4 Polynomial and Rational Functions

Outcome/Performance Criteria:

4. Understand polynomial and rational functions.

   (a) Identify the degree, lead coefficient and constant term of a polynomial function from its equation.

   (b) Given the graph of a polynomial function, determine its possible degrees and the signs of its lead coefficient and constant term. Given the degree of a polynomial function and the signs of its lead coefficient and constant term, sketch a possible graph of the function.

   (c) Give the end behavior of a polynomial function, from either its equation or its graph, using "as $x \to a$, $f(x) \to b$ notation.

   (d) Graph a polynomial function from the factored form of its equation; given the graph of a polynomial function with its x-intercepts and one other point, give the equation of the polynomial function.

   (e) Solve a polynomial inequality.

   (f) Give the equations of the vertical and horizontal asymptotes of a rational function from either the equation or the graph of the function.

   (g) Graph a rational function from its equation, without using a calculator.

   (h) Give the end behavior of a rational function from its graph, using "as $x \to a$, $f(x) \to b$ notation.

4.1 Introduction to Polynomial Functions

Performance Criteria:

4. (a) Identify the degree, lead coefficient and constant term of a polynomial function from its equation.

   (b) Given the graph of a polynomial function, determine its possible degrees and the signs of its lead coefficient and constant term. Given the degree of a polynomial function and the signs of its lead coefficient and constant term, sketch a possible graph of the function.

   (c) Give the end behavior of a polynomial function, from either its equation or its graph, using "as $x \to a$, $f(x) \to b$ notation.

Polynomial functions are simply a special class of functions, just like mammals are a special class of animals. Our study of polynomial functions will proceed in the same way that we might study a particular class of animals: First we must clearly define the class of functions that we are concerned with, so that we know when we are in fact considering a polynomial function. After that we will develop some specialized vocabulary for dealing with polynomial functions and their graphs. This is necessary so that we can actually talk about them in a way that is clear and precise. Finally, we will consider what interests us the most - the behavior of polynomial functions.

What IS a Polynomial Function?

It is perhaps easiest to get the idea of what a polynomial function is by considering some functions that are polynomial functions, and some that are not. These *are* polynomial functions:

$$P(x) = -7x^{14} + 2x^{11} + x^6 - \tfrac{2}{3}x^2 - 3 \qquad\qquad g(x) = x^2 - x - 6$$

$$y = 5x - 9 \qquad\qquad h(x) = 2 + 3x - 5x^2 + 16x^5$$

These *are not* polynomial functions:

$$f(x) = \sqrt{x^3 - 5x^2 + 7x - 1} \qquad\qquad y = 7^x \qquad\qquad R(x) = \frac{3x^2 - 5x + 2}{x^2 - 9}$$

As you can probably see, a polynomial function consists of various whole number (0, 1, 2, 3, ...) powers of x, each possibly multiplied by some number, all added (or subtracted) together. Most polynomial functions include a number that appears not to be multiplying a power of x, but we could say that such a number is multiplying x^0, since $x^0 = 1$. Note that the powers of x can be ordered in any way, although we will generally order them from highest to lowest, as in P above.

The last function above is a fraction made up of two polynomials. It is not a polynomial function, but is a type of function called a **rational function**, which is the ratio of two polynomials. We'll work with rational functions in Sections 4.3 and 4.4.

Vocabulary of Polynomial Functions

The numbers multiplying the powers of x in a polynomial function are called **coefficients** of the polynomial. For example, the coefficients of $f(x) = 5x^4 - 7x^3 + 3x - 4$ are $5,\ -7,\ 0,\ 3$ and -4. Note that the signs in front of any of the coefficients are actually part of the coefficients themselves. Each coefficient and its power of x together are called **terms** of the polynomial. The terms of this polynomial are $5x^4, -7x^3, 3x$ and -4.

The coefficient of the highest power of x is called the **lead coefficient** of the polynomial, and if there is a number without a power of x, it is called the **constant coefficient** or **constant term** of the polynomial. For our example of $f(x) = 5x^4 - 7x^3 + 3x - 4$, the lead coefficient is 5 and the constant coefficient is -4. We will see that both the lead coefficient and the constant coefficient give us valuable information about the behavior of a polynomial.

We will also find that the exponent of the highest power of x gives us some information as well, so we have a name for that number. It is called the **degree** of the polynomial. The polynomial $f(x) = 5x^4 - 7x^3 + 3x - 4$ has degree 4. (We also say this as "f is a fourth degree polynomial.") *Note that the degree is simply a whole number!* Let us make special note of polynomials of degrees zero, one, and two. A polynomial of degree zero is something like $f(x) = -2$, which we also call a **constant function**. The term *constant* is used because the output $f(x)$ remains the same (in this case -2) no matter what the input value x is. An example of a first degree polynomial would be a linear function $f(x) = \frac{2}{3}x + 1$, which we have already studied. You should know that when we graph it we will get a line with a slope of $\frac{2}{3}$ and a y-intercept of 1. A second degree polynomial like $f(x) = 2 + 5x - 4x^2$ is just a quadratic function, and its graph is of course a parabola (in this case, opening down).

⋄ **Example 4.1(a):** For the polynomial function $P(x) = -7x^{14} + 2x^{11} + x^6 - \frac{2}{3}x^2 - 3$, give

(a) the terms of the polynomial,
(b) the coefficients of the polynomial,
(c) the lead coefficient of the polynomial,
(d) the constant term of the polynomial,
(e) the degree of the polynomial.

(a) The terms of the polynomial are $-7x^{14}$, $2x^{11}$, x^6, $-\frac{2}{3}x^2$ and -3.
(b) The coefficients of the polynomial are $-7,\ 2,\ 1,\ -\frac{2}{3}$ and -3.
(c) The lead coefficient of the polynomial is -7.
(d) The constant term of the polynomial is -3.
(e) The degree of the polynomial is 14.

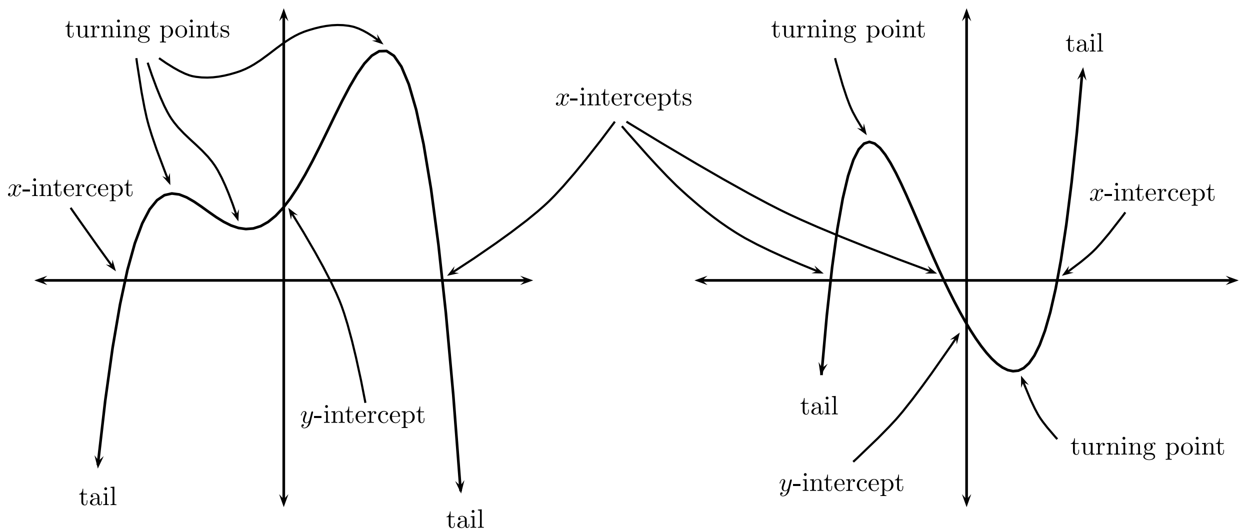
At times we will want to discuss a general polynomial, without specifying the degree or coefficients. To do that we write the general polynomial as

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 \ .$$

The symbols $a_n, a_{n-1}, ..., a_2, a_1$ represent the coefficients of $x^n, x^{n-1}, ..., x^2$ and x, and a_0 is (always) the constant coefficient. If, for example, a polynomial has no x^7 term, then $a_7 = 0$. Finally, let us mention that we sometimes use $P(x)$ instead of $f(x)$ to denote a polynomial function.

Graphs of Polynomial Functions

There is also some vocabulary associated with the graphs of polynomial functions. The graphs below are for two polynomial functions; we will use them to illustrate some terminology associated with the graph of a polynomial.



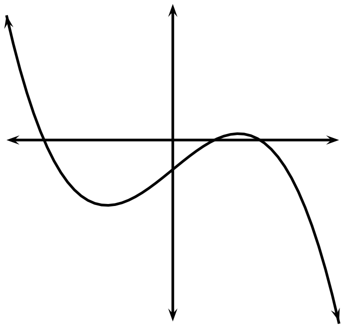
- Your eye may first be drawn to the "humps" of each graph (both "right side up" and "upside down"). The top of a "right side up" hump or the bottom of an "upside down" hump are called **turning points** of the graph. Note that turning points are also relative maxima and minima.

- The point where the graph crosses the y-axis is called (surprise!) the y-intercept. Every polynomial function has exactly one y-intercept. This is because the function can be evaluated at $x = 0$ since the domain of a polynomial function is all real numbers, so there has to be at least one y-intercept. There can't be more than one y-intercept or we would not have a function.

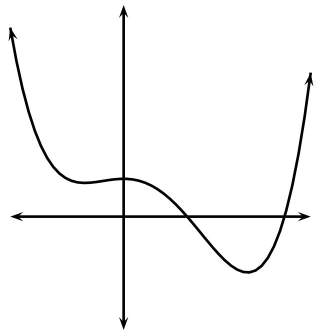
- The points where the graph crosses the x-axis are of course x-intercepts. Remember that x-intercepts are located at x values where $f(x) = 0$, where the function has an output of zero.

- The two ends of the graph are generally drawn with arrowheads to indicate that the graph keeps going. These ends are called "tails" of the graph. Note that graphs of polynomials spread forever to the left and right, since the domain of a polynomial function is all real numbers.

