. The only point on a parabola that is any more noteworthy than all the others is the vertex and we saw in this example that -1 is in fact the x-coordinate of the vertex. The y-coordinate of the vertex is the corresponding value of y, which is -2. None of this is a fluke, and the facts can be summarized like this:

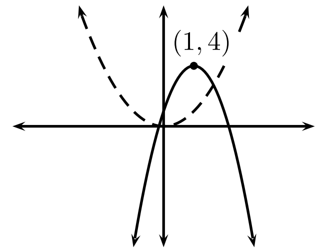
The Equation $y = a(x-h)^2 + k$

The graph of $y = a(x-h)^2 + k$ is a parabola. The x-coordinate of the vertex of the parabola is the value of x that makes $x - h$ equal to zero, and the y-coordinate of the vertex is k. The number a is the same as in the form $y = ax^2 + bx + c$ and has the following effects on the graph:

- If a is positive, the parabola opens upward, and if a is negative, the parabola opens downward.
- If the absolute value of a is less than one, the parabola is "smashed flatter" than the parabola $y = x^2$.
- If the absolute value of a is greater than one, the parabola is stretched vertically, to become "narrower" than the parabola $y = x^2$.

- **Example 3.6(b):** Sketch the graph of $y = -3(x-1)^2 + 4$ on the same grid with a dashed graph of $y = x^2$. Label the vertex with its coordinates and make sure the shape of the graph compares correctly with that of $y = x^2$.

  The value that makes the quantity $x-1$ zero is $x = 1$, so that is the x-coordinate of the vertex, and the y-coordinate is $y = 4$. Because $a = -3$, the parabola opens downward and is narrower than $y = x^2$. From this we can graph the parabola as shown to the right.



Finding the Equation of a Parabola

If we are given the coordinates of the vertex of a parabola and the coordinates of one other point on the parabola, it is quite straightforward to find the equation of the parabola using the equation $y = a(x-h)^2 + k$. The procedure is a bit like what is done to find the equation of a line through two points, and we'll demonstrate it with an example.

- **Example 3.6(c):** Find the equation of the parabola with vertex $(-3, -4)$ with one x-intercept at $x = -7$.

  Since the x-coordinate of the vertex is -3 , the factor $x - h$ must be $x + 3$ so that $x = -3$ will make it zero. Also, k is the y-coordinate of the vertex, -4. From these two things we know the equation looks like $y = a(x+3)^2 - 4$. Now we need values of x and y for some other point on the parabola; if we substitute them in we can solve for a. Since -7 is an x-intercept, the y-coordinate that goes with it is zero, so we have the point $(-7, 0)$. Substituting it in we get

$$\begin{aligned} 0 &= a(-7+3)^2 - 4 \\ 4 &= (-4)^2 a \\ 4 &= 16a \\ \tfrac{1}{4} &= a \end{aligned}$$

  The equation of the parabola is then $y = \frac{1}{4}(x+3)^2 - 4$.

Putting an Equation in $y = a(x-h)^2 + k$ form

What we have seen is that if we have the equation of a parabola in the form $y = a(x-h)^2 + k$ it is easy to get a general idea of what the graph of the parabola looks like. Suppose instead that we have a quadratic function in the form $y = ax^2 + bx + c$. If we could put it in the standard form $y = a(x-h)^2 + k$, then we would be able to graph it easily. That can be done by completing the square; here's how to do this when $a = 1$:

$y = x^2 - 6x - 3$	
$y = (x^2 - 6x + \underline{\quad}) - 3$	pull constant term away from the x^2 and x terms
$y = (x^2 - 6x + 9) - 9 - 3$	multiply 6 by $\frac{1}{2}$ and square to get 9, add and subtract
$y = (x-3)^2 - 12$	factor the trinomial, combine "extra constants"

And here's how to do it when a is not one:

◇ **Example 3.6(d):** Put the equation $y = -2x^2 + 12x + 5$ into the form $y = a(x - h)^2 + k$ by completing the square.

$y = -2x^2 + 12x + 5$	
$y = -2(x^2 - 6x) + 5$	factor -2 out of the x^2 and x terms
$y = -2(x^2 - 6x + 9 - 9) + 18 + 5$	add and subtract 9 inside the parentheses
$y = -2(x^2 - 6x + 9) + 18 + 5$	bring the -9 out of the parentheses - it comes out as 18 because of the -2 factor outside the parentheses
$y = -2(x - 3)^2 + 23$	factor the trinomial, combine 18 and 5

Section 3.6 Exercises

1. For each of the following, sketch the graph of $y = x^2$ and the given function on the same coordinate grid. The purpose of sketching $y = x^2$ is to show how "open" the graph of the given function is, *relative to* $y = x^2$.

   (a) $f(x) = 3(x - 1)^2 - 7$ (b) $g(x) = (x + 4)^2 - 2$

   (c) $h(x) = -\frac{1}{3}(x + 2)^2 + 5$ (d) $y = -(x - 5)^2 + 3$

2. For each of the following, find the equation of the parabola meeting the given conditions. Give your answer in the standard form $y = a(x - h)^2 + k$.

   (a) Vertex $V(-5, -7)$ and through the point $(-3, -25)$.

   (b) Vertex $V(6, 5)$ and y-intercept 29.

   (c) Vertex $V(4, -3)$ and through the point $(-5, 6)$.

   (d) Vertex $V(-1, -4)$ and y-intercept 1.

3. For each of the following, use completing the square to write the equation of the function in the standard form $f(x) = a(x-h)^2+k$. Check your answers by graphing the original equation and your standard form equation together on your calculator - they should of course give the same graph if you did it correctly! Hint for (c): Divide (or multiply) both sides by -1.

