Boundary Layers

Or

The Mathematics of WhoVille

From: *Computational Aerodynamics and Fluid Dynamics* - J.J.Chattot

The following ODE

$$\frac{d}{dx}\left(\frac{u^2}{2}\right) = u, \qquad 0 \leq x \leq 1$$

with two boundary conditions

$$u(0) = 1, \quad u(1) = -1$$

has the exact solution:

$$\begin{cases} u(x) = x + 1, & 0 \leq x < \frac{1}{2} \\ u(x) = x - 2, & \frac{1}{2} < x \leq 1 \end{cases}$$

???

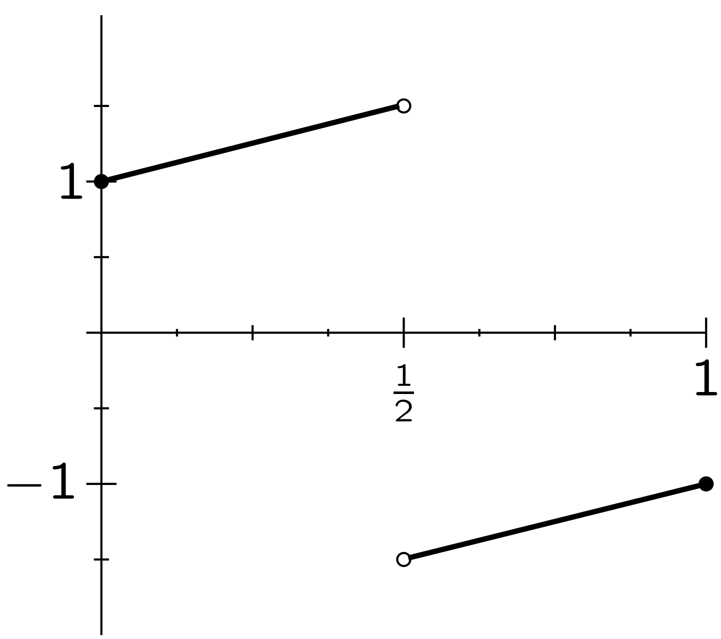
$$
\begin{aligned}
\left(\frac{u^2}{2}\right)' &= u \\
\frac{2uu'}{2} &= u \\
\Rightarrow (u'-1)u &= 0
\end{aligned}
$$

$$u' = 1 \quad \text{or} \quad u = 0$$

$$u = x + C \quad \text{or} \quad u = 0$$

$$u(0) = 1 \Rightarrow \; u = x + 1$$

$$u(1) = -1 \Rightarrow \; u = x - 2$$



Problems:

1. u is not continuous at $x = \frac{1}{2}$
   (and thus not differentiable)

2. The ODE is first order and yet is expected to satisfy **two** boundary conditions!

3. Why break at $x = \frac{1}{2}$?
   Why not some other point(s)?

Consider the 2nd order ODE

$$\epsilon u'' + 2u' + u = 0$$

with the boundary conditions

$$u(0) = 0 \quad u(1) = 1$$

for small parameter ϵ.

Letting $\epsilon \to 0$ leads to the 1st order ODE:

$$2u' + u = 0 \qquad \Rightarrow \quad u = Ce^{-x/2}$$

But

$$u(0) = 0 \quad \Rightarrow \quad C = 0$$

while

$$u(1) = 1 \quad \Rightarrow \quad C = e^{\frac{1}{2}}$$

Solving directly we have characteristic equation:

$$\epsilon r^2 + 2r + 1 = 1$$

$$r_{\pm} = \frac{-2 \pm \sqrt{4 - 4\epsilon}}{2\epsilon} = \frac{-1 \pm \sqrt{1 - \epsilon}}{\epsilon}$$