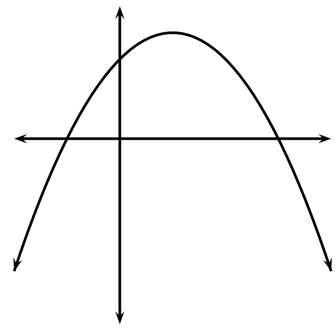
We'll now look at some polynomial functions and their graphs, in order to try to see how the graph of a polynomial function is related to the degree of the polynomial and to its coefficients. Six polynomial functions and their graphs are shown at the top of the next page.



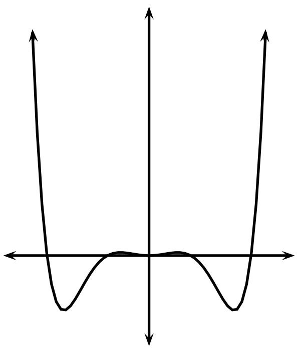
$f(x) = -2x^3 + 15x - 13$



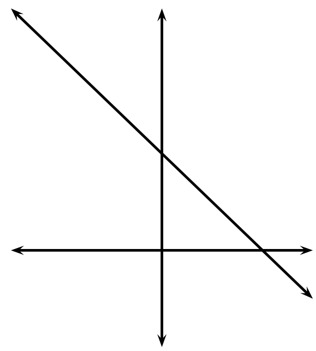
$P(x) = x^4 - 2x^3 - 3x^2 + 10$



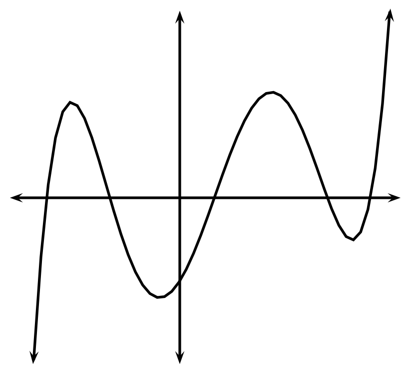
$y = -x^2 + 2x + 3$



$g(x) = x^6 - 7x^4 + 6x^2$



$y = -\frac{3}{2}x + 4$



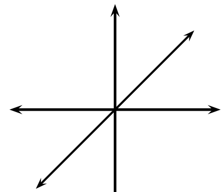
$$Q(x) = x^5 - 5x^4 - 23x^3 + 93x^2 + 130x - 200$$

The general shape of the graph of a polynomial function is dictated by its degree and the sign of its lead coefficient, and the vertical positioning is determined by the constant term. Study the graphs just given to see how they illustrate the following.

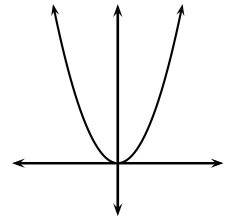
Graphs of Polynomial Functions:

- If the degree of the polynomial is even, both tails either go upward or both go downward. If the degree is odd, one tail goes up and the other goes down.
- If the lead coefficient of an even degree polynomial is positive, both tails go up; if the lead coefficient is negative, both tails go down.
- If the lead coefficient of an odd degree polynomial is positive, the left tail goes down and the right tail goes up. If the lead coefficient is negative, the left tail goes up and the right tail goes down.
- The number of turning points is *at most* one less than the degree of the polynomial. The number of x-intercepts is at most the degree of the polynomial.
- The y-intercept is the constant term.

These things can be remembered by thinking of the odd and even functions whose graphs you are most familiar with, $y = x^1$ and $y = x^2$, shown to the right. $y = x^1$ has an odd degree and positive lead coefficient, and you can see that its left tail goes down and right tail up, as is the case for all polynomials with odd degree and positive lead coefficient. $y = x^2$ has even degree and positive lead coefficient and, like all such polynomials, both tails go up.



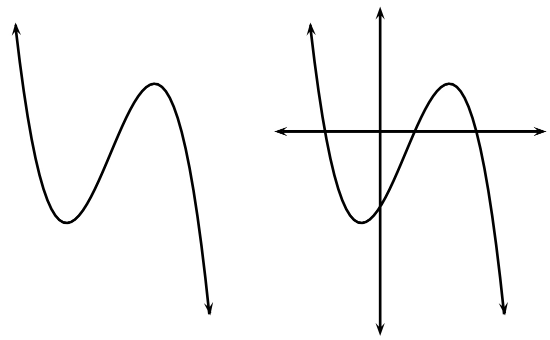
$y = x$



$y = x^2$

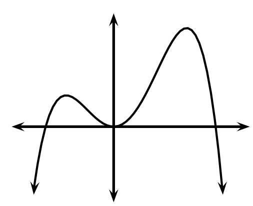
◇ **Example 4.1(b):** Sketch the graph of a polynomial function with degree three, negative lead coefficient and negative constant term.

Because the degree is odd, one tail goes up and the other goes down and, because of the negative lead coefficient the left tail goes up and the right tail goes down. Therefore we know the graph looks something like the first one shown to the right. We then simply need to put in coordinate axes in such a way that the y-intercept is negative, as shown in the second picture. We know this must be the case because the constant term is negative.



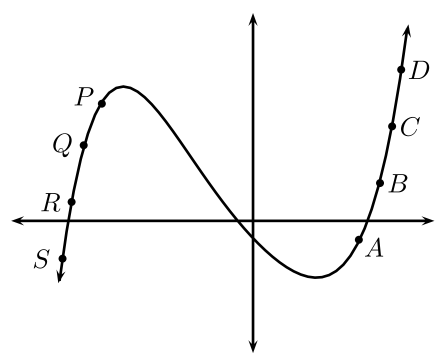
◇ **Example 4.1(c):** Give the smallest possible degree, the sign of the lead coefficient, and the sign of the constant term for the polynomial function whose graph is shown below and to the right.

Because both tails go the same way the degree must be even. To have three turning points the degree must be at least four. For both tails to go down the lead coefficient must be negative. The y-intercept is zero, so the constant term must be zero also (or we could just say there is no constant term).



End Behavior of Polynomial Functions

Given the graph of a function on a coordinate grid, we can get reasonably good values of $f(x)$ for any value of x, and vice versa. Our goal now is to give some sort of written description of the function's behavior at the edges of the graph and beyond. We'll use the graph to the right to explain how this is done.



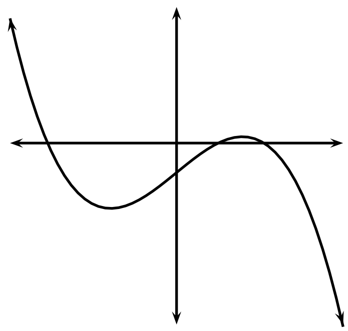
Consider the points A, B, C and D. Note that as we progress from one point to the next, in that order, the values of x are getting larger and larger. We will indicate this by the notation $x \to \infty$. That is, there is a place "infinity" at the right end of the x-axis, and the values of x are headed toward it. At the same time the corresponding values of y are also getting larger as we proceed from point A to point D. We summarize all this by writing

$$\text{As } x \to \infty, \; y \to \infty.$$

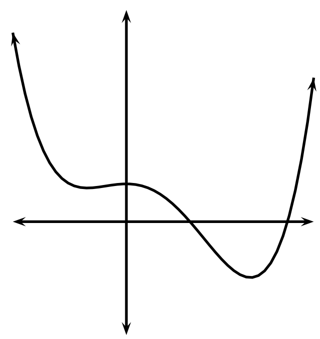
Similarly, as we proceed from point P to Q, R and S, the values of x are getting smaller and smaller, and the corresponding y values are as well. In this case we write

$$\text{As } x \to -\infty, \; y \to -\infty.$$

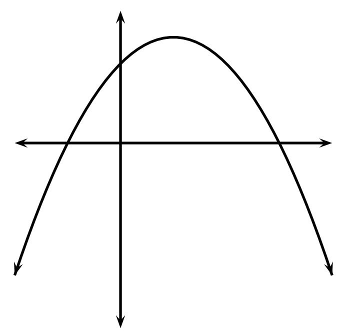
◇ **Example 4.1(d):** Describe the end behavior of each of the polynomial functions graphed below using "as $x \to a$, $y \to b$" statements.



$f(x) = -2x^3 + 15x - 13$



$P(x) = x^4 - 2x^3 - 3x^2 + 10$



$y = -x^2 + 2x + 3$

For the first function, f, we have

$$\text{as } x \to -\infty, \; y \to \infty \text{ and as } x \to \infty, \; y \to -\infty.$$

For P, the second function,

$$\text{as } x \to -\infty, \; y \to \infty \text{ and as } x \to \infty, \; y \to \infty.$$

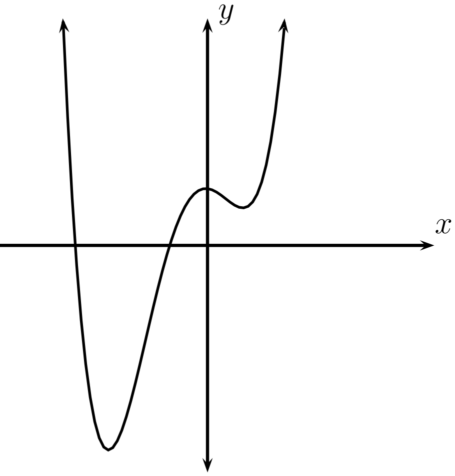
Finally, for the last function,

$$\text{as } x \to -\infty, \; y \to -\infty \text{ and as } x \to \infty, \; y \to -\infty.$$

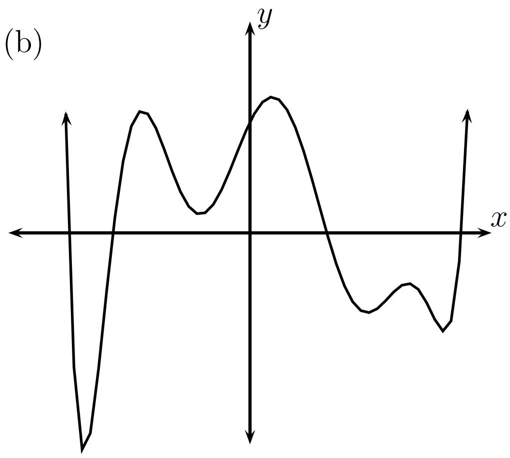
Section 4.1 Exercises

1. For each of the following graphs of polynomial functions, tell what you can about the lead coefficient, the constant coefficient, and the degree of the polynomial.