   (a) $f(x) = x^2 - 6x + 14$ (b) $g(x) = x^2 + 12x + 35$

   (c) $h(x) = -x^2 - 8x - 13$ (d) $y = 3x^2 + 12x + 50$

4. Write the quadratic function $y = 3x^2 - 6x + 5$ in $y = a(x - h)^2 + k$ form.

5. Write the quadratic function $h(x) = -\frac{1}{4}x^2 + \frac{5}{2}x - \frac{9}{4}$ in $y = a(x - h)^2 + k$ form.

6. Sketch the graphs of $y = x^2$ and $f(x) = -\frac{3}{4}(x - 1)^2 + 6$ without using your calculator, check with your calculator.

7. Find the equation of the parabola with vertex $V(4, 1)$ and y-intercept -7. Give your answer in $y = a(x - h)^2 + k$ form.

8. Write the quadratic function $y = -3x^2 + 6x - 10$ in $y = a(x - h)^2 = k$ form.

9. Find the equation of the quadratic function whose graph has vertex $V(3, 2)$ and passes through $(-1, 10)$. Write your answer in $y = a(x - h)^2 + k$ form.

10. Write the quadratic function $y = 5x^2 + 20x + 17$ in $y = a(x - h)^2 + k$ form.

3.7 Average Rates of Change, Secant Lines and Difference Quotients

Performance Criteria:

3. (j) Find the average rate of change of a function over an interval, from either the equation of the function or the graph of the function. Include units when appropriate.

   (k) Find and simplify a difference quotient.

Average Rate of Change

◇ **Example 3.7(a):** Suppose that you left Klamath Falls at 10:30 AM, driving north on on Highway 97. At 1:00 PM you got to Bend, 137 miles from Klamath Falls. How fast were you going on this trip?

Since the trip took two and a half hours, we need to divide the distance 137 miles by 2.5 hours, with the units included in the operation:

$$\text{speed} = \frac{137 \text{ miles}}{2.5 \text{ hours}} = 54.8 \text{ miles per hour}$$

Of course we all know that you weren't going 54.8 miles per hour the entire way from Klamath Falls to Bend; this speed is the *average* speed during your trip. When you were passing through all the small towns like Chemult and La Pine you likely slowed down to 30 or 35 mph, and between towns you may have exceeded the legal speed limit. 54.8 miles per hour is the speed you would have to go to make the trip in two and a half hours driving at a constant speed for the entire distance. The speed that you see any moment that you look at the speedometer of your car is called the *instantaneous* speed.

Speed is a quantity that we call a **rate of change**. It tells us the change in distance (in the above case, measured in miles) for a given change in time (measured above in hours). That is, for every change in time of one hour, an additional 54.8 miles of distance will be gained. In general, we consider rates of change when one quantity depends on another; we call the first quantity the **dependent variable** and the second quantity the **independent variable**. In the above example, the distance traveled depends on the time, so the distance is the dependent variable and the time is the independent variable. We find the average rate of change as follows:

$$\text{average rate of change} = \frac{\text{change in dependent variable}}{\text{change in independent variable}}$$

The changes in the variables are found by subtracting. The order of subtraction does not matter for one of the variables, but *the values of the other variable must be subtracted in the corresponding order*. This should remind you of the process for finding a slope. In fact, we will see soon that every average rate of change can be interpreted as the slope of a line.

The following example will illustrate what we have just talked about.

◇ **Example 3.7(b):** Again you were driving, this time from Klamath Falls to Medford. At the top of the pass on Highway 140, which is at about 5000 feet elevation, the outside thermometer of your car registered a temperature of 28°F. In Medford, at 1400 feet of elevation, the temperature was 47°. What was the average rate of change of temperature with respect to elevation?

The words "temperature with respect to elevation" implies that temperature is the dependent variable and elevation is the independent variable. From the top of the pass to Medford we have

$$\text{average rate of change} = \frac{28°\text{F} - 47°\text{F}}{5000 \text{ feet} - 1400 \text{ feet}} = \frac{-19°\text{F}}{3600 \text{ feet}} = -0.0053°\text{F per foot}$$

Note that if instead we were to consider going from Medford to the top of the pass we would have

$$\text{average rate of change} = \frac{47°\text{F} - 28°\text{F}}{1400 \text{ feet} - 5000 \text{ feet}} = \frac{19°\text{F}}{-3600 \text{ feet}} = -0.0053°\text{F per foot}$$

How do we interpret the negative sign with our answer? Usually, when interpreting a rate of change, its value is *the change in the dependent variable for each INCREASE in one unit of the independent variable.* So the above result tells us that the temperature (on that particular day, at that time and in that place) *decreases* by 0.0053 degrees Fahrenheit for each foot of elevation *gained.*

Average Rate of Change of a Function

You probably recognized that what we have been calling independent and dependent variables are the "inputs" and "outputs" of a function. This leads us to the following:

> **Average Rate of Change of a Function**
>
> For a function $f(x)$ and two values a and b of x with $a < b$, the **average rate of change of** f **with respect to** x over the interval $[a, b]$ is
>
> $$\left.\frac{\Delta f}{\Delta x}\right|_{[a,b]} = \frac{f(b) - f(a)}{b - a} = \frac{f(a) - f(b)}{a - b}$$

There is no standard notation for average rate of change, but it is standard to use the capital Greek letter delta (Δ) for "change." The notation $\left.\frac{\Delta f}{\Delta x}\right|_{[a,b]}$ could be understood by any mathematician as change in f over change in x, over the interval $[a, b]$. Because it is over an interval it must necessarily be an average, and the fact that it is one change divided by another indicates that it is a *rate* of change.

