Math and Magic Squares

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Leonhard Euler’s French Officers Problem:

Arrange thirty-six officers in a six-by-six square

- from six regiments: 1st, 2nd, 3rd, 4th, 5th, 6th
- with six ranks: Recruit, Lieutenant, Captain, Major, Brigadier, General
- so that each row and column has one representative from each regiment and rank.
Easy for 25 officers in a $5 \times 5$ square:

<table>
<thead>
<tr>
<th></th>
<th>1L</th>
<th>2C</th>
<th>3M</th>
<th>4B</th>
<th>5G</th>
</tr>
</thead>
<tbody>
<tr>
<td>5C</td>
<td>1M</td>
<td>2B</td>
<td>3G</td>
<td>4L</td>
<td></td>
</tr>
<tr>
<td>4M</td>
<td>5B</td>
<td>1G</td>
<td>2L</td>
<td>3C</td>
<td></td>
</tr>
<tr>
<td>3B</td>
<td>4G</td>
<td>5L</td>
<td>1C</td>
<td>2M</td>
<td></td>
</tr>
<tr>
<td>2G</td>
<td>3L</td>
<td>4C</td>
<td>5M</td>
<td>1B</td>
<td></td>
</tr>
</tbody>
</table>
What’s the pattern?

Can be done for:

- $3 \times 3$, $5 \times 5$, $7 \times 7$
- All $n \times n$
  - for odd $n$
- $4 \times 4$,
- even $8 \times 8$!
What’s the pattern?

Can be done for:
- $3 \times 3$, $5 \times 5$, $7 \times 7$
- All $n \times n$ for odd $n$
- $4 \times 4$
- even $8 \times 8$

But Not for:
- $2 \times 2$
- $6 \times 6$
No Solution to the $2 \times 2$ French Officers Problem

Then what?

1L  2C

2?  1?
No Solution to the $2 \times 2$ French Officers Problem

Then what?

<table>
<thead>
<tr>
<th>1L</th>
<th>2C</th>
</tr>
</thead>
<tbody>
<tr>
<td>2C</td>
<td>1L</td>
</tr>
</tbody>
</table>

No! Not Allowed!

Two First Lieutenants and Two Second Captains, but No Second Lieutenants or First Captains!
No Solution to the $2 \times 2$ French Officers Problem

Then what?

![2x2 grid with officers]

No! Not Allowed!

Two officers of the same rank in the same column!
Euler’s Conjecture (1782):
No Solutions for $n = 2, 6, 10, 14, 18, \ldots$
Conjectures and Theorems

Euler’s Conjecture (1782):
No Solutions for \( n = 2, 6, 10, 14, 18, \ldots \)

...is partly right...

Tarry (1900), Stimson (1984):
No Solutions for \( n = 6 \)
Euler’s Conjecture (1782):

No Solutions for $n = 2, 6, 10, 14, 18, \ldots$

...is partly right...

Tarry (1900), Stimson (1984):

No Solutions for $n = 6$

...but mostly wrong.

Bose, Parker, Shrikhande (1960):

Solutions for all $n$ except $n = 2$ or $n = 6$. 
Definition:

A Latin Square is an $n \times n$ square with $n$ symbols arranged so that each row and column has each symbol exactly once.
Example Latin Square: Puppies!
Example Latin Square: Sudoku

```
1 6 4
5 7 2
3 9 8
8 4 5
6 2 9
7 1 3
4 5 1
2 8 6
9 3 7
```
Building Latin Squares: Top-left to middle
Building Latin Squares: Top-left to middle
Building Latin Squares: Top-left to middle
Building Latin Squares: Top-left to middle
Building Latin Squares: Top-left to far-right
Building Latin Squares: Top-left to far-right
Building Latin Squares: Top-left to far-right
Building Latin Squares: Top-left to far-right
Orthogonal Latin Squares

**Definition:**
Lay one Latin square over another. These squares are **orthogonal** if each pair appears exactly once.

**Definition:**
The resulting square of pairs is called a **Greco-Latin Square**.

**Example:**
```
<table>
<thead>
<tr>
<th>d</th>
<th>s</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>d</td>
<td>s</td>
</tr>
<tr>
<td>s</td>
<td>y</td>
<td>d</td>
</tr>
</tbody>
</table>
```
**Definition:**
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Orthogonal Latin Squares

**Definition:**
Lay one Latin square over another. These squares are **orthogonal** if each pair appears exactly once.

**Definition:**
The resulting square of pairs is called a **Greco-Latin Square**.
Two $4 \times 4$ Latin Squares...
French Officers Problem
Latin Squares
Greco-Latin Squares
Magic Squares