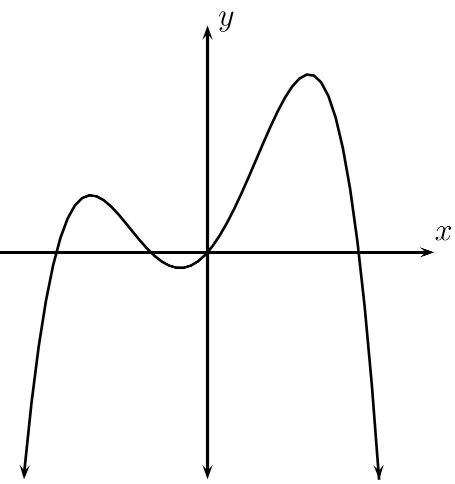
(a)



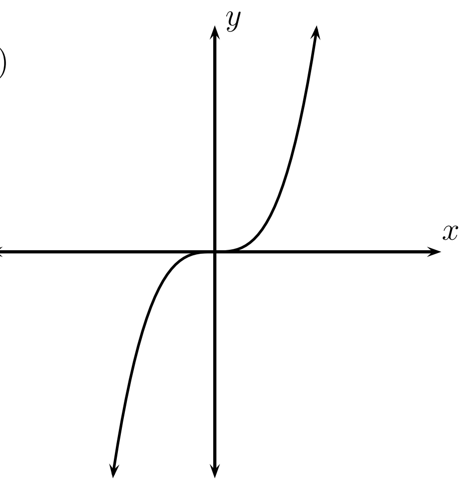
(b)



(c)



(d)



2. (a) - (d) For each graph in Exercise 1, give the end behaviors in the manner used in Example 4.1(d).

3. Sketch the graph of a polynomial function having the following characteristics. Some are not possible; in those cases, write "not possible".

   (a) Degree three, negative lead coefficient, one x-intercept.

   (b) Degree three, negative lead coefficient, two x-intercepts.

   (c) Degree three, positive lead coefficient, four x-intercepts.

   (d) Degree three, negative lead coefficient, no x-intercepts.

   (e) Degree four, positive lead coefficient, three x-intercepts.

   (f) Degree four, positive lead coefficient, one x-intercept.

   (g) Degree four, negative lead coefficient, two x-intercepts.

4. For what degrees is it possible for a polynomial to not have any x-intercepts?

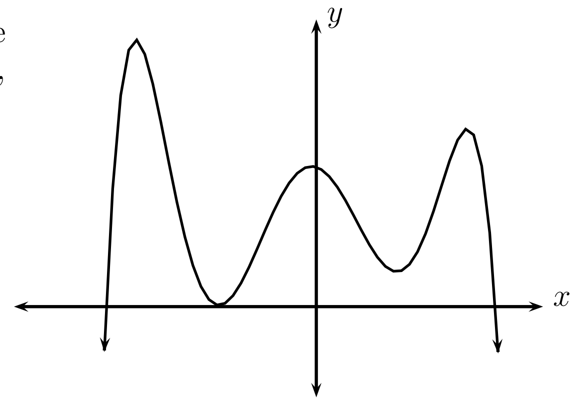
5. A polynomial has exactly 3 turning points. Which of the following are possible numbers of x-intercepts of the polynomial: 0, 1, 2, 3, 4, 5?

6. A polynomial has exactly three x-intercepts. Which of the following are possible numbers of turning points for the polynomial: 0, 1, 2, 3, 4, 5?

7. How many x-intercepts does a polynomial of degree one have?

8. How many x-intercepts does a polynomial of degree zero have?

9. Which of the polynomials below could have the graph shown to the right? For any that couldn't, tell why.



   (a) $f(x) = 5x^6 - 7x^3 + x + 3$

   (b) $f(x) = -5x^6 - 7x^3 + x + 3$

   (c) $f(x) = -5x^6 - 7x^3 + x - 3$

   (d) $f(x) = -5x^4 - 7x^3 + x + 3$

4.2 Polynomial Functions in Factored Form

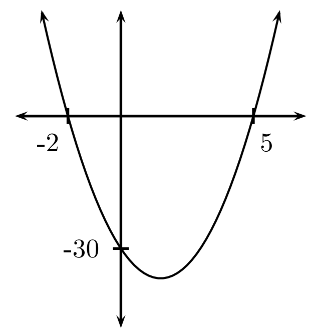
Performance Criteria:

4. (d) Graph a polynomial function from the factored form of its equation; given the graph of a polynomial function with its x-intercepts and one other point, give the equation of the polynomial function.

Let's once again begin with an example.

◇ **Example 4.2(a):** Find the intercepts of $f(x) = 3x^2 - 9x - 30$ and sketch the graph using only those points.

First we note that the graph is a parabola opening upward, and can see that the y-intercept is -30. Next we factor the right side to get



$$f(x) = 3x^2 - 9x - 30 = 3(x^2 - 3x - 10) = 3(x + 2)(x - 5).$$

From this we can easily see that $x = -2$ and $x = 5$ are the zeros (x-intercepts) of the function. It was easy to see from the original that $f(0) = -30$, but it isn't too hard to get that from the factored form:

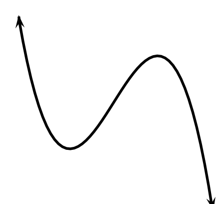
$$f(0) = 3(0 + 2)(0 - 5) = 3(2)(-5) = -30.$$

We now plot the intercepts and draw the graph, shown above and to the right.

In this section we will see how to get the graph of a polynomial function from its factored form, and vice versa, using a method just like that used in the example above.

◇ **Example 4.2(b):** Sketch a graph of the function $f(x) = -\frac{1}{2}(x + 1)(x - 1)(x - 4)$, indicating clearly the intercepts.

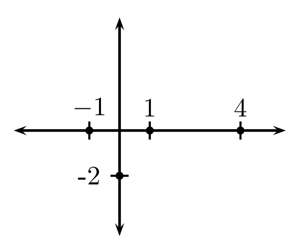
If we were to imagine multiplying the right side of this equation out, the first term of the result would be $-\frac{1}{2}x^3$, so we would expect a graph with the left tail going up and the right tail going down. We would also expect our graph to probably have two turning points. See the diagram to the right for what we would expect the graph to look like.



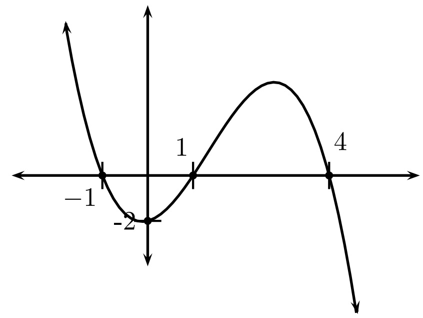
Next we would observe that the graph has x-intercepts -1, 1 and 4. (These are obtained by noting that these values of x result in $f(x) = 0$.) If we then let $x = 0$ and compute $f(0)$ we get

$$f(0) = -\tfrac{1}{2}(1)(-1)(-4) = -2,$$

which is the y-intercept of our graph. The graph to the right shows the intercepts plotted on a set of coordinate axes.



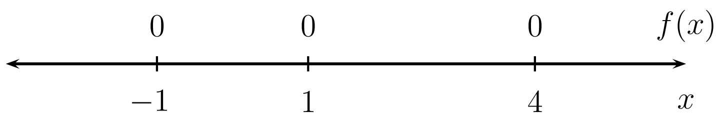
Now we just have to sketch a graph that looks like what we expected, and passing through the intercepts we have found. The final graph is shown to the right.



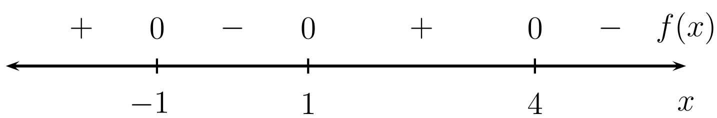
We can use sign charts (see Section 3.4) to help us graph a polynomial function as well. The next example will remind you of the process for creating a sign chart for a function.

◇ **Example 4.2(c):** Create the sign chart for $f(x) = -\frac{1}{2}(x+1)(x-1)(x-4)$.

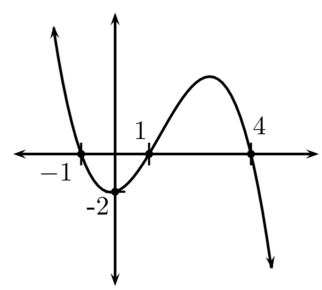
We know that $f(x) = 0$ when $x = -1,\ 1,\ 4,$ so we draw and label a number line indicating this:



When $x = -2$, $-\frac{1}{2} < 0$, $x+1 < 0$, $x-1 < 0$ and $x-4 < 0$, so the sign of $f(x)$ is $(-)(-)(-)(-) = +$. When $x = 0$, the sign of $f(x)$ is $(-)(+)(-)(-) = -$. Checking the signs for x values in the remaining two intervals and labeling the signs of $f(x)$ above the axis, we end up with this:



Compare this last sign chart with the graph of the function, shown to the right, to see how the sign chart relates to the graph. The sign chart gives us the intervals where the function is positive or negative - remember when looking at the graph that "the function" means the y values.

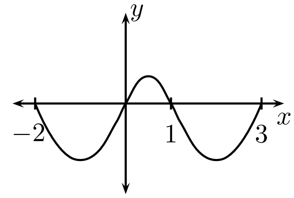


◇ **Example 4.2(d):** The sign chart for $g(x) = \frac{1}{3}x(x+2)(x-1)(x-3)$ is

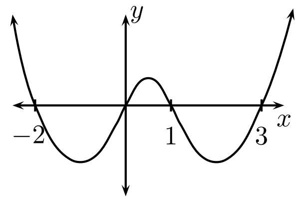
$+$	0	$-$	0	$+$	0	$-$	0	$+$	$g(x)$
	-2		0		1		3		x

Use this sign chart to sketch the graph of the function.