◇ **Example 3.7(c):** For the function $h(t) = -16t^2 + 48t$, find the average rate of change of h with respect to t from $t = 0$ to $t = 2.5$.

$$\left.\frac{\Delta h}{\Delta t}\right|_{[0,2.5]} = \frac{h(2.5) - h(0)}{2.5 - 0} = \frac{20 - 0}{2.5 - 0} = 8$$

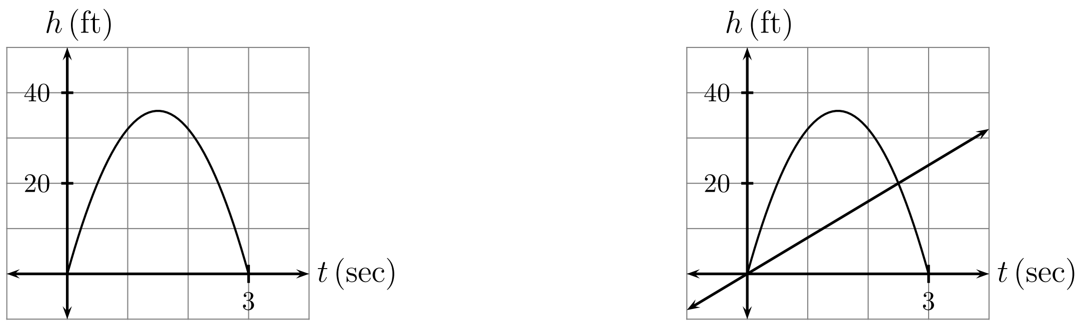
You might recognize $h(t) = -16t^2 + 48t$ as the height h of a projectile at any time t, with h in feet and t in seconds. The units for our answer are then feet per second, indicating that over the first 2.5 seconds of its flight, the height of the projectile increases at an average rate of 8 feet per second.

Secant Lines

The graph below and to the left is for the function $h(t) = -16t^2 + 48t$ of Example 3.7(c). To the right is the same graph, with a line drawn through the two points $(0, 0)$ and $(2.5, 20)$. Recall that the average rate of change in height with respect to time from $t = 0$ to $t = 2.5$ was determined by

$$\left.\frac{\Delta h}{\Delta t}\right|_{[0,2.5]} = \frac{h(2.5) - h(0)}{2.5 - 0} = \frac{20 - 0}{2.5 - 0} = 8 \text{ feet per second}$$

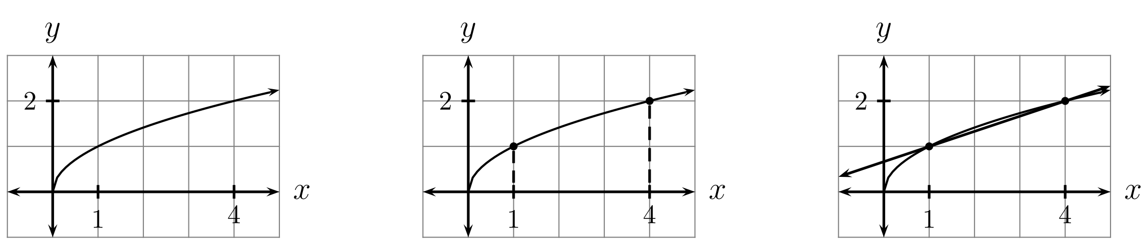
With a little thought, one can see that the average rate of change is simply the slope of the line in the picture to the right below.



A line drawn through two points on the graph of a function is called a **secant line**, and the slope of a secant line represents the rate of change of the function between the values of the independent variable where the line intersects the graph of the function.

◇ **Example 3.7(d):** Sketch the secant line through the points on the graph of the function $y = \sqrt{x}$ where $x = 1$ and $x = 4$. Then compute the average rate of change of the function between those two x values.

The graph below and to the left is that of the function. In the second graph we can see how we find the two points on the graph that correspond to $x = 1$ and $x = 4$. The last graph shows the secant line drawn in through those two points.



The average rate of change is $\frac{2-1}{4-1} = \frac{1}{3}$, the slope of the secant line.

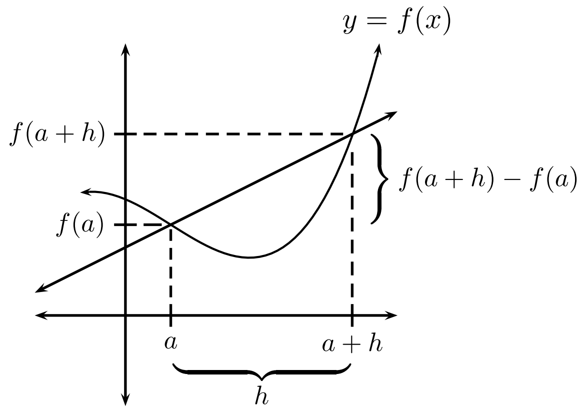
Difference Quotients

The following quantity is the basis of a large part of the subject of calculus:

> **Difference Quotient for a Function**
>
> For a function $f(x)$ and two values a and h of x, the **difference quotient** for f is
>
> $$\frac{f(a+h)-f(a)}{h}$$

Let's rewrite the above expression to get a better understanding of what it means. With a little thought you should agree that

$$\frac{f(a+h)-f(a)}{h} = \frac{f(a+h)-f(a)}{(a+h)-a}$$



This looks a lot like a slope, and we can see from the picture to the right that it is. The quantity $f(a+h)-f(a)$ is a "rise" and $(a+h)-a=h$ is the corresponding "run." The difference quotient is then the slope of the tangent line through the two points $(a, f(a))$ and $(a+h, f(a+h))$.

