Math and Magic Squares

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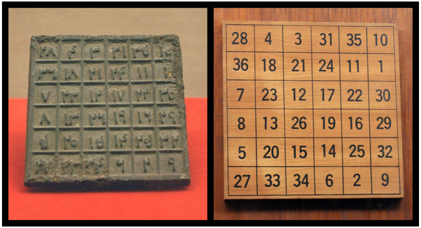

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Leonhard Euler's French Officers Problem:

Arrange thirty-six officers in a six-by-six square

- from six regiments: 1st,2nd,3rd,4th,5th,6th
- with six ranks: Recruit, Lieutenant, Captain, Major, Brigadier, General
- so that each row and column has one representative from each regiment and rank.

Easy for 25 officers in a 5×5 square:

1L	2C	3M	4B	5G
5C	1M	2B	3G	4L
4M	5B	1G	2L	3C
3B	4G	5L	1C	2M
2G	3L	4C	5M	1B

What's the pattern?

Can be done for:

- 3×3, 5×5, 7×7
- All $n \times n$ for **odd** n
- 4×4,
- even 8×8!

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- even 8×8!

But Not for:

- 2×2
- 6×6

No Solution to the 2×2 French Officers Problem

Then what?

1L	2C
2?	1?

No Solution to the 2×2 French Officers Problem

Then what?

1L	2C
2C	1L

No! Not Allowed!

Two First Lieutenants and Two Second Captains, but No Second Lieutenants or First Captains!

No Solution to the 2 × 2 French Officers Problem

Then what?

1L	2C
2L	1C

No! Not Allowed!

Two officers of the same rank in the same column!

Conjectures and Theorems

Euler's Conjecture (1782):

No Solutions for $n = 2, 6, 10, 14, 18, \ldots$

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Conjectures and Theorems

Euler's Conjecture (1782):

No Solutions for $n = 2, 6, 10, 14, 18, \ldots$

. . . is partly right. . .

Tarry (1900), Stimson (1984):
No Solutions for $n = 6$

. . . but mostly wrong.

Bose, Parker, Shrikhande (1960):
Solutions for **all** n **except** $n = 2$ or $n = 6$.

Latin Squares

Definition:

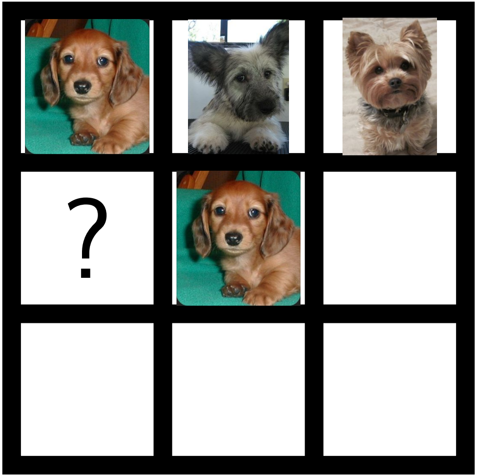
A Latin Square is an $n \times n$ square with n symbols arranged so that each row and column has each symbol **exactly once**.

Example Latin Square: Puppies!

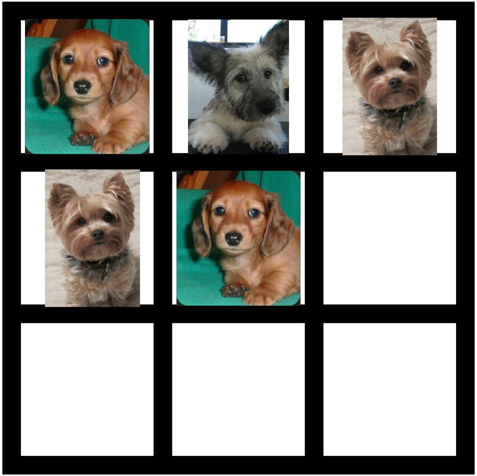
Example Latin Square: Sudoku

1	6	4	3	8	5	9	2	7
5	7	2	9	4	6	8	1	3
3	9	8	1	7	2	5	6	4
8	4	5	2	9	1	3	7	6
6	2	9	7	5	3	4	8	1
7	1	3	4	6	8	2	9	5
4	5	1	6	2	9	7	3	8
2	8	6	5	3	7	1	4	9
9	3	7	8	1	4	6	5	2