...which are Not Orthogonal

\[
\begin{array}{ccc}
d & s & y & p \\
p & d & s & y \\
y & p & d & s \\
s & y & p & d \\
\end{array}
\]
... which are Not Orthogonal

<table>
<thead>
<tr>
<th></th>
<th>dO</th>
<th>sA</th>
<th>yB</th>
<th>pG</th>
</tr>
</thead>
<tbody>
<tr>
<td>pA</td>
<td>dB</td>
<td>sG</td>
<td>yO</td>
<td></td>
</tr>
<tr>
<td>yB</td>
<td>pG</td>
<td>dO</td>
<td>sA</td>
<td></td>
</tr>
<tr>
<td>sG</td>
<td>yO</td>
<td>pA</td>
<td>dB</td>
<td></td>
</tr>
</tbody>
</table>

Too Many

- ‘dO’
- ‘sA’

But No

- ‘dA’
- ‘sO’
...which are Not Orthogonal
### First Challenge:

**Can You Find:**

1. A Latin Square orthogonal to Puppies?
2. Can you find two which are orthogonal to Puppies and each other?

**Puppies**

<table>
<thead>
<tr>
<th></th>
<th>d</th>
<th>s</th>
<th>y</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>y</td>
<td>s</td>
<td>d</td>
<td></td>
</tr>
<tr>
<td>y</td>
<td>p</td>
<td>d</td>
<td>s</td>
<td></td>
</tr>
<tr>
<td>s</td>
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<td>p</td>
<td>y</td>
<td></td>
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Can You Find:
1. A Latin Square orthogonal to Puppies?
2. Can you find two which are orthogonal to Puppies and each other?
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2. Can you find two which are orthogonal to Puppies and each other?

Puppies

<table>
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<th>dO</th>
<th>sA</th>
<th>yB</th>
<th>pG</th>
</tr>
</thead>
<tbody>
<tr>
<td>pX</td>
<td>yO?</td>
<td>sO?</td>
<td>dX</td>
</tr>
<tr>
<td>y</td>
<td>p</td>
<td>d</td>
<td>s</td>
</tr>
<tr>
<td>s</td>
<td>d</td>
<td>p</td>
<td>y</td>
</tr>
</tbody>
</table>
One Possible Solution:

<table>
<thead>
<tr>
<th>O</th>
<th>A</th>
<th>B</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>O</td>
<td>G</td>
<td>B</td>
</tr>
<tr>
<td>G</td>
<td>B</td>
<td>A</td>
<td>O</td>
</tr>
<tr>
<td>B</td>
<td>G</td>
<td>O</td>
<td>A</td>
</tr>
</tbody>
</table>
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<td>yO</td>
<td>sG</td>
<td>dB</td>
</tr>
<tr>
<td>yG</td>
<td>pB</td>
<td>dA</td>
<td>sO</td>
</tr>
<tr>
<td>sB</td>
<td>dG</td>
<td>pO</td>
<td>yA</td>
</tr>
</tbody>
</table>
One Possible Solution:

<table>
<thead>
<tr>
<th></th>
<th>α</th>
<th>β</th>
<th>γ</th>
<th>δ</th>
</tr>
</thead>
<tbody>
<tr>
<td>γ</td>
<td>δ</td>
<td>α</td>
<td>β</td>
<td></td>
</tr>
<tr>
<td>β</td>
<td>α</td>
<td>δ</td>
<td>γ</td>
<td></td>
</tr>
<tr>
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<td>γ</td>
<td>β</td>
<td>α</td>
<td></td>
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</table>
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<th>s</th>
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<th>y</th>
<th>γ</th>
<th>p</th>
<th>δ</th>
</tr>
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<tr>
<td>p</td>
<td>γ</td>
<td>y</td>
<td>δ</td>
<td>s</td>
<td>α</td>
<td>d</td>
<td>β</td>
</tr>
<tr>
<td>y</td>
<td>β</td>
<td>p</td>
<td>α</td>
<td>d</td>
<td>δ</td>
<td>s</td>
<td>γ</td>
</tr>
<tr>
<td>s</td>
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<td>d</td>
<td>γ</td>
<td>p</td>
<td>β</td>
<td>y</td>
<td>α</td>
</tr>
</tbody>
</table>
One Possible Solution:

<table>
<thead>
<tr>
<th></th>
<th>O α</th>
<th>A β</th>
<th>B γ</th>
<th>G δ</th>
</tr>
</thead>
<tbody>
<tr>
<td>A γ</td>
<td>O δ</td>
<td>G α</td>
<td>B β</td>
<td></td>
</tr>
<tr>
<td>G β</td>
<td>B α</td>
<td>A δ</td>
<td>O γ</td>
<td></td>
</tr>
<tr>
<td>B δ</td>
<td>G γ</td>
<td>O β</td>
<td>A α</td>
<td></td>
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</tbody>
</table>
Theorem:
The maximum number of mutually orthogonal $n \times n$ Latin squares is $n - 1$. 
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The maximum number of mutually orthogonal $n \times n$ Latin squares is $n - 1$.