◇ **Example 3.7(e):** For the function $f(x) = 3x-5$, find and simplify the difference quotient $\dfrac{f(2+h)-f(2)}{h}$.

$$\frac{f(2+h)-f(2)}{h} = \frac{[3(2+h)-5]-[3(2)-5]}{h} = \frac{[6+3h-5]-[6-5]}{h} = \frac{3h}{h} = 3$$

Note that in the above example, the h in the denominator eventually canceled with one in the numerator. *This will always happen if you carry out all computations correctly!*

◇ **Example 3.7(f):** Find and simplify the difference quotient $\dfrac{g(x+h)-g(x)}{h}$ for the function $g(x) = 3x-x^2$.

With more complicated difference quotients like this one, it is sometimes best to compute $g(x+h)$, then $g(x+h)-g(x)$, before computing the full difference quotient:

$$g(x+h) = 3(x+h)-(x+h)^2 = 3x+3h-(x^2+2xh+h^2) = 3x+3h-x^2-2xh-h^2$$

$$g(x+h)-g(x) = (3x+3h-x^2-2xh-h^2)-(3x-x^2) = 3h-2xh-h^2$$

$$\frac{g(x+h)-g(x)}{h} = \frac{3h-2xh-h^2}{h} = \frac{h(3-2x-h)}{h} = 3-2x-h$$

Section 3.7 Exercises

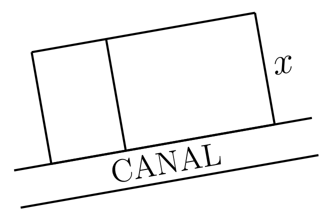
1. In Example 2.4(a) we examined the growth of a rectangle that started with a width of 5 inches and a length of 8 inches. At time zero the width started growing at a constant rate of 2 inches per minute, and the length began growing by 3 inches per minute. We found that the equation for the area A (in square inches) as a function of time t (in minutes) was $A = 6t^2 + 31t + 40$. Determine the average rate of increase in area with respect to time from time 5 minutes to time 15 minutes. Give units with your answer!

2. The height h (feet) of a projectile at time t (seconds) is given by $h = -16t^2 + 144t$.

   (a) Find the average rate of change in height, with respect to time, from 4 seconds to 7 seconds. Give your answer as a sentence that includes these two times, your answer, and whichever of the words *increasing* or *decreasing* that is appropriate.

   (b) Find the average rate of change in height, with respect to time, from time 2 seconds to time 7 seconds. Explain your result *in terms of the physical situation*.

3. A farmer is going to create a rectangular field with two compartments against a straight canal, as shown to the right, using 1000 feet of fence. No fence is needed along the side formed by the canal.



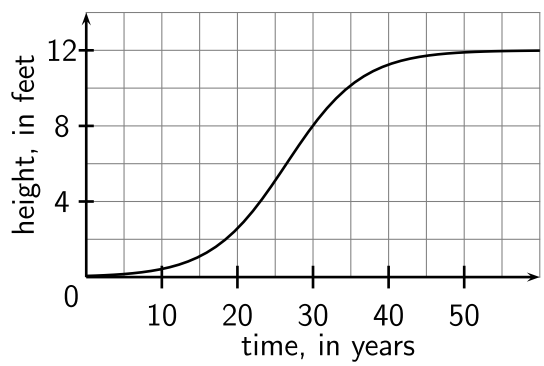
   (a) Find the total area of the field when x is 200 feet and again when x is 300 feet.

   (b) Find the average rate of change in area, with respect to x, between when x is 200 feet and when it is 300 feet. Give units with your answer.

4. The graph to the right shows the heights of a certain kind of tree as it grows. Use it to find the average rate of change of height with respect to time from

   (a) 10 years to 50 years

   (b) 25 years to 30 years



5. In a previous exercise you found the revenue equation $R = 20000p - 100p^2$ for sales of Widgets, where p is the price of a widget and R is the revenue obtained at that price.

   (a) Find the average change in revenue, with respect to price, from a price of \$50 to \$110. Include units with your answer.

   (b) Sketch a graph of the revenue function and draw in the secant line whose slope represents your answer to (a). Label the relevant values on the horizontal and vertical axes.

6. (a) Sketch the graph of the function $y = 2^x$. Then draw in the secant line whose slope represents the average rate of change of the function from $x = -2$ to $x = 1$.

   (b) Determine the average rate of change of the function from $x = -2$ to $x = 1$. Give your answer in (reduced) fraction form.

7. Find the average rate of change of $f(x) = x^3 - 5x + 1$ from $x = 1$ to $x = 4$.

8. Find the average rate of change of $g(x) = \dfrac{3}{x-2}$ from $x = 4$ to $x = 7$.

9. Consider the function $h(x) = \frac{2}{3}x - 1$.

   (a) Find the average rate of change from $x = -6$ to $x = 0$.

   (b) Find the average rate of change from $x = 1$ to $x = 8$.

   (c) You should notice two things about your answers to (a) and (b). What are they?

10. For the function $h(x) = \frac{2}{3}x - 1$ from the previous exercise, find and simplify the difference quotient $\dfrac{h(x+s) - h(s)}{s}$.

11. For the function $f(x) = 3x^2$, find and simplify the difference quotient $\dfrac{f(5+h) - f(5)}{h}$.

12. (a) Compute and simplify $(x+h)^3$

    (b) For the function $g(x) = x^3 - 5x + 1$, find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$.

    (c) In Exercise 7 you found the average rate of change of $f(x) = x^3 - 5x + 1$ from $x = 1$ to $x = 4$. That is equivalent to computing the difference quotient with $x = 1$ and $x + h = 4$. Determine the value of h, then substitute it and $x = 1$ into your answer to part (b) of this exercise. You should get the same thing as you did for Exercise 7!