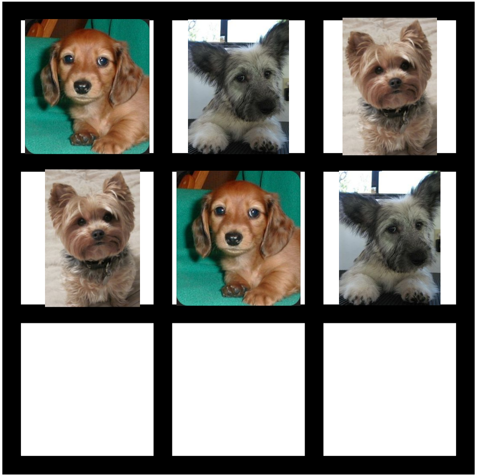
Building Latin Squares: Top-left to middle



Building Latin Squares: Top-left to middle

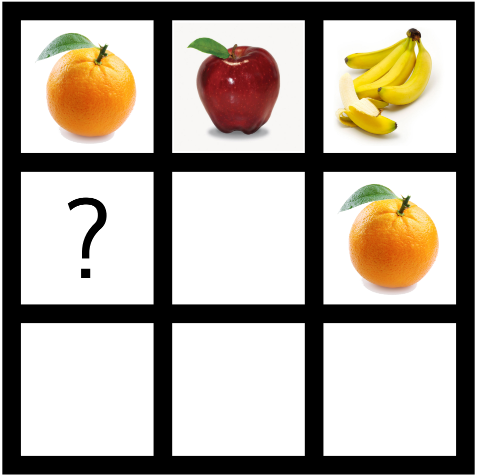


Building Latin Squares: Top-left to middle

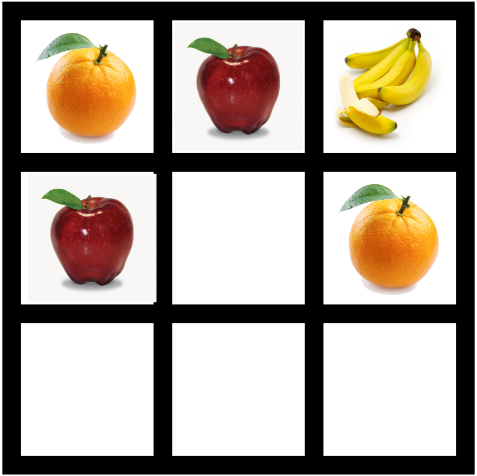


Building Latin Squares: Top-left to middle

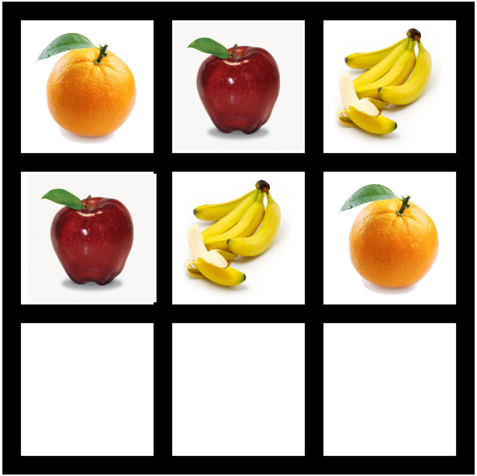
Building Latin Squares: Top-left to far-right



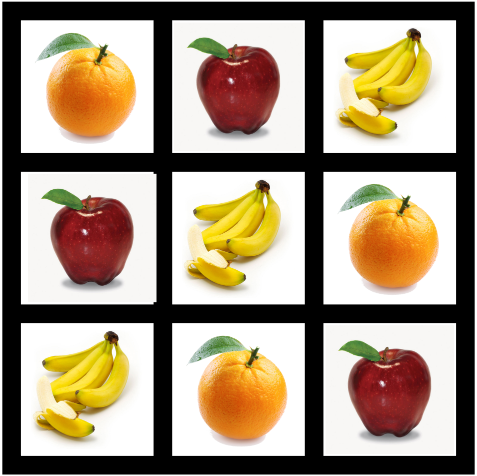
Building Latin Squares: Top-left to far-right



Building Latin Squares: Top-left to far-right



Building Latin Squares: Top-left to far-right



Orthogonal Latin Squares

Definition:

Lay one Latin square over another. These squares are **orthogonal** if each pair appears exactly once.

Definition:

The resulting square of pairs is called a **Greco-Latin Square**.