Example:
The maximum number of mutually orthogonal $4 \times 4$ Latin squares is 3 (and we found them).
Theorem:
The maximum number of mutually orthogonal $n \times n$ Latin squares is $n - 1$.

Example:
The maximum number of mutually orthogonal $4 \times 4$ Latin squares is 3 (and we found them).

Sometimes there are many fewer.
The French Officers Problem shows there are not even two $6 \times 6$ orthogonal Latin squares.
Theorem:

There are \( n - 1 \) orthogonal \( n \times n \) Latin squares if \( n \) is prime or the power of a prime.
When Do You Have the Maximum?

**Theorem:**
There are \( n - 1 \) orthogonal \( n \times n \) Latin squares \textbf{if} \( n \) is prime or the power of a prime.

**Example:**
4 is the \( 2^2 \), so there should be (and are) three mutually orthogonal \( 4 \times 4 \) Latin squares.
When Do You Have the Maximum?

**Theorem:**
There are \( n - 1 \) orthogonal \( n \times n \) Latin squares if \( n \) is prime or the power of a prime.

**Example:**
4 is the \( 2^2 \), so there should be (and are) three mutually orthogonal \( 4 \times 4 \) Latin squares.

**Example:**
6 is not prime or the power of a prime, so there do not have to be five mutually orthogonal \( 6 \times 6 \) Latin squares (and there aren’t).
Part-magic Squares

Definition:

A Part-magic square is an $n \times n$ square of $n^2$ numbers (usually 1, 2\ldots n^2) where each row and column add up to the same sum.
Definition:
A Part-magic square is an $n \times n$ square of $n^2$ numbers (usually 1, 2, ..., $n^2$) where each row and column add up to the same sum.

Definition:
A Magic square is a part-magic square where the main diagonals also add up to the same sum as the rows and columns.
Part-magic, but not Magic

\[
\begin{array}{ccc}
1 & 5 & 9 \\
6 & 7 & 2 \\
8 & 3 & 4 \\
\end{array}
\]

Rows
\[
1 + 5 + 9 = 6 + 7 + 2 = 8 + 3 + 4 = 15
\]

Columns
\[
1 + 6 + 8 = 5 + 7 + 3 = 9 + 2 + 4 = 15
\]

Diagonals
\[
1 + 7 + 4 = 12 \neq 15 \\
9 + 7 + 8 = 24 \neq 15
\]
Two orthogonal Latin squares \(\Rightarrow\) Part-magic square

\[
\begin{array}{ccc}
dO & sA & yB \\
yA & dB & sO \\
sB & yO & dA \\
\end{array}
\]

\[
\begin{align*}
d &= 1 & O &= 0 \\
s &= 2 & A &= 3 \\
y &= 3 & B &= 6
\end{align*}
\]

\(\ldots\) and just add.
Two orthogonal Latin squares
⇒ Part-magic square

\[
\begin{array}{ccc}
1 + 0 & 2 + 3 & 3 + 6 \\
3 + 3 & 1 + 6 & 2 + 0 \\
2 + 6 & 3 + 0 & 1 + 3 \\
\end{array}
\]

\[
\begin{array}{c|c}
d & 1 \\
O & 0 \\
s & 2 \\
A & 3 \\
y & 3 \\
B & 6 \\
\end{array}
\]

...and just add.
Two orthogonal Latin squares $\Rightarrow$ Part-magic square

\[
\begin{array}{ccc}
1 & 5 & 9 \\
6 & 7 & 2 \\
8 & 3 & 4 \\
\end{array}
\]

\[
\begin{array}{c|c}
d = 1 & O = 0 \\
s = 2 & A = 3 \\
y = 3 & B = 6 \\
\end{array}
\]

... and just add.
Part-magic square $\Rightarrow$ Magic

Each row and column is:

$$d + s + y + O + A + B =$$

$$6 + 9 = 15$$
Part-magic square $\Rightarrow$ Magic

Each row and column is:

\[
d + s + y + O + A + B = \\
6 + 9 = 15
\]

But to be Magic

\[
3d + O + A + B = 15 \\
d + s + y + 3B = 15
\]
Making Magic...