The $-$ sign between -2 and 0 tells us that the graph "loops" below the x-axis there. It loops above the x-axis from 0 to 1 and below again from 1 to 3. At this point we know the graph looks like figure (a) to the right



To the left of -2 and to the right of 3 the graph is above the x-axis. From our understanding of graphs of polynomial functions, we would guess the tails are like figure (b) to the right.



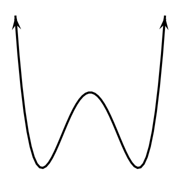
Note that before making the sign chart for the above function we could have determined that it is a fourth degree polynomial with positive lead coefficient, so both tails must go up and it could have as many as three humps. We could also get the y-intercept from

$$g(0) = \frac{1}{3}(0)(0+2)(0-1)(0-3) = 0$$

Of course we already knew the graph went through the origin, but if the y-intercept was not zero, we could determine it this way.

◇ **Example 4.2(e):** Use a sign chart to sketch the graph of the polynomial function $f(x) = \frac{1}{4}(x+1)(x-1)(x-4)^2$.

First we note that the polynomial is fourth degree with a negative lead coefficient, so the graph will look something like the one to the right. We know that $f(x) = 0$ at $x = -1,\ 1,\ 4.$ so our sign chart starts like this:

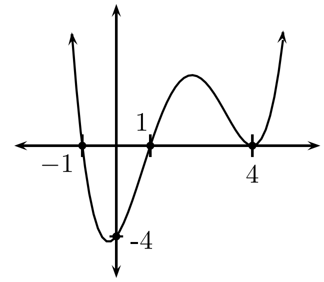


0	0	0	$f(x)$
-1	1	4	x

When $x = -2$, $\frac{1}{4} > 0$, $x + 1 < 0$, $x - 1 < 0$ and $(x-4)^2 > 0$, so the sign of $f(x)$ is $(+)(-)(-)(+) = +$. When $x = 0$, the sign of $f(x)$ is $(+)(+)(-)(+) = -$. Checking the signs for x values in the remaining two intervals, we end up with

$+$	0	$-$	0	$+$	0	$+$	$f(x)$
	-1		1		4		x

This sign chart indicates that y is zero at $x = 4$, and on either side of 4 the y values are positive. This means that the lower right hand turning point in our original sketch of the graph has its bottom at the point $(4, 0)$. Because the y values are negative between -1 and 1, the lower left turning point is below the x-axis. Let's find the y-intercept:



$$f(0) = \tfrac{1}{4}(0+1)(0-1)(0-4)^2 = \tfrac{1}{4}(-16) = -4$$

The final graph is shown to the right, with the intercepts labeled.

Section 4.2 Exercises

1. Sketch the graph of $g(x) = x(x+3)^2$, indicating all the intercepts clearly. Check yourself with your calculator.

2. Consider the function $h(x) = \frac{1}{4}(x+1)(x-1)(x-4)^3$, which is very similar to the function f from Example 4.2(e).

   (a) Try to graph this function without your calculator. Check your answer with your calculator.

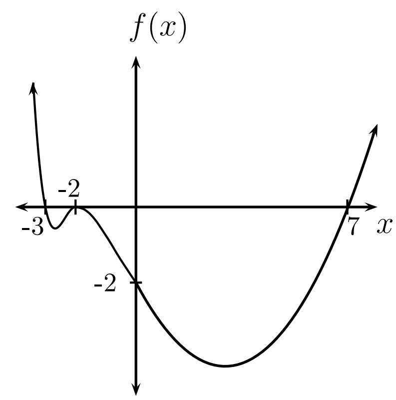
   (b) Both f and h have x-intercepts at $x = 4$, but their behaviors are different there. Describe the difference. Can you *explain WHY* they are different? (**Hint:** Consider the sign of $(x-4)^n$ at values close to $x = 4$, on either side of 4.)

3. Consider the function whose graph is shown below and to the right.

   (a) What are the factors of the polynomial having the graph shown?

   (b) One of your factors from (a) should be squared - which one is it?

   (c) Create the equation of a polynomial function by multiplying your factors from (a) together, with a squared on the term that should have it. (**DO NOT** multiply it out!) What are the degree and lead coefficient of your polynomial? Based on those, should the graph of your equation look something like the one given?

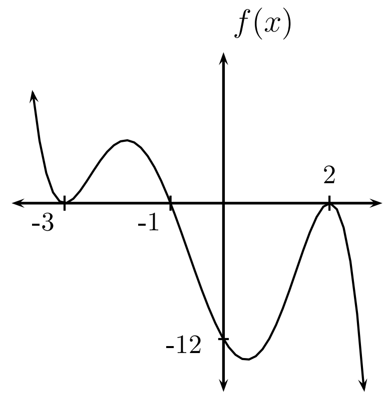


   (d) Find the y-intercept for your equation. It should *NOT* match the graph.

   (e) Here's how to make your equation have the correct y-intercept: Put a factor of a on the front of your polynomial function from (c), then substitute in x and y values for the y-intercept and solve for a.

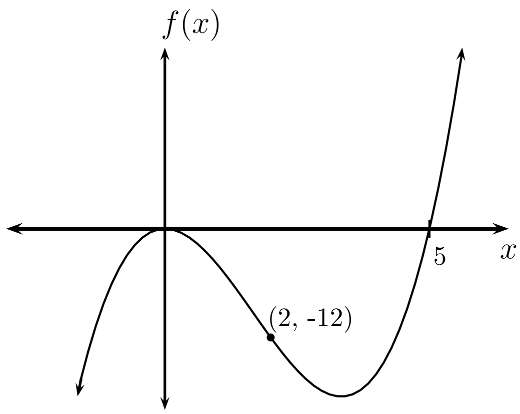
   (f) What is the equation of the polynomial function that will have a graph like the one shown?

4. Use a process like what you did in Exercise 3 to determine the factored form of a polynomial function whose graph is the one shown to the right. Check yourself with your calculator. The graph will not look quite like this one, but it should have the same shape and intercepts.

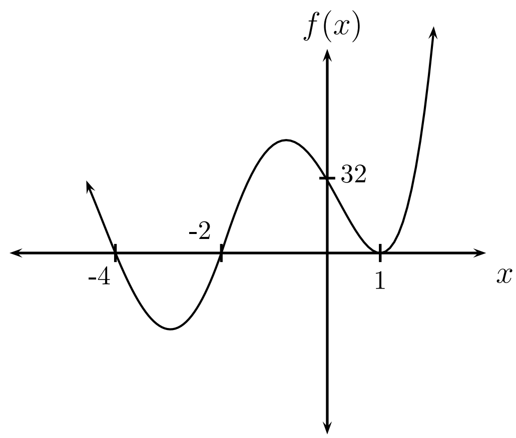


5. (a) Find the factored form of a polynomial function that gives the graph shown below and to the left.

   (b) Find the factored form of a polynomial function that gives the graph shown below and to the right.



Exercise 5(a)



Exercise 5(b)

6. Find the graph of each of the following, without using your calculator. Check yourself with your calculator. Your graph may not look quite like the graph, but it should have the same shape and intercepts.

   (a) $f(x) = \frac{1}{2}(x-1)^2(x+2)^2$

   (b) $f(x) = -x^3(x+5)^2$

   (c) $f(x) = -\frac{1}{6}(x-1)(x+2)(x-3)(x+4)$

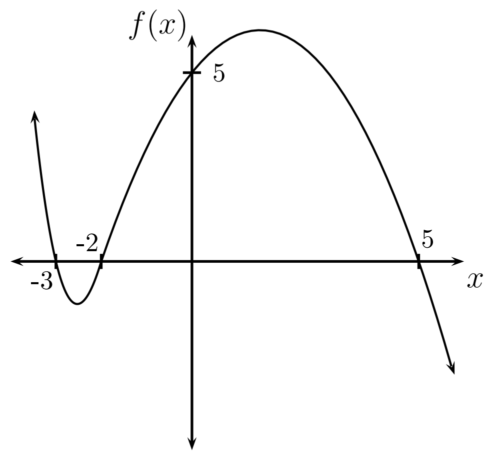
   (d) $f(x) = x(x-2)(x+1)^3$

   (e) $f(x) = \frac{1}{3}(x+1)(x-1)(x+3)(x-3)$

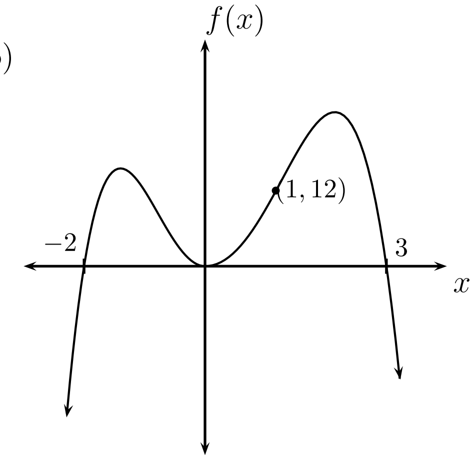
   (f) $f(x) = -(x-4)^3$

7. For each of the following, try to find the factored form of a polynomial function having the given graph. As usual, check yourself with your calculator.

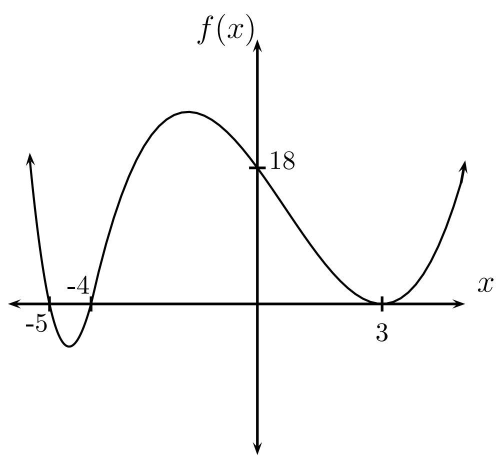
(a)



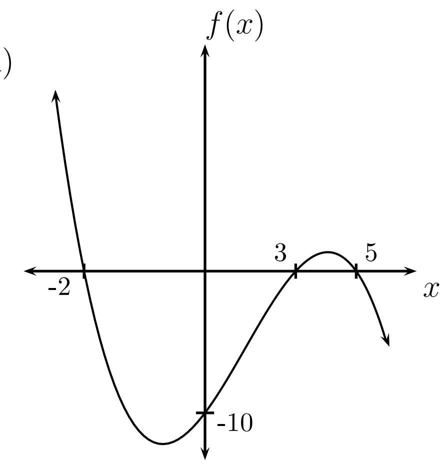
(b)



(c)



(d)



8. (a) - (d) Give the end behavior for each of the functions from Exercise 7.