Since

$$\sqrt{1 - \epsilon} = 1 - \frac{\epsilon}{2} + O(\epsilon^2)$$

we have

$$r_- \approx -\frac{2}{\epsilon} \qquad\qquad r_+ \approx -\frac{1}{2}$$

thus the general solution is

$$u = C_1\, e^{-\frac{2x}{\epsilon}} + C_2\, e^{-\frac{x}{2}}$$

Applying the boundary conditions we have

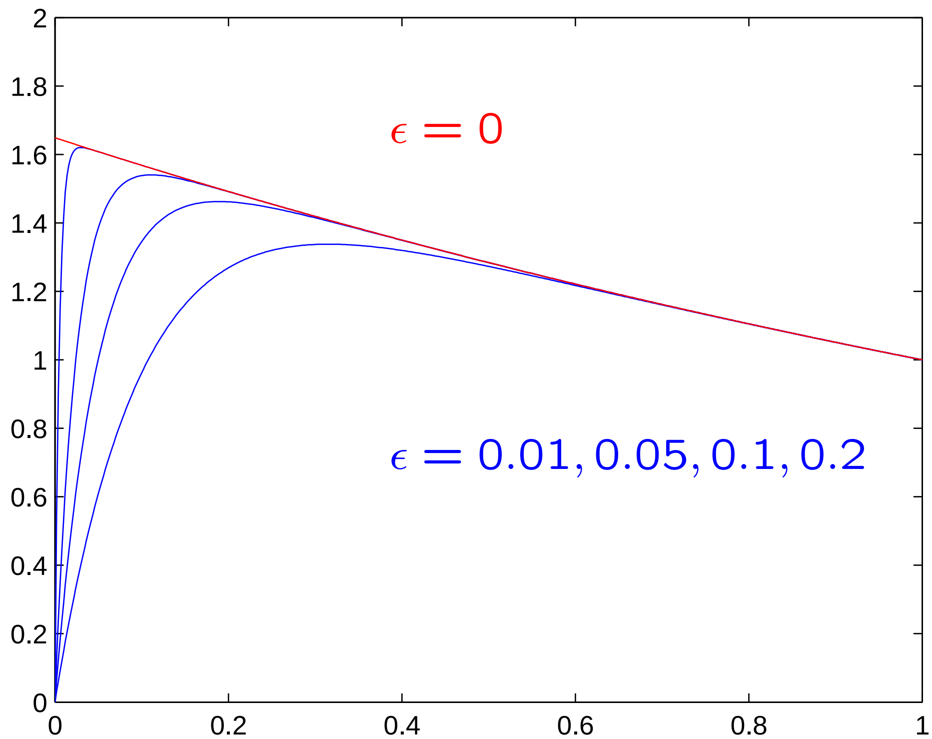
$$u = \frac{e^{-\frac{2x}{\epsilon}} - e^{-\frac{x}{2}}}{e^{-\frac{2}{\epsilon}} - e^{-\frac{1}{2}}} \approx \; e^{\frac{1}{2}}\, e^{-\frac{x}{2}} - e^{\frac{1}{2}}\, e^{-\frac{2x}{\epsilon}}$$

Note the first term is the $\epsilon = 0$ solution with the $u(1) = 1$ boundary condition.

This is sensible since for any $x > 0$

$$\lim_{\epsilon \to 0} e^{-\frac{2x}{\epsilon}} = 0$$

Nevertheless if $x = 0$ then $e^{-\frac{2x}{\epsilon}} = 1$ for any $\epsilon > 0$.



For small ϵ the $\epsilon = 0$ solution dominates most of the interval.

Only close to $x = 0$ is $u'' \gg 1$ so that the $\epsilon u''$ term may contribute to the ODE.

$$\epsilon u'' + 2u' + u = 0$$

This region is called the **Boundary Layer**.

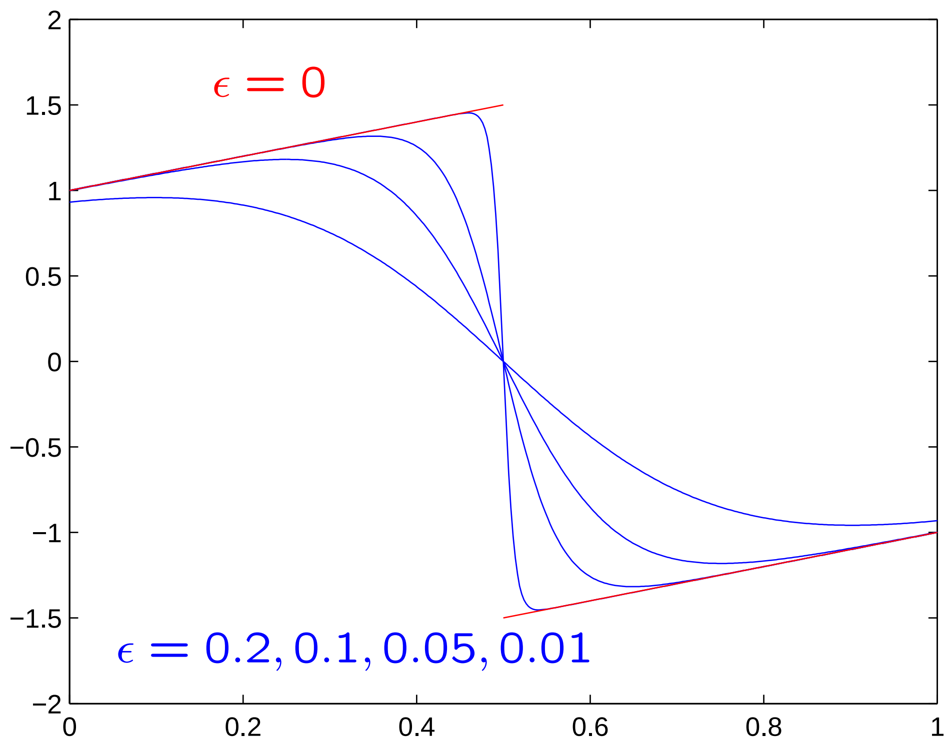
Similarly, it can be shown that the ODE

$$\epsilon\frac{d^2u}{dx^2} - \frac{d}{dx}\left(\frac{u^2}{2}\right) + u = 0$$

with two boundary conditions

$$u(0) = 1, \quad u(1) = -1$$

has a **boundary layer** at $x = \frac{1}{2}$



How do we find a boundary layer in a system we can't solve explicitly?