13. For this exercise you will be working with the function $f(x) = \dfrac{1}{x}$.

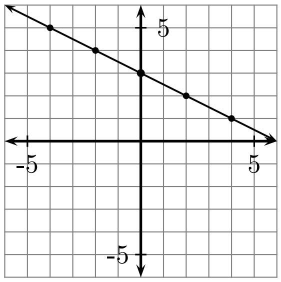
    (a) Find and simplify $f(x+h) - f(x)$ by carefully obtaining a common denominator and combining the two fractions.

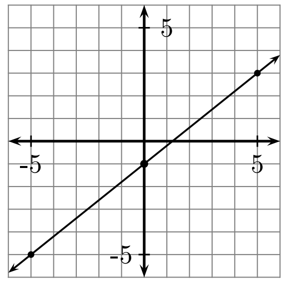
    (b) Using the fact that dividing by h is the same as multiplying by $\frac{1}{h}$, find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$.

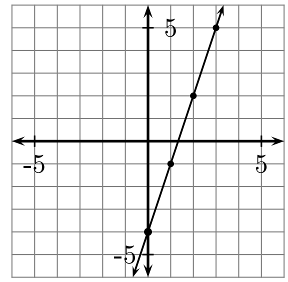
Chapter 3 Solutions

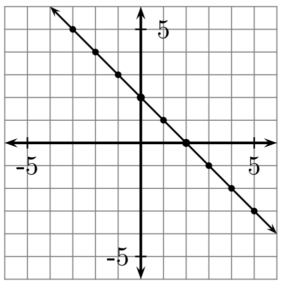
Section 3.1

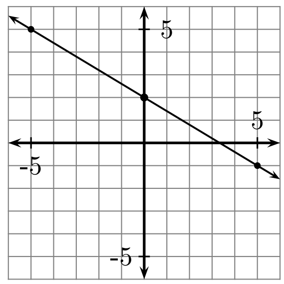
1. (a) $-\frac{2}{3}$ (b) $\frac{2}{5}$ (c) 0 (d) undefined (e) 3 (f) $-\frac{1}{2}$
2. (a) $-\frac{5}{2}$ (b) 0 (c) $\frac{3}{4}$ (d) undefined (e) $\frac{4}{3}$ (f) $-\frac{1}{3}$
3. (a) $y = -\frac{5}{2}x + 3$ (b) $y = 3$ (c) $y = \frac{3}{4}x - 2$
   (d) $x = 3$ (e) $y = \frac{4}{3}x$ (f) $y = -\frac{1}{3}x + 2$
4. 3
5. $\frac{1}{2}$

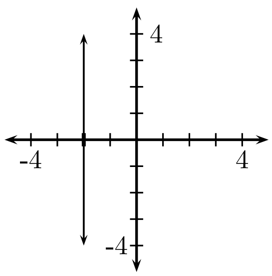
6. (a) 

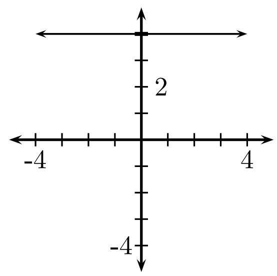
(b) 

(c) 

(d) 

(e) 

7. (a) 

(b) 

8. (a) $y = \frac{4}{3}x + 2$ (b) $y = -\frac{2}{3}x + 7$ (c) $y = 2$
(d) $y = \frac{1}{4}x - 5$ (e) $y = -\frac{3}{4}x + \frac{1}{2}$ (f) $y = 4$
(g) $y = 3x + 10$ (h) $x = 1$ (i) $y = 4x - 1$
(j) $y = 2x + 5$ (k) $y = -\frac{1}{11}x - \frac{28}{11}$ (l) $y = \frac{5}{2}x$

9. $3x - 4y = 7 \Rightarrow y = \frac{3}{4}x - \frac{7}{4}$, $8x - 6y = 1 \Rightarrow y = \frac{4}{3}x - \frac{1}{6}$ The lines are neither parallel nor perpendicular.

10. $y = \frac{3}{2}x - 2$ 11. $y = \frac{3}{4}x + 3$ 12. $y = -5$ 13. $y = -\frac{4}{5}x - \frac{7}{5}$

14. The slope of the line through A and B is $\frac{3}{4}$ and the slope of the line through B and C is $\frac{6}{8} = \frac{3}{4}$. The three points lie on the same line.

15. $y = \frac{3}{4}x + 6\frac{1}{4}$

Section 3.2

1. (a) 17.4 mpg, 17.3 mpg, The mileage decreased by 0.1 mpg as the speed increased from 45 to 50 mph.

   (b) 16.5 mpg, 16.4 mpg, The mileage decreased by 0.1 mpg as the speed increased from 90 to 95 mph.

   (c) The slope of the line is -0.02, and when you multiply it by 5 you get -0.1. This the change in mileage (negative indicating a decrease) in mileage as the speed is increased by 5 mph.

   (d) 40 mph

   (e) The slope of the line is $-0.02\ \frac{\text{mpg}}{\text{mph}}$. This tells us that the car will get 0.02 miles per gallon less fuel economy for each mile per hour that the speed increases.

   (f) We should not interpret the value of the m-intercept because it should represent the mileage when the speed is zero miles per hour. We are told in the problem that the equation only applies for speeds between 30 and 100 miles per hour. Furthermore, when the speed is zero mph the mileage should be infinite, since no gas is being used!

2. (a) The weight of a lizard that is 3 cm long is -18 grams. This is not reasonable because a weight cannot be negative. The problem is that the equation is only valid for lizards between 12 and 30 cm in length.

   (b) The w-intercept is -84, which would be the weight of a lizard with a length of zero inches. It is not meaningful for the same reason as given in (a).