4.3 Polynomial Inequalities

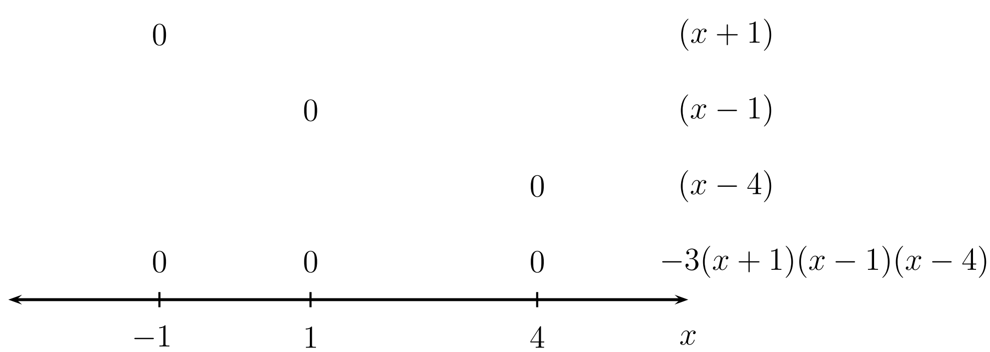
> **Performance Criteria:**
>
> 4. (e) Solve a polynomial inequality.

We'll illustrate solving a polynomial inequality with an example, using a sign chart created slightly differently than we did before.

◇ **Example 4.3(a):** Solve the inequality $-3(x+1)(x-1)(x-4) \leq 0$.

To set up our sign chart, we draw a number line with no numbers marked on it. Below the right hand end of the number line we put x, and above the right hand end we put the non-zero side of our inequality. Above that we put the individual factors of the no-zero side. The whole thing looks like this:

Now we know that the values $x = -1, 1, 4$ will be of importance, so we mark them on our number line. We know that the factor $x+1$ will be zero when $x = -1$, so we indicate that by placing a zero above $x = -1$ and in the row for the factor $(x+1)$. We do similarly for the other two factors, and we also know that the entire quantity $-3(x+1)(x-1)(x-4)$ will be zero when any of the other factors is zero, so we indicate that as well at this point our sign chart looks like this:



If we evaluate the factor $x+1$ for any number less than -1, it should be clear that the result will be negative. If we evaluate it for any number greater than negative one the result will be positive. We indicate this by putting a negative sign to the left of the zero in the $(x+1)$ row, and a few $+$ signs to the right of the zero. We do the same for the other two factors, getting something like the chart at the top of the next page.

$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$(x+1)$
$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$(x-1)$
$-$	$-$	$-$	$-$	$-$	$-$	0	$+$	$(x-4)$
	0		0			0		$-3(x+1)(x-1)(x-4)$
	-1		1			4		x

We are now ready to finish up the sign chart and solve the inequality. Consider the interval from $x=-1$ to $x=1$. From the sign chart we can see that in that interval $(x+1)$ is positive, $(x-1)$ is negative, and $(x-4)$ is negative, so their product is positive. But the quantity we are interested in is $-3(x+1)(x-1)(x-4)$, and the negative three gives us a sign of negative in that interval. This is shown by putting a negative sign or two in the row of the sign chart for $-3(x+1)(x-1)(x-4)$ above the interval from $x=-1$ to $x=1$. We then do the same for all four intervals of the number line to get the final sign chart:

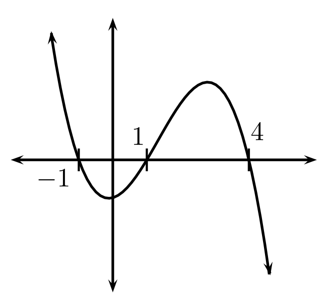
$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$(x+1)$
$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$(x-1)$
$-$	$-$	$-$	$-$	$-$	$-$	0	$+$	$(x-4)$
$+$	0	$-$	0	$+$	$+$	0	$-$	$-3(x+1)(x-1)(x-4)$
	-1		1			4		x

We are finally ready to solve the inequality $-3(x+1)(x-1)(x-4) \leq 0$. From the sign chart we can see that the left side is negative from $x=-1$ to $x=1$ and from $x=4$ to ∞. Since the inequality is less than *or equal to*, the solution set to the inequality is then any value in the intervals $[-1,1]$ and $[4,\infty)$.

We could have included a row in the sign chart for the factor -3, but it would just be negative signs all the way across; we took that into account above. Note that if we were to graph the function

$$y = -3(x+1)(x-1)(x-4)$$

we would get the graph to the right. The solution to the inequality is then all the intervals where the function is negative. One can see that those intervals are precisely the ones we arrived at from the sign chart.



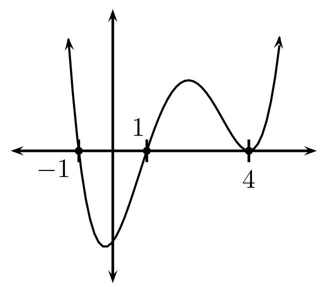
◇ **Example 4.3(b):** Use a sign chart to solve the inequality $\frac{1}{4}(x+1)(x-1)(x-4)^2 > 0$.

The sign chart for this inequality is shown below. Note that $(x-4)^2$ is positive for all values of x except four, where it is zero.

$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$(x+1)$
$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$(x-1)$
$+$	$+$	$+$	$+$	$+$	$+$	0	$+$	$(x-4)^2$
$+$	0	$-$	0	$+$	$+$	0	$+$	$\frac{1}{4}(x+1)(x-1)(x-4)^2$
	-1		1			4		x

Keeping in mind that this is a strict inequality (NOT greater than or equal to), we are looking for only the values where $\frac{1}{4}(x+1)(x-1)(x-4)^2$ is positive, not zero. The solution is then the intervals $(-\infty,-1)$, $(1,4)$ and $(4,\infty)$. Note that $x=4$ itself is not included.

The graph of $y=\frac{1}{4}(x+1)(x-1)(x-4)^2$ is shown to the right. From it we can see that y is positive on precisely the intervals found above. How would our results change for the inequality $\frac{1}{4}(x+1)(x-1)(x-4)^2 \geq 0$? Well, now we include the values $x=-1,1,4$, where $\frac{1}{4}(x+1)(x-1)(x-4)^2$ equals zero. This "plugs the hole" at $x=4$, giving the solution $(-\infty,-1]$ and $[1,\infty)$.



Section 4.3 Exercises

1. Solve the inequality $x(x-4)(x+1) \leq 0$. You will want to include a row for the factor x in your sign chart.

2. Solve each of the following polynomial inequalities.

   (a) $(x+3)(x-1)(x-5) < 0$

   (b) $(x+2)(x-2)(x-5)(x+3) \leq 0$

   (c) $0 < (x+5)(x+1)^2$

   (d) $x^2(x-2)(x+2) \geq 0$

   (e) $0 \geq -x(x-4)(x-6)$

   (f) $(x+2)^2(x-3)^2 > 0$

4.4 Rational Functions, Part I

Performance Criteria:

4. (f) Give the equations of the vertical and horizontal asymptotes of a rational function from either the equation or the graph of the function.

   (g) Graph a rational function from its equation, without using a calculator.

What IS a Rational Function?

Rational functions are functions that consist of one polynomial over another. Recognizing also that linear functions and constants qualify as polynomials, here are a few examples of rational functions:

$$f(x)=\frac{3x^2-14x-5}{x^2-9}, \qquad g(x)=\frac{3x^2}{(x-2)^2}, \qquad y=-\frac{3}{x+5}.$$

In this section we will investigate the behaviors of rational functions through their graphs. Even though rational functions are quotients of two polynomial functions, *their graphs are significantly different than polynomial functions!* Before beginning, let's make note of two facts:

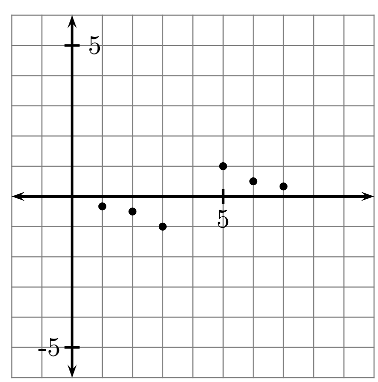
- A fraction is not allowed to have a denominator of zero.
- A fraction *CAN* have a numerator of zero, as long as the denominator is not zero at the same time; in that case the value of the entire fraction is zero. In fact, *the only way that a fraction can have value zero is if its numerator is zero.*

It is very important that you think about these two things and commit them to memory if you are to work comfortably with rational functions.

Graphs of Rational Functions, a First Look

Let's begin our investigation of rational functions with the rational function $h(x)=\frac{1}{x-4}$. The first observation we should make is that $\text{Dom}(f)=\{x\mid x\neq 4\}$. This means that above or below every point on the x-axis except four there will be a point on the graph of this function. The graph will then have two parts - a part to the left of where x is four and a part to the right of where x is four. Let's begin by finding a few function values for choices of x on either side of four and graphing them, as shown to the right.

