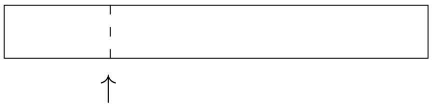
Folding Fifths

from 'Project Origami' by Thomas Hull

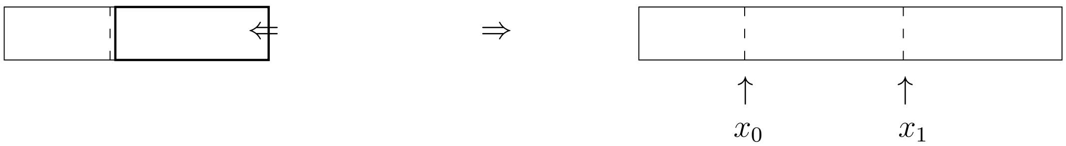
How do you divide a strip into 5ths?

Fujimoto's Approximation Method:

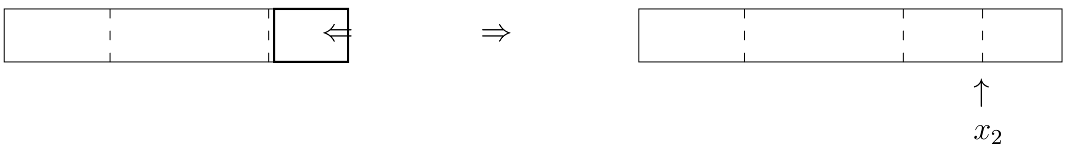


Guess x_