   (c) The slope is 22 grams per centimeter. This says that for each centimeter of length gained by a lizard, the weight gain will be 22 grams.

3. (a) \$2300, \$3800 (b) $P = 0.03S + 800$

   (c) the slope of the line is 0.03 dollars of pay per dollar of sales, the commission rate.

   (a) The P-intercept of the line is \$800, the monthly base salary.

4. (a) The F-intercept is $32°$, which is the Fahrenheit temperature when the Celsius temperature is 0 degrees.

   (b) The slope of the line is $\frac{9}{5}$ degrees Fahrenheit per degree Celsius; it tells us that each degree increase in Celsius temperature is equivalent to $\frac{9}{5}$ degrees increase in Fahrenheit temperature.

5. (a) The C-intercept is \$5000. This represents costs that Acme has regardless of how many Widgets they make. These are usually referred to as *fixed costs*.

   (b) The slope of the line is \$7 per Widget. It says that each additional Widget produced adds \$7 to the total cost. Business people often refer to this as "marginal cost."

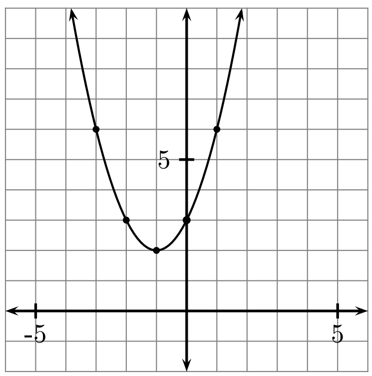
   (c) The total cost is \$15,437, the cost per Widget is \$10.35

6. The slope of \$0.24 per mile is the additional cost for each additional mile driven. The intercept of \$30 is the the base charge for one day, without mileage.

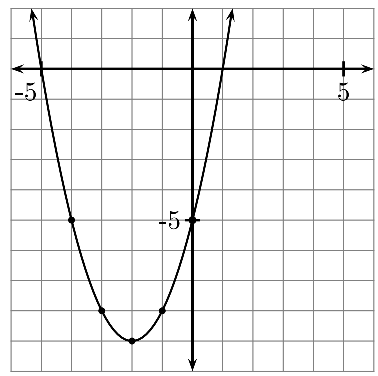
7. (a) $W = 6t + 8$
   (b) The slope is 6 pounds per year, and represents the additional weight the baby will gain each year after birth.
   (c) The W-intercept of 8 pounds is the weight at birth, which is time zero.
   (d) The child weighed 26 pounds at 3 years of age.

Section 3.3

1. See graph below and to the left.



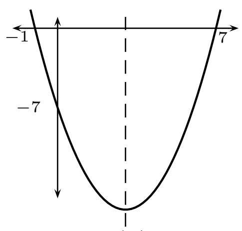
Exercise 1



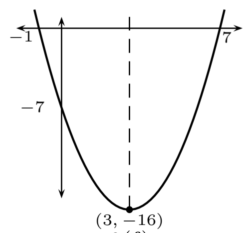
Exercise 2

2. The graph has the same x-intercepts as $f(x) = -x^2 - 4x + 5$, but the the y-intercept and y-coordinate of the vertex are the opposites of what they are for f.
3. The graph of a quadratic function can have 0, 1 or 2 x-intercepts.
4. (a) The x-intercept cannot be found. When we try to use the quadratic formula to find the x-intercepts, we get a negative number under a square root.
   (b) The parabola opens upward.
   (c) The vertex is $(3, 4)$.
   (d) Because the vertex is above the x-axis and the parabola opens upward, there are no x-intercepts.
   (e) $(2, 5)$, $(4, 5)$, $(1, 8)$, $(5, 8)$, $(0, 13)$, $(6, 13)$, etc.
5. Every quadratic function has exactly one absolute maximum (if the graph of the function opens downward) or one absolute minimum (if the graph opens upward), but not both. The are no relative maxima or minima other than the absolute maximum or minimum.
6. (a) upward   (b) -7 (c) -1, 7
   (d) See below. The x-coordinate of the vertex is 3.
   (e) The y-coordinate of the vertex is -16.   (f) See below.
   (d) The minimum value of the function is -16, which occurs at $x = 3$.

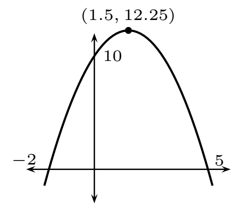
7. 6.5 8. (a) $a = -1,\ b = 3,\ c = 10$ (b) See below.



6(d)



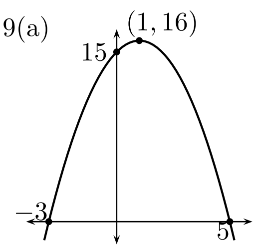
6(f)



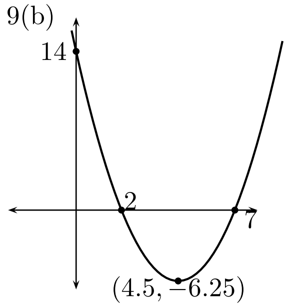
8(b)

9. (a) Parabola opening downward, x-intercepts $(-3, 0)$, $(5, 0)$, y-intercept $(0, 15)$, vertex $(1, 16)$. See graph below. The function has an absolute maximum of 16 at $x = 1$.
   (b) Parabola opening upward, x-intercepts $(2, 0)$, $(7, 0)$, y-intercept $(0, 14)$, vertex $(4\frac{1}{2}, -6\frac{1}{4})$. See graph below. The function has an absolute minimum of $-6\frac{1}{4}$ at $x = 4\frac{1}{2}$.
   (c) Parabola opening downward, x-intercepts $(1, 0)$, $(9, 0)$, y-intercept $(0, -2\frac{1}{4})$, vertex $(5, 4)$. See graph below. The function has an absolute maximum of 5 at $x = 4$.
   (d) Parabola opening upward, x-intercept $(-3, 0)$, y-intercept $(0, 4\frac{1}{2})$, vertex $(-3, 0)$. See graph below. The function has an absolute minimum of 0 at $x = -3$.