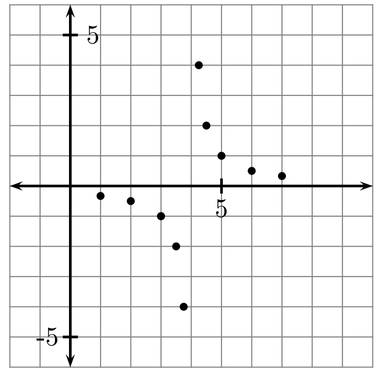
x	$h(x)$
1	$-\frac{1}{3}$
2	$-\frac{1}{2}$
3	-1
5	1
6	$\frac{1}{2}$
7	$\frac{1}{3}$



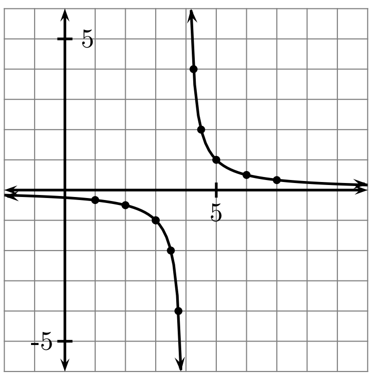
This picture is a bit baffling. There are two sets of points, each seeming to indicate a portion of a graph, but it is not clear what is happening between them. Lets consider some values of x on either side of four, getting "closer and closer" to four itself, working with decimals now. Such values of the function are shown in the two tables below and to the left, and below and to the right is our graph with some of the new points.

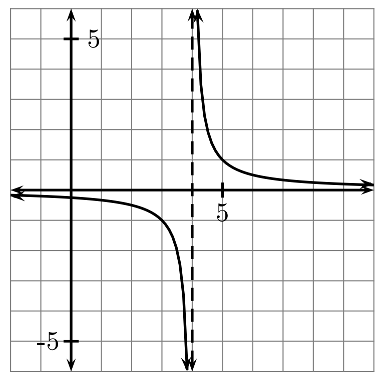
x	$h(x)$
3.5	-2
3.75	-4
3.9	-10
3.99	-100

x	$h(x)$
4.5	2
4.25	4
4.1	10
4.01	100



We can now see that our graph seems to consist of two parts, one on each side of where x has the value four. The graph is shown below and to the left; below and to the right we have added as a dashed line the graph of the vertical line with equation $x = 4$.

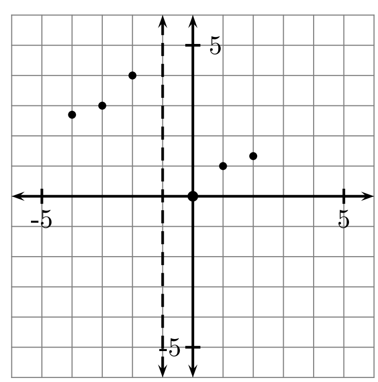




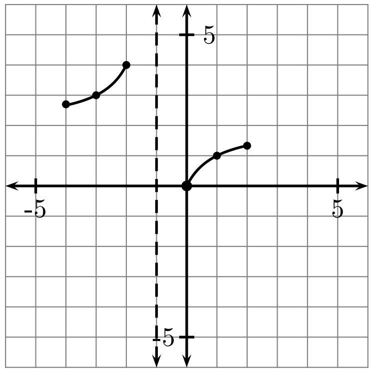
A line that a graph gets closer and closer to is called an **asymptote**. In this case the line $x = 4$ is what we will call a **vertical asymptote**. We can also see that as the graph goes off to the left or the right it gets closer and closer to the x-axis, which is the line $y = 0$. That line is of course a **horizontal asymptote**. It should be clear that the vertical asymptote is caused by the fact that x cannot have the value four, but the reason for the horizontal asymptote might not be as clear.

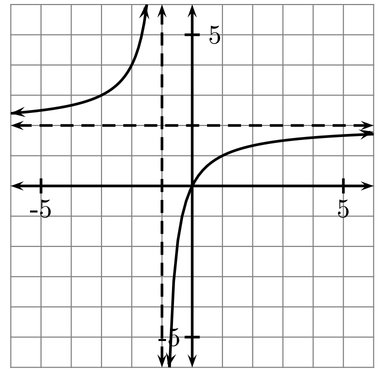
Let's look at another example to try to see how the equation might give us the horizontal asymptote; we'll use $g(x) = \dfrac{2x}{x+1}$. Because $\mathrm{Dom}(g) = \{x \mid x \neq -1\}$ we expect a vertical asymptote with equation $x = -1$. To the right we plot the vertical asymptote, and a few points on either side of it.

x	$g(x)$
-4	2.7
-3	3
-2	4
0	0
1	1
2	1.3



If we connect the points we have so far we get the graph shown below and to the left. It appears that the horizontal asymptote might be the line $y = 2$, and we will see that this is in fact the case. The final graph of $g(x) = \dfrac{2x}{x+1}$ is shown below and to the right.





Finding Asymptotes

By now it should be clear how to find the locations and equations of vertical asymptotes of a rational function; a vertical asymptote will occur at any value of x for which the denominator is zero. (This is a tiny lie - there is one unusual situation in which that is not the case, but for our purpose you can take it to be true.) To understand horizontal asymptotes let's begin by looking at the function $g(x) = \dfrac{2x}{x+1}$. Note that if the value of x is fairly large, $x+1 \approx x$ and $g(x) \approx \dfrac{2x}{x} = 2$. The same thing occurs if x is a "large negative," meaning a negative number that is large in absolute value. Therefore, as we follow the ends of the graph toward negative and positive infinity for x, the values of y get closer and closer to two.

Consider the rational function $y = \dfrac{-3x^2+12x}{2x^2-2x-12}$. In this case, the values of $12x$ and $-x$ are not small as x gets large (negative or positive), but they *ARE* small *relative to the values of* $-3x^2$ and $2x^2$. For example, if $x = 1000$, $12x = 12,000$ but $-3x^2 = -3,000,000$. The absolute value of $-3x^2$ is much larger than the absolute value of $12x$. (If you don't see this, just imagine the difference between having \$12,000 and three million dollars. \$12,000 wouldn't even buy one new automobile, whereas three million dollars would buy 150 cars that cost \$20,000 each. A similar disparity exists between $2x^2$ and $-2x$ in the denominator. Because of this, for the absolute value of x large (that is, if x is a large positive or negative number),

$$y = \frac{-3x^2+12x}{2x^2-2x-12} \approx \frac{-3x^2}{2x^2} = -\frac{3}{2}$$

The rational function $y = \dfrac{-3x^2+12x}{2x^2-2x-12}$ then has a horizontal asymptote of $y = -\frac{3}{2}$. Note that the numerator value -3 came from the lead coefficient of the numerator, and the denominator value of 2 is the lead coefficient of the denominator.

So what about our first function, $h(x) = \dfrac{1}{x-4}$? In this case, for $|x|$ large we have $h(x) = \dfrac{1}{x-4} \approx \dfrac{1}{x} \approx 0$. This is why our graph had a horizontal asymptote of $y = 0$. Here is a summary of how we find the asymptotes of a rational function:

Finding Vertical and Horizontal Asymptotes

For the rational function R,

- if the number a causes the denominator of R to be zero, then $x = a$ is a vertical asymptote of R. It is generally best to factor both the numerator and denominator of a rational function if possible; the values that make the denominator zero can then be easily seen.
- if the degree of the numerator of R is less than the degree of the denominator, R will have a horizontal asymptote of $y = 0$.
- if the degrees of the numerator and denominator of R are the same, then the horizontal asymptote of R will be $y = \frac{A}{B}$, where A and B are the lead coefficients of the numerator and denominator of R, respectively (including their signs!).

You might notice that we left out the case where the degree of the numerator is greater than the degree of the denominator. The appearance of the graph of such a rational function is a bit more complicated, and we won't address that issue.

◇ **Example 4.4(a):** Give the equations of the vertical and horizontal asymptotes of $y = \dfrac{3x-3}{x^2-2x-3} = \dfrac{3(x-1)}{(x+1)(x-3)}$.

We can see that if x was -1 or 3 the denominator of the function would be zero. Therefore the vertical asymptotes of the function are the vertical lines $x = -1$ and $x = 3$. Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y = 0$.

The following example shows something that a person needs to be a little cautious about.

◇ **Example 4.4(b):** Give the equations of the vertical and horizontal asymptotes of $y = -\dfrac{x^2-4x+3}{x^2-4x+4} = -\dfrac{(x-1)(x-3)}{(x-2)^2}$.

Since only the value 2 for x results in a zero denominator, there is just one vertical asymptote, $x = 2$. The degree of the numerator is the same as the degree of the denominator, so the horizontal asymptote will be the *negative of* the lead coefficient of the numerator over the lead coefficient of the denominator, because of the negative sign in front of the fraction. Thus the horizontal asymptote is $y = -1$.

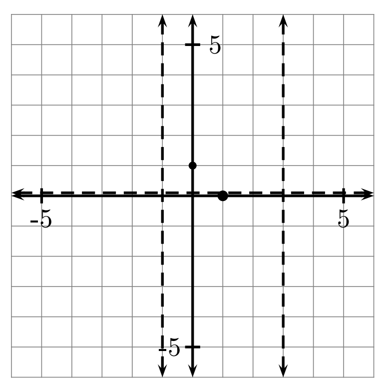
Graphing Rational Functions

To graph a rational function,

1) Find all values of x that cause the denominator to be zero, and graph the corresponding vertical asymptotes.
2) Find and graph the horizontal asymptote.
3) Find and plot the x-intercepts, keeping in mind that a fraction can be zero only if its *numerator* is zero.
4) Find and plot the y-intercept.
5) Sketch in each part of the graph, either by using reasoning and experience with what such graphs look like, or by plotting a few more points in each part of the domain of the function.

◇ **Example 4.4(c):** Sketch the graph of $y = \dfrac{3x-3}{x^2-2x-3} = \dfrac{3(x-1)}{(x+1)(x-3)}$, indicating all asymptotes and intercepts clearly and accurately.

We can see that if x was -1 or 3 the denominator of the function would be zero. Therefore the vertical asymptotes of the function are $x = -1$ and $x = 3$. Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y = 0$.