Example:

d	s	y
y	d	s
s	y	d

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yA	dB	sO
sB	yO	dA

Orthogonal Latin Squares

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Example:

O	A	B
A	B	O
B	O	A

Two 4×4 Latin Squares...

d	s	y	p
p	d	s	y
y	p	d	s
s	y	p	d

O	A	B	G
A	B	G	O
B	G	O	A
G	O	A	B

...which are Not Orthogonal

d	s	y	p
p	d	s	y
y	p	d	s
s	y	p	d

. . . which are Not Orthogonal

dO	sA	yB	pG
pA	dB	sG	yO
yB	pG	dO	sA
sG	yO	pA	dB

Too Many

'dO''s and 'sA''s

But No

'dA''s or 'sO''s

. . . which are Not Orthogonal

O	A	B	G
A	B	G	O
B	G	O	A
G	O	A	B

First Challenge:

Can You Find:

1. A Latin Square orthogonal to Puppies?
2. Can you find two which are orthogonal to Puppies **and each other?**

Puppies

d	s	y	p
p	y	s	d
y	p	d	s
s	d	p	y

First Challenge:

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Puppies

dO	sA	yB	pG
p?	y?	s?	d?
y	p	d	s
s	d	p	y

First Challenge:

Can You Find:

1. A Latin Square orthogonal to Puppies?
2. Can you find two which are orthogonal to Puppies **and each other?**

Puppies

dO	sA	yB	pG
pX	yO?	sO?	dX
y	p	d	s
s	d	p	y

One Possible Solution:

O	A	B	G
A	O	G	B
G	B	A	O
B	G	O	A

One Possible Solution:

dO	sA	yB	pG
pA	yO	sG	dB
yG	pB	dA	sO
sB	dG	pO	yA

One Possible Solution:

α	β	γ	δ
γ	δ	α	β
β	α	δ	γ
δ	γ	β	α

One Possible Solution:

d α	s β	y γ	p δ
p γ	y δ	s α	d β
y β	p α	d δ	s γ
s δ	d γ	p β	y α

One Possible Solution:

O α	A β	B γ	G δ
A γ	O δ	G α	B β
G β	B α	A δ	O γ
B δ	G γ	O β	A α

Maximum Number of Orthogonal Squares?

Theorem:

The maximum number of mutually orthogonal $n \times n$ Latin squares is $n - 1$.

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Example:

The maximum number of mutually orthogonal 4×4 Latin squares is 3 (and we found them).

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The maximum number of mutually orthogonal $n \times n$ Latin squares is $n - 1$.

Example:

The maximum number of mutually orthogonal 4×4 Latin squares is 3 (and we found them).

Sometimes there are many fewer.

The French Officers Problem shows there are **not even two** 6×6 orthogonal Latin squares.

When Do You Have the Maximum?

Theorem:

There are $n - 1$ orthogonal $n \times n$ Latin squares **if** n is prime or the power of a prime.

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There are $n-1$ orthogonal $n \times n$ Latin squares **if** n is prime or the power of a prime.

Example:

4 is the 2^2, so there should be (and are) three mutually orthogonal 4×4 Latin squares.

When Do You Have the Maximum?

Theorem:

There are $n-1$ orthogonal $n \times n$ Latin squares **if** n is prime or the power of a prime.

Example:

4 is the 2^2, so there should be (and are) three mutually orthogonal 4×4 Latin squares.

Example:

6 is not prime or the power of a prime, so there do not have to be five mutually orthogonal 6×6 Latin squares (and there aren't).

Part-magic Squares

Definition:

A Part-magic square is an $n \times n$ square of n^2 numbers (usually $1, 2 \ldots n^2$) where each row and column add up to the same sum.

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A Part-magic square is an $n \times n$ square of n^2 numbers (usually $1, 2 \ldots n^2$) where each row and column add up to the same sum.

Definition:

A Magic square is a part-magic square where the main diagonals also add up to the same sum as the rows and columns.

Part-magic, but not Magic

1	5	9
6	7	2
8	3	4

Rows

$$\begin{aligned} 1+5+9 &= 6+7+2 \\ &= 8+3+4 = 15 \end{aligned}$$

Columns

$$\begin{aligned} 1+6+8 &= 5+7+3 \\ &= 9+2+4 = 15 \end{aligned}$$

Diagonals

$$\begin{aligned} 1+7+4 &= 12 \neq 15 \\ 9+7+8 &= 24 \neq 15 \end{aligned}$$

Two orthogonal Latin squares $\Rightarrow$ Part-magic square

dO	sA	yB
yA	dB	sO
sB	yO	dA

$$\begin{array}{rcl|rcl} d & = & 1 & O & = & 0 \\ s & = & 2 & A & = & 3 \\ y & = & 3 & B & = & 6 \end{array}$$

. . . and just add.

Two orthogonal Latin squares $\Rightarrow$ Part-magic square

$1+0$	$2+3$	$3+6$
$3+3$	$1+6$	$2+0$
$2+6$	$3+0$	$1+3$

$$\begin{array}{ccc|ccc} d & = & 1 & O & = & 0 \\ s & = & 2 & A & = & 3 \\ y & = & 3 & B & = & 6 \end{array}$$

…and just add.