Notice in our $\epsilon u'' + 2u' + u = 0$ example there were two functions:

1. A function dominant on most of the domain, corresponding to $\epsilon = 0$

$$u_0(x) = e^{\frac{1}{2}}\, e^{-\frac{x}{2}}$$

2. A function which is only significant in the narrow region of the boundary layer

$$u_{bl}(x) = -e^{\frac{1}{2}}\, e^{-\frac{2x}{\epsilon}}$$

Define the "WhoVille" variable for the boundary layer

$$\chi = \frac{x}{\epsilon} \quad \Leftrightarrow \quad x = \epsilon\chi$$

$$\frac{d}{d\chi}u(x) = \frac{d}{d\chi}u(\epsilon\chi) = \epsilon u'(x)$$

If we let $\bar{u}(\chi) = u(\epsilon\chi) = u(x)$ then

$$u' = \frac{1}{\epsilon}\,\bar{u}' \quad \text{and} \quad u'' = \frac{1}{\epsilon^2}\,\bar{u}''$$

Substituting into the ODE,

$$\begin{aligned} \epsilon u'' + 2u' + u &= 0 \\ \epsilon\left(\tfrac{1}{\epsilon^2}\,\bar{u}''\right) + 2\left(\tfrac{1}{\epsilon}\,\bar{u}'\right) + \bar{u} &= 0 \\ \bar{u}'' + 2\bar{u}' + \epsilon\bar{u} &= 0 \end{aligned}$$

Letting $\epsilon \to 0$ gives us the "WhoVille" ODE

$$\bar{u}'' + 2\bar{u}' = 0 \quad \Rightarrow \quad \bar{u} = C_1 e^{-2\chi} + C_2$$

Boundaries of WhoVille are:

$$\bar{u}(0) = 0, \qquad \lim_{\chi\to\infty} \bar{u} = \lim_{x\to 0} u_0$$

The second equation is Horton hearing the Whos!

$$\lim_{\chi\to\infty} \bar{u} = \lim_{\chi\to\infty} C_1 e^{-2\chi} + C_2 = C_2$$

$$\lim_{x\to 0} u_0 = \lim_{x\to 0} e^{\frac{1}{2}}\, e^{-\frac{x}{2}} = e^{\frac{1}{2}}$$

So $C_2 = \sqrt{e}$. Applying the other boundary condition gives us

$$\bar{u} = e^{\frac{1}{2}}(1 - e^{-2\chi})$$

The final solution is found by adding the two and subtracting the "overlap"

$$\begin{aligned}
u &= u_0 + \bar{u} - C_2 \\
&= e^{\frac{1}{2}}\, e^{-\frac{x}{2}} + e^{\frac{1}{2}}(1 - e^{-2\chi}) - e^{\frac{1}{2}} \\
&= e^{\frac{1}{2}}\, e^{-\frac{x}{2}} - e^{\frac{1}{2}} e^{-2\frac{x}{\epsilon}}
\end{aligned}$$

For our original problem the "WhoVille" variable is

$$\chi = \frac{x - x_0}{\epsilon} \quad \Leftrightarrow \quad x = \epsilon\chi + x_0$$

The "WhoVille" ODE is then

$$\begin{aligned}
\epsilon u'' - \left(\frac{u^2}{2}\right)' + u &= 0 \\
\epsilon u'' - u u' + u &= 0 \\
\epsilon\left(\frac{1}{\epsilon^2}\,\bar{u}''\right) - \bar{u}\left(\frac{1}{\epsilon}\,\bar{u}'\right) + \bar{u} &= 0 \\
\bar{u}'' - \bar{u}\bar{u}' + \epsilon\bar{u} &= 0 \\
\Rightarrow \bar{u}'' - \left(\frac{\bar{u}^2}{2}\right)' &= 0
\end{aligned}$$

$$\bar{u} = -C_1 \tanh\left(\frac{C_1\,\chi}{2} + C_2\right)$$

The boundaries of WhoVille are now

$$\lim_{\chi\to\infty} \bar{u} = \lim_{x\to x_0^+} u_0, \qquad \lim_{\chi\to-\infty} \bar{u} = \lim_{x\to x_0^-} u_0$$

Now,

$$\lim_{\chi\to\infty} -C_1 \tanh\left(\frac{C_1\,\chi}{2} + C_2\right) = -C_1$$

while

$$\lim_{\chi\to-\infty} -C_1 \tanh\left(\frac{C_1\,\chi}{2} + C_2\right) = C_1$$

This forces "WhoVille" to be at $x_0 = \frac{1}{2}$, since

$$\lim_{x\to\frac{1}{2}^+} x - 2 = -\frac{3}{2} = -C_1$$

$$\lim_{x\to\frac{1}{2}^-} x + 1 = \frac{3}{2} = C_1$$

From symmetry we may argue $C_2 = 0$, so

$$u = x - \frac{1}{2} - \frac{3}{2}\tanh\left(\frac{3(x-1/2)}{4\epsilon}\right)$$