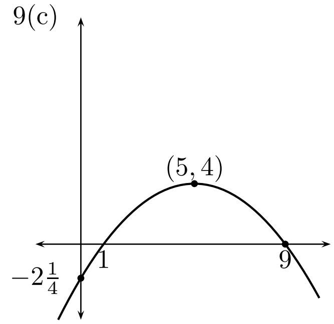
9(a)



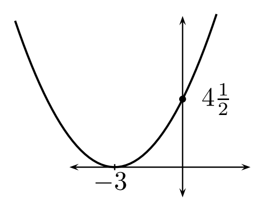
9(b)



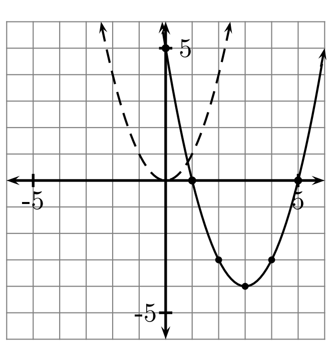
9(c)



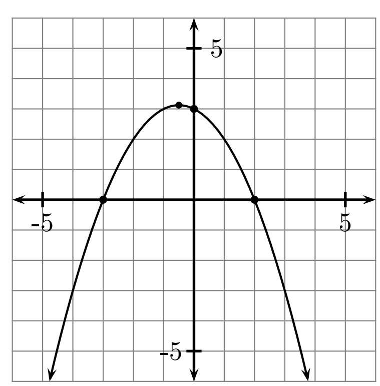
9(d)



10. (a) $x^2 - 6x + 5$
    (b) $(3, -4)$ The x-coordinate is the value that makes $(x-3)$ zero, and the y-coordinate is the -4 at the end of the equation.
    (c) The x-intercepts are 1 and 5, and the y-intercept is 5.
    (d) See on next page.



Exercise 10



Exercise 11

11. (a) a would be $-\frac{1}{2}$, so the parabola opens down.
    (b) The y-intercept is $y = -\frac{1}{2}(0-2)(0+3) = 3$.
    (c) The x-intercepts are 2 and -3.
    (d) The x-coordinate of the vertex is $x = \dfrac{2+(-3)}{2} = -\dfrac{1}{2}$.
    The y-coordinate of the vertex is $y = -\frac{1}{2}(-\frac{1}{2}-2)(-\frac{1}{2}+3) = -\frac{1}{2}(-\frac{5}{2})(\frac{5}{2}) = \frac{25}{8} = 3\frac{1}{8}$.
    (e) See above.

Section 3.4

1. (a) $x \geq \frac{14}{5}$ (b) $x < 11$ (c) $x \geq 2$ (d) $x > -3$
   (e) $x > 9$ (f) $y \geq -1$ (g) $x < 9$ (h) $x \leq \frac{4}{7}$
   (i) $x \geq -1$ (j) $x > 2$ (k) $x \geq -\frac{1}{5}$ (l) $x \geq -3$
2. (a) $1 < x < 8$, OR $(1, 8)$
   (b) $-2 \leq x \leq \frac{5}{2}$, OR $[-2, \frac{5}{2}]$
   (c) $x \leq -3$ or $x \geq 7$, OR $(-\infty, -3] \cup [7, \infty)$
   (d) $0 < x < 2$, OR $(0, 2)$
   (e) $-5 < x < 3$, OR $(-5, 3)$
   (f) $x \leq -6$ or $x \geq -1$, OR $(-\infty, -6] \cup [-1, \infty)$
   (g) $-5 \leq x \leq \frac{3}{2}$, OR $[-5, \frac{3}{2}]$
   (h) all real numbers
   (i) $x < -\frac{1}{5}$ or $x > 0$, OR $(-\infty, -\frac{1}{5}) \cup (0, \infty)$
   (j) $1 \leq x \leq \frac{3}{2}$, OR $[1, \frac{3}{2}]$
3. (a) $-2 \leq x \leq -1$ or $x \geq 0$, OR $[-2, -1] \cup [0, \infty)$
   (b) $x < -\frac{3}{5}$ or $0 < x < \frac{7}{2}$, OR $(-\infty, -\frac{3}{5}) \cup (0, \frac{7}{2})$
4. (a) $x < -4$ or $-1 < x < 3$, OR $(-\infty, -4) \cup (-1, 3)$
   (b) $-2 \leq x \leq 2$ or $x \geq 5$, OR $[-2, -2] \cup [5, \infty)$
   (c) $-2 < x < 3$ or $x > 3$, OR $(-2, 3) \cup (3, \infty)$
   (d) $x \leq 2$, OR $(-\infty, 2]$

Section 3.5

1. (a) 34.6 feet
   (b) The rock will be at a height of 32 feet at 1 second (on the way up) and 2 seconds (on the way down).
   (c) 0.6 seconds, 2.4 seconds
   (d) The rock hits the ground at 3 seconds.

(e) The mathematical domain is all real numbers. The feasible domain is [0,3]

(f) The rock reaches a maximum height of 36 feet at 1.5 seconds.

(g) The rock is at a height of less than 15 feet on the time intervals $[0, 0.35]$ and $[2.65, 3]$.

2. We can see that $f(x) = x - x^2$, so its graph is a parabola opening downward. From the form $f(x) = x(1-x)$ it is easy to see that the function has x-intercepts zero and one, so the x-coordinate of the vertex is $x = \frac{1}{2}$. Because $f(\frac{1}{2}) = \frac{1}{4}$, the maximum value of the function is $\frac{1}{4}$ at $x = \frac{1}{2}$.