Two orthogonal Latin squares $\Rightarrow$ Part-magic square

1	5	9
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…and just add.

Part-magic square $\Rightarrow$ Magic

Each row and column is:

$$\begin{array}{ccccc} d+s+y & + & O+A+B & = & \\ 6 & + & 9 & = & 15 \end{array}$$

Part-magic square $\Rightarrow$ Magic

Each row and column is:

$$\begin{array}{ccccc} d+s+y & + & O+A+B & = & \\ 6 & + & 9 & = & 15 \end{array}$$

But to be **Magic**

$$\begin{aligned} 3d+O+A+B &= 15 \\ d+s+y+3B &= 15 \end{aligned}$$

Making Magic. . .

. . . with a little math.

$$
\begin{aligned}
3d &+ O + A + B &= 15 \\
3d &+ 9 &= 15 \\
& \Rightarrow d &= 2 \\
d + s + y &+ 3B &= 15 \\
6 &+ 3B &= 15 \\
& \Rightarrow B &= 3
\end{aligned}
$$

Magic!

dO	sA	yB
yA	dB	sO
sB	yO	dA

$$\begin{array}{ccc|ccc} d & = & 2 & O & = & 0 \\ s & = & 3 & A & = & 6 \\ y & = & 1 & B & = & 3 \end{array}$$

Magic!

2+O	s+A	y+3
y+A	2 + 3	s+O
s+3	y+O	2+A

$$\begin{array}{ccc|ccc} d & = & 2 & O & = & 0 \\ s & = & 3 & A & = & 6 \\ y & = & 1 & B & = & 3 \end{array}$$

Magic!

$2+0$	$3+6$	$1+3$
$1+6$	$2+3$	$3+0$
$3+3$	$1+0$	$2+6$

$$\begin{array}{ccc|ccc} d & = & 2 & O & = & 0 \\ s & = & 3 & A & = & 6 \\ y & = & 1 & B & = & 3 \end{array}$$

Magic!

2	9	4
7	5	3
6	1	8

$$\begin{array}{ccc|ccc} d & = & 2 & O & = & 0 \\ s & = & 3 & A & = & 6 \\ y & = & 1 & B & = & 3 \end{array}$$

Second Challenge: Find a 4×4 Magic Square

dO	sA	yB	pG
pA	yO	sG	dB
yG	pB	dA	sO
sB	dG	pO	yA

$$\{d, s, y, p\} = \{1, 2, 3, 4\}$$

$$\{O, A, B, G\} = \{0, 4, 8, 12\}$$

Possible Solution:

For each row and column:

$$
\begin{array}{rcl}
d + s + y + p & = & 10 \\
+O + A + B + G & = & 24 \\
\hline
 & = & 34
\end{array}
$$

Possible Solution:

For each row and column:

$$\begin{array}{rcl} d+s+y+p & = & 10 \\ +O+A+B+G & = & 24 \\ \hline & = & 34 \end{array}$$

So the diagonals satisfy:

$$\begin{array}{rcl} 2(d+y)+2(O+A) & = & 34 \\ 2(s+p)+2(B+G) & = & 34 \\ \Rightarrow (d+y)+(O+A) & = & 17 \\ \Rightarrow (s+p)+(B+G) & = & 17 \end{array}$$

One possible choice:

dO	sA	yB	pG
pA	yO	sG	dB
yG	pB	dA	sO
sB	dG	pO	yA

$$\begin{aligned} d + y = &\quad s + p = 5 \\ A + O = &\quad B + G = 12 \end{aligned}$$

$$\begin{aligned} d &= 1 & O &= 0 \\ s &= 2 & A &= 12 \\ y &= 4 & B &= 8 \\ p &= 3 & G &= 4 \end{aligned}$$

One possible choice:

$1+0$	$2+12$	$4+8$	$3+4$
$3+12$	$4+0$	$2+4$	$1+8$
$4+4$	$3+8$	$1+12$	$2+0$
$2+8$	$1+4$	$3+0$	$4+12$

$$\begin{aligned} d+y = & \quad s+p = 5 \\ A+O = & \quad B+G = 12 \end{aligned}$$

$$\begin{array}{ll} d=1 & O=0 \\ s=2 & A=12 \\ y=4 & B=8 \\ p=3 & G=4 \end{array}$$

One possible choice:

1	14	12	7
15	4	6	9
8	11	13	2
10	5	3	16

$$\begin{aligned} d + y = \quad & s + p = 5 \\ A + O = \quad & B + G = 12 \end{aligned}$$

$$\begin{array}{ll} d = 1 & O = 0 \\ s = 2 & A = 12 \\ y = 4 & B = 8 \\ p = 3 & G = 4 \end{array}$$

Magic without a Greco-Latin Square