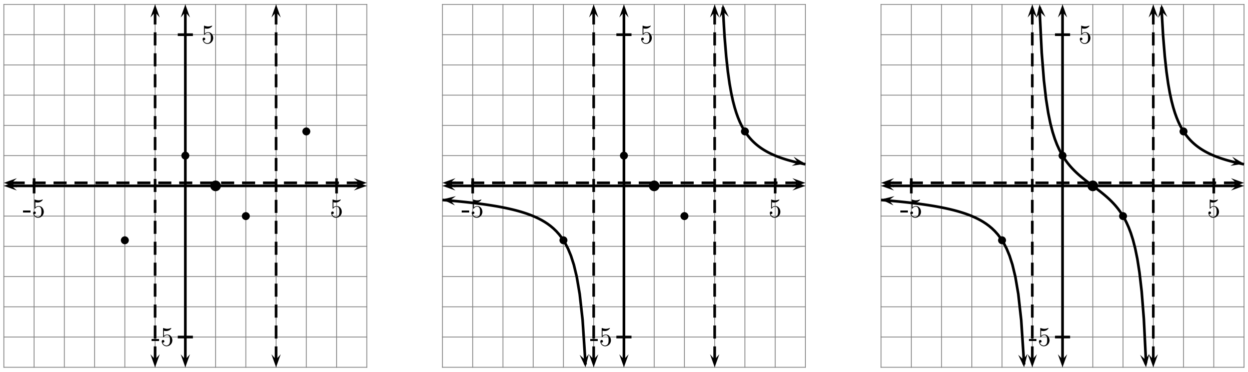


Next we find the intercepts. The only way that y can be zero is if $x = 1$, So we have an x-intercept of $(1, 0)$. If $x = 0$ then $y = 1$, so our y-intercept is $(0, 1)$. We now begin constructing our graph by sketching in the asymptotes and plotting these two intercepts, as shown to the right.

To fill in the rest of the graph we might wish first to plot a few more points. Good locations to have more information would be "outside" the two vertical asymptotes, and between the x-intercept and the vertical asymptote $x = 3$. Computing y for $x = -2$, 2 and 4 gives us the points $(-2, -\frac{9}{5})$, $(2, -1)$ and $(4, \frac{9}{5})$. The graph to the left at the top of the next page shows the above graph with those points added.

Now consider the point $(4, \frac{9}{5})$. As the graph goes leftward from that point it will be "pushed" either up or down by the vertical asymptote at $x = 3$. Because there is no x-intercept between $x = 3$ and $x = 4$, the graph must go upward as it moves left from $(4, \frac{9}{5})$. As it moves right from that point it must approach the horizontal asymptote of $y = 0$, But the graph won't cross it because there are no x-intercepts to the right of

$x = 4$. Similar reasoning can be used to the left of the vertical asymptote $x = -1$; from these things we get the graph shown in the middle below.



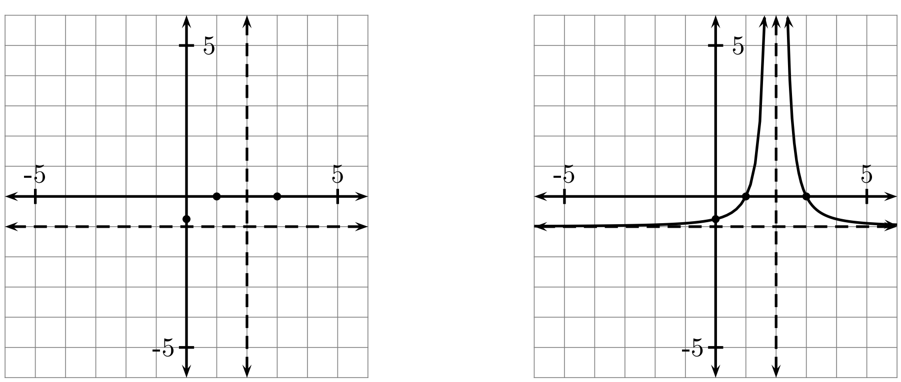
We now need only to complete the middle portion of the graph, between the two asymptotes. The three points are on a line, and to the left of the y-intercept the graph must bend upward because if it bent downward there would have to be another x-intercept between $x = -1$ and $x = 0$. By the same reasoning the graph must bend downward from $(2, -1)$, so the final graph has the appearance shown above and to the right.

The above example illustrates something important:

> The graph of a rational function *CAN* cross a *horizontal* asymptote, and it often does.

◇ **Example 4.4(d):** Sketch the graph of $y = -\dfrac{x^2 - 4x + 3}{x^2 - 4x + 4} = -\dfrac{(x-1)(x-3)}{(x-2)^2}$, indicating all asymptotes and intercepts clearly and accurately.

In this case there is just one vertical asymptote, $x = 2$, and the horizontal asymptote is $y = -1$. (See Example 4.4(b).) $y = 0$ when $x = 1$ or 3, so the x-intercepts are 1 and 3. The y-intercept is $-\frac{3}{4}$. Plotting all of this information gives the graph shown below and to the left. Using the same kinds of reasoning as in the previous example we can then complete the graph as shown below and to the right.



Section 4.4 Exercises

1. For each of the following rational functions,

   (i) give all x-intercepts (ii) give all y-intercepts

   (iii) give the *equations* of all vertical asymptotes

   (iv) give the *equation* of the horizontal asymptote

   (a) $y = \dfrac{6}{x-3}$ (b) $y = \dfrac{6x}{x-3}$ (c) $f(x) = -\dfrac{3}{x+5}$

   (d) $h(x) = \dfrac{6-2x}{x-4}$ (e) $g(x) = \dfrac{6-2x}{x^2-2x-8}$ (f) $y = \dfrac{2x^2+7x+5}{x^2+3x-4}$

   (g) $y = \dfrac{x+1}{x^2-5x}$ (h) $f(x) = \dfrac{3x^2-14x-5}{x^2-9}$

2. Graph each of the rational functions from Exercise 1. Check your answers with a graphing calculator or an online grapher like the one at *http://rechneronline.de/function-graphs/*

3. Recall that the cost (in dollars) for the Acme Company to produce x Widgets in a week was given by the equation $C = 7x + 5000$.

   (a) What is the cost of producing 5,000 Widgets in one week? What is the *average* cost per widget when 5,000 Widgets are produced in a week?
   (b) What is the average cost per Widget if 10,000 are produced in one week?
   (c) Explain any difference in your answers to (a) and (b).
   (d) Write an equation that gives the average cost $\bar{C}$ per widget produced in one week, again using x to represent the number of Widgets produced in that week.
   (e) Your answer to (d) is a rational function. Discuss its domain (both the mathematical domain and the realistic domain).
   (f) Give any vertical asymptotes of the function and describe their meaning(s) in terms of costs and Widgets.
   (g) Give any horizontal asymptotes of the function and describe their meaning(s) in terms of costs and Widgets.

4.5 Rational Functions, Part II

Performance Criteria:

4. (h) Give the end behavior and behavior of a rational function from its graph, using “as $x \to a$, $f(x) \to b$ notation.

When working with a polynomial function like $y = x^3 + 5x^2 - 7$, we acknowledge that the domain of the function is all real numbers. A consequence of this is that if we want to know something about the value of the function at a value of x we can simply find the y value that corresponds to that x value, regardless of what it is. On the other hand, consider the function $f(x) = \dfrac{1}{(x-3)^2}$, whose graph is shown below and to the left. It should be clear that the domain of the function is all real numbers except for three; because of this, the value of $f(3)$ cannot be found. In spite of this, we would like to be able to describe the behavior of the function when x gets “near” three. To do this, we need to recall that there are “places” at the left and right ends of the x-axis that we call negative infinity and infinity (denoted by $-\infty$ and ∞), and similarly for the ends of the y-axis. These “places” are beyond all numbers, no matter how large or small. You saw this when we were using interval notation to define domains and ranges of functions and, more recently, when we described end behavior of polynomial functions in Section 4.1.

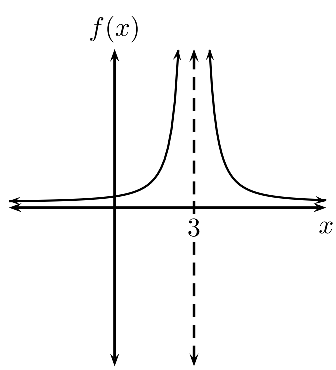
Given this idea, we are now ready to describe the behavior of the function f near $x = 3$. What we say is that “as x approaches three, $f(x)$ approaches infinity.” Symbolically we will write the same thing this way:

$$\text{As } x \to 3,\ f(x) \to \infty$$

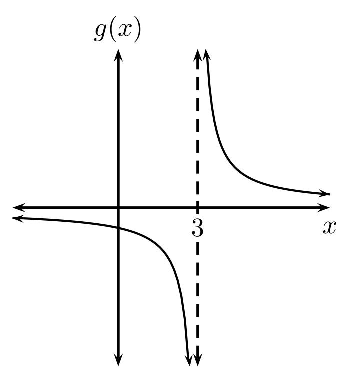
We also write similar statements to describe what happens as the graph leaves the picture to the left and right:

$$\text{As } x \to -\infty,\ f(x) \to 0 \qquad \text{and} \qquad \text{As } x \to \infty,\ f(x) \to 0$$

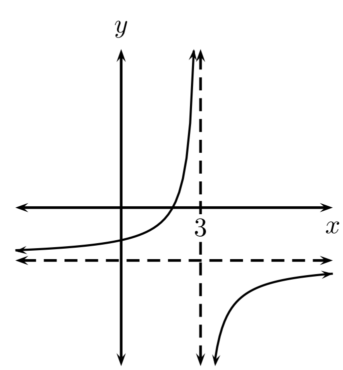
Since the function does the same thing regardless of which infinity x approaches, these two statements can be combined into one: As $x \to \pm\infty$, $f(x) \to 0$.



$f(x) = \dfrac{1}{(x-3)^2}$



$g(x) = \dfrac{1}{x-3}$



$y = \dfrac{2x}{3-x}$

Now look at the graph of $g(x) = \dfrac{1}{x-3}$ on the previous page. Hopefully you can see that there is a problem with the statement "As $x \to 3$, $f(x) \to \infty$," since the function goes to either ∞ or $-\infty$, depending on which side we come at three from. When x approaches three and is larger than three, we say that it approaches three "from above," or "from the positive side."This is denoted boy $x \to 3^+$. Similarly, when x approaches three from the left, or from below, we write $x \to 3^-$. Using these notations, we can now solve our dilemma by writing

$$\text{As } x \to 3^-,\ g(x) \to -\infty \qquad \text{and} \qquad \text{As } x \to 3^+,\ g(x) \to \infty$$

Also, as $x \to \pm\infty$, $g(x) \to 0$.