3. $t = \dfrac{v_0 \pm \sqrt{v_0^2 - 2gs + 2gs_0}}{g}$

4. (a) The P-intercept of the function is -800 dollars, which means that the company loses \$800 if zero Geegaws are sold.

   (b) Acme must make and sell between 34 and 236 Geegaws to make a true profit.

   (c) The maximum profit is \$1022.50 when 135 Geegaws are made and sold. We know it is a maximum because the graph of the equation is a parabola that opens downward.

5. (a) Maximum revenue is \$1,000,000 when the price is $\$100$ per Widget.

   (b) The price needs to be between $\$29.29$ and $\$170.71$.

6. (a) $A = x(1000 - 2x) = 1000x - 2x^2$

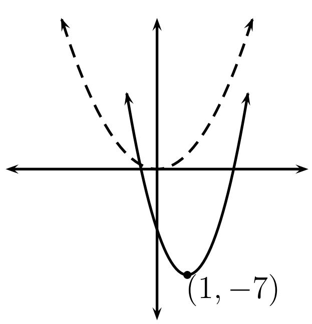
   (b) $0 < x < 500$ or $(0, 500)$

   (c) The maximum area of the field is $125{,}000$ square feet when $x = 250$ feet.

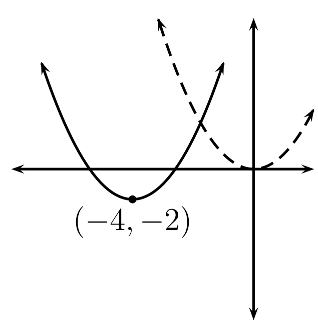
   (d) x must be between 117.7 and 382.3 feet.

Section 3.6

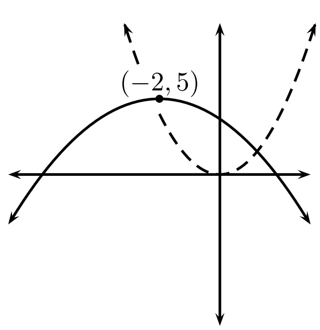
1. (a)



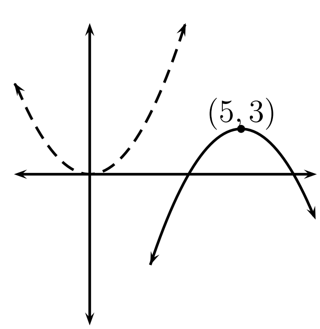
(b)



(c)



(d)



2. (a) $y = -\frac{9}{2}(x+5)^2 - 7$ (b) $y = \frac{2}{3}(x-6)^2 + 5$

(c) $y = \frac{1}{9}(x-4)^2 - 3$ (d) $y = 5(x+1)^2 - 4$

4. $y = 3(x-1)^2 + 2$

5. $h(x) = -\frac{1}{4}(x-5)^2 + 4$

7. $y = -\frac{1}{2}(x-4)^2 + 1$

8. $y = -3(x-1)^2 - 7$

9. $y = \frac{1}{2}(x-3)^2 + 2$

10. $y = 5(x+2)^2 - 3$

Section 3.7

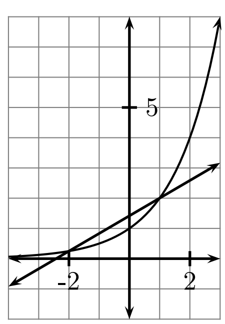
1. 151 square inches per minute

2. (a) From 4 seconds to 7 seconds the height decreased at an average rate of 26.7 feet per second.
   (b) The average rate of change is 0 feet per second. This is because at 2 seconds the projectile is at a height of 224 feet, on the way up, and at 7 seconds it is at 224 feet again, on the way down.

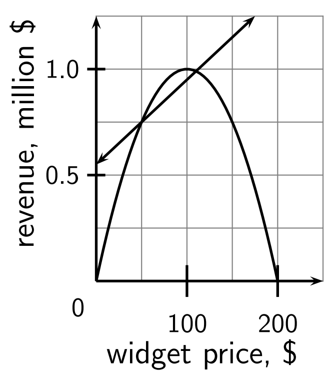
3. (a) When $x = 200$ feet, the area is 80,000 square feet, and when $x = 300$ feet, the area is 30,000 square feet.
   (b) -500 square feet per foot

4. (a) $\frac{11.5}{40} = 0.29$ feet per year (b) $\frac{3}{5} = 0.6$ feet per year

5. (a) \$4000 of revenue per dollar of price (b) See graph below.



Exercise 6(a)



Exercise 5(b)

6. (a) See graph above. (b) $\frac{7}{12}$

7. 16

8. $-\frac{9}{10} = -0.9$

9. (a) $\frac{2}{3}$ (b) $\frac{2}{3}$
   (c) The answers are the same, and they are both equal to the slope of the line.

10. $\frac{2}{3}$

11. $30 + 3h$

12. (a) $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

(b) $f(x+h) = x^3 + 3x^2h + 3xh^2 + h^3 - 5x - 5h + 1$

$f(x+h) - f(x) = 3x^2h + 3xh^2 + h^3 - 5h$

$$\frac{f(x+h) - f(x)}{h} = 3x^2 + 3xh + h^2 - 5$$

(c) $3(1)^2 + 3(1)(3) + 3^2 - 5 = 16$

13. (a) $f(x+h) - f(x) = \dfrac{-h}{x(x+h)}$ (b) $\dfrac{f(x+h) - f(x)}{h} = \dfrac{-1}{x(x+h)}$