...with a little math.

\[
\begin{align*}
3d + O + A + B &= 15 \\
3d + 9 &= 15 \\
\Rightarrow d &= 2 \\
d + s + y + 3B &= 15 \\
6 + 3B &= 15 \\
\Rightarrow B &= 3
\end{align*}
\]
### Magic!

<table>
<thead>
<tr>
<th>dO</th>
<th>sA</th>
<th>yB</th>
</tr>
</thead>
<tbody>
<tr>
<td>yA</td>
<td>dB</td>
<td>sO</td>
</tr>
<tr>
<td>sB</td>
<td>yO</td>
<td>dA</td>
</tr>
</tbody>
</table>

| \( d \) = 2 | \( O \) = 0 |
| \( s \) = 3 | \( A \) = 6 |
| \( y \) = 1 | \( B \) = 3 |
Magic!

Magic Squares

\[
\begin{array}{ccc}
2+O & s+A & y+3 \\
y+A & 2+3 & s+O \\
s+3 & y+O & 2+A \\
\end{array}
\]

\[
\begin{align*}
d &= 2 & O &= 0 \\
s &= 3 & A &= 6 \\
y &= 1 & B &= 3
\end{align*}
\]
Magic!

\[
\begin{array}{ccc}
2 + 0 & 3 + 6 & 1 + 3 \\
1 + 6 & 2 + 3 & 3 + 0 \\
3 + 3 & 1 + 0 & 2 + 6 \\
\end{array}
\]

\[
\begin{array}{cc}
d = 2 & O = 0 \\
s = 3 & A = 6 \\
y = 1 & B = 3 \\
\end{array}
\]
Magic!

\[
\begin{array}{ccc}
2 & 9 & 4 \\
7 & 5 & 3 \\
6 & 1 & 8 \\
\end{array}
\]

\[
\begin{align*}
d &= 2 & O &= 0 \\
s &= 3 & A &= 6 \\
y &= 1 & B &= 3 \\
\end{align*}
\]
Second Challenge: Find a $4 \times 4$ Magic Square

\[
\begin{array}{cccc}
  dO & sA & yB & pG \\
  pA & yO & sG & dB \\
  yG & pB & dA & sO \\
  sB & dG & pO & yA \\
\end{array}
\]

\[
\{d, s, y, p\} = \{1, 2, 3, 4\}
\]

\[
\{O, A, B, G\} = \{0, 4, 8, 12\}
\]
Possible Solution:

For each row and column:

\[
\begin{align*}
    d + s + y + p &= 10 \\
    + O + A + B + G &= 24 \\
    &= 34
\end{align*}
\]
Possible Solution:

For each row and column:

\[
\begin{align*}
d + s + y + p &= 10 \\
O + A + B + G &= 24 \\
\text{Total} &= 34
\end{align*}
\]

So the diagonals satisfy:

\[
\begin{align*}
2(d + y) + 2(O + A) &= 34 \\
2(s + p) + 2(B + G) &= 34 \\
\Rightarrow (d + y) + (O + A) &= 17 \\
\Rightarrow (s + p) + (B + G) &= 17
\end{align*}
\]
One possible choice:

\[
\begin{array}{cccc}
  d & y & s & p \\
  p & y & s & d \\
  y & p & d & s \\
  s & d & p & y \\
\end{array}
\]

\[
\begin{align*}
  d + y &= s + p = 5 \\
  A + O &= B + G = 12 \\
  d &= 1 \quad O = 0 \\
  s &= 2 \quad A = 12 \\
  y &= 4 \quad B = 8 \\
  p &= 3 \quad G = 4 \\
\end{align*}
\]
One possible choice:

\[
\begin{array}{cccc}
1 + 0 & 2 + 12 & 4 + 8 & 3 + 4 \\
3 + 12 & 4 + 0 & 2 + 4 & 1 + 8 \\
4 + 4 & 3 + 8 & 1 + 12 & 2 + 0 \\
2 + 8 & 1 + 4 & 3 + 0 & 4 + 12 \\
\end{array}
\]

\[d + y = 5, \quad s + p = 5\]

\[A + O = 12, \quad B + G = 12\]

\[
\begin{align*}
d &= 1 & O &= 0 \\
s &= 2 & A &= 12 \\
y &= 4 & B &= 8 \\
p &= 3 & G &= 4 \\
\end{align*}
\]
One possible choice:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>14</th>
<th>12</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>4</td>
<td>6</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>11</td>
<td>13</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>3</td>
<td>16</td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{align*}
  d + y &= s + p = 5 \\
  A + O &= B + G = 12 \\
  d &= 1 \quad O = 0 \\
  s &= 2 \quad A = 12 \\
  y &= 4 \quad B = 8 \\
  p &= 3 \quad G = 4
\end{align*}
\]
Magic without a Greco-Latin Square