Finally, look at the graph of $y = \dfrac{2x}{3-x}$ on the previous page and try to write statement like we have been, for it. You should come up with

$$\text{As } x \to 3^-,\ y \to \infty, \qquad \text{as } x \to 3^+,\ y \to -\infty \qquad \text{and} \qquad \text{as } x \to \pm\infty,\ y \to -2.$$

You should think about how these three examples compare with each other. Note in particular that when the function does different things as x approaches a value from different directions we must write two separate statements indicating this. But when the function does the same thing regardless of which side x approaches a value from we can write just one statement, like the statement "as $x \to 3$, $f(x) \to \infty$" we wrote for the function f on the previous page.

Section 4.5 Exercises

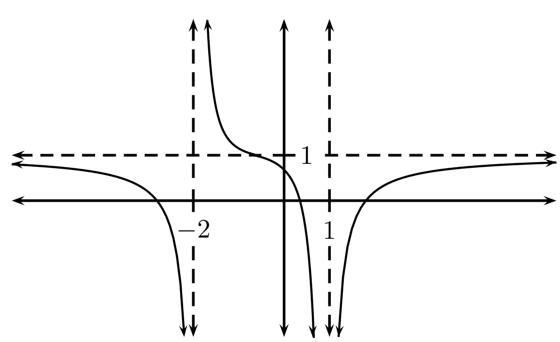
1. Fill in the blanks for the function whose graph is shown below and to the right.

   As $x \to -2^-$, $y \to$ ______

   As $x \to -2^+$, $y \to$ ______

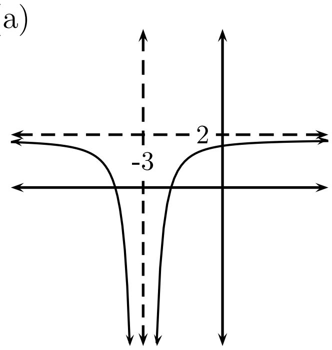
   As $x \to 1$, $y \to$ ______

   As $x \to \pm\infty$, $y \to$ ______

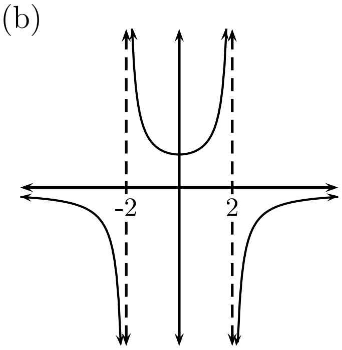


2. Statements of the form "as $x \to a$, $y \to b$" are called **limit statements**. For each of the functions whose graphs are shown, write all appropriate limit statements.

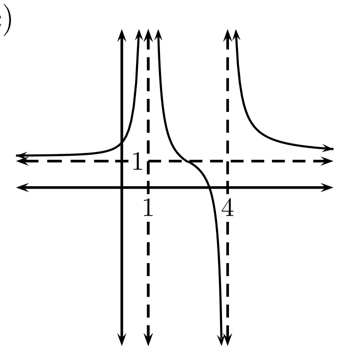
(a)



(b)



(c)



Chapter 4 Solutions

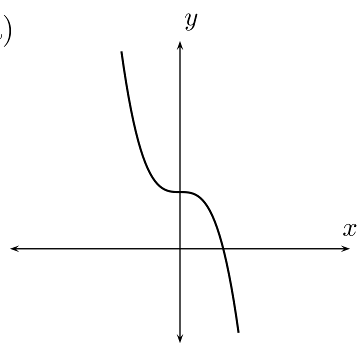
Section 4.1

1. (a) Positive lead coefficient and constant term, even degree of four or more.
   (b) Positive lead coefficient and constant term, even degree of eight or more.
   (c) Negative lead coefficient, no constant term, even degree of four or more.
   (d) Positive lead coefficient, no constant coefficient, odd degree of three or more.

2. (a) As $x \to -\infty,\ y \to \infty$ and as $x \to \infty,\ y \to \infty$
   (b) As $x \to -\infty,\ y \to \infty$ and as $x \to \infty,\ y \to \infty$
   (c) As $x \to -\infty,\ y \to -\infty$ and as $x \to \infty,\ y \to -\infty$
   (d) As $x \to -\infty,\ y \to -\infty$ and as $x \to \infty,\ y \to \infty$

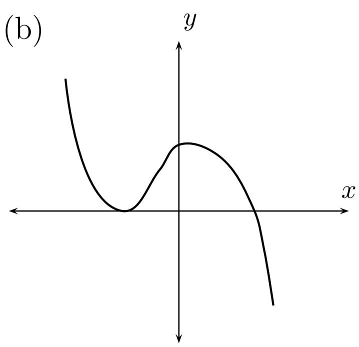
3. For those graphs that are possible, answers may vary.

   (a)



or...

   (b)

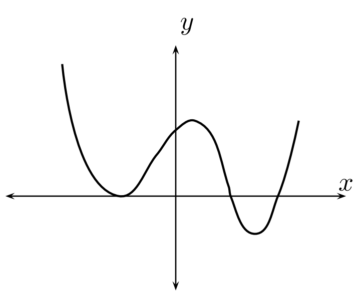


or...

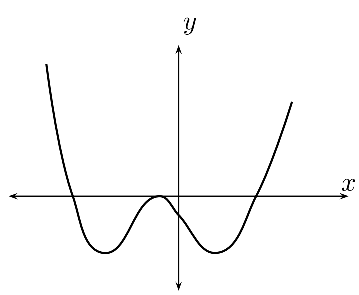
   (c) not possible

   (d) not possible

   (e)



or...



4. Any even degree.

5. 0, 1, 2, 3, 4

6. 3, 4, 5

7. one

8. none ($y = a,\ a \neq 0$) or infinitely many ($y = 0$)

9. (a) could not, lead coefficient for graph is negative
   (b) could
   (c) could not, constant coefficient for graph must be positive
   (d) could not, degree for graph must be even and at least six

Section 4.2

2. (b) When x is on either side of 4, $(x-4)^2$ is positive, so the graph of f "bounces off" the x-axis at $x = 4$. That is, it touches the axis but does not pass through. When x is less than 4, $(x-4)^3$ is negative, and when x is greater than 4, $(x-4)^3$ is positive. Therefore the graph of h passes through the x-axis at $x = 4$.

3. (a) The factors are $x+3$, $x+2$, and $x-7$.
   (b) $x+2$ should be squared
   (c) $P(x) = (x+3)(x+2)^2(x-7)$ The degree of this polynomial is four, and its lead coefficient is -1.
   (d) The y-intercept is $P(0) = (3)(2)^2(-7) = -84$.
   (f) $P(x) = \frac{1}{42}(x+3)(x+2)^2(x-7)$

5. (a) $f(x) = x^2(x-5)$ (b) $f(x) = 4(x+4)(x+2)(x-1)^2$

7. (a) $f(x) = -\frac{1}{6}(x+3)(x+2)(x-5)$ (b) $f(x) = -2x^2(x+2)(x-3)$
   (c) $f(x) = \frac{1}{10}(x+5)(x+4)(x-3)^2$ (d) $f(x) = -\frac{1}{3}(x+2)(x-3)(x-5)$

8. (a) As $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to -\infty$
   (b) As $x \to -\infty$, $y \to -\infty$ and as $x \to \infty$, $y \to -\infty$
   (c) As $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to \infty$
   (d) As $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to -\infty$

Section 4.3

1. $(-\infty, -1]$ and $[0, 4]$

2. (a) $(-\infty, -3)$ and $(1, 5)$ (b) $[-3, -2]$ and $[2, 5]$
   (c) $(-5, -1)$ and $(-1, \infty)$ (d) $(-\infty, -2]$, $\{0\}$ and $[2, \infty)$
   (e) $[0, 4]$ and $[6, \infty)$ (f) $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$ OR $\{x \mid x \neq -2, 3\}$

Section 4.4

1. (a) (i) none (ii) -2 (iii) $x=3$ (iv) $y=0$
   (b) (i) 0 (ii) 0 (iii) $x=3$ (iv) $y=6$
   (c) (i) none (ii) $-\frac{3}{5}$ (iii) $x=-5$ (iv) $y=0$
   (d) (i) 3 (ii) $-\frac{3}{2}$ (iii) $x=4$ (iv) $y=-2$
   (e) (i) 3 (ii) $-\frac{3}{4}$ (iii) $x=4,\ x=-2$ (iv) $y=0$
   (f) (i) $-1, -\frac{5}{2}$ (ii) $-\frac{5}{4}$ (iii) $x=-4,\ x=1$ (iv) $y=2$
   (g) (i) -1 (ii) none (iii) $x=0,\ x=5$ (iv) $y=0$
   (h) (i) $-\frac{1}{3}, 5$ (ii) $\frac{5}{9}$ (iii) $x=-3,\ x=3$ (iv) $y=3$

3. (a) The cost is \$40,000, the average cost per Widget is \$8.00 per Widget.
   (b) The cost is \$75,000, the average cost per Widget is \$7.50 per Widget.
   (c) When more Widgets are produced, the fixed costs of \$5000 are "spread around more."
   (d) $\bar{C} = \dfrac{7x + 5000}{x}$
   (e) The mathematical domain is all real numbers except zero. The feasible domain is $(0, \infty)$.
   (f) The vertical asymptote is $x = 0$, indicating that the number of Widgets produced each week cannot be zero.
   (g) The horizontal asymptote is $y = 7$, indicating that the average cost cannot be less than \$7.00 per Widget.

Section 4.5

1. From top blank to bottom, $-\infty$, ∞, $-\infty$, 1

2. (a) As $x \to -3$, $y \to -\infty$, as $x \to \pm\infty$, $y \to 2$
   (b) As $x \to -2^-$, $y \to -\infty$, as $x \to -2^+$, $y \to \infty$
   As $x \to 2^-$, $y \to \infty$, as $x \to 2^+$, $y \to -\infty$
   As $x \to \pm\infty$, $y \to 0$
   (c) As $x \to 4^-$, $y \to -\infty$, as $x \to 4^+$, $y \to \infty$
   As $x \to 1$, $y \to \infty$, as $x \to \pm\infty$, $y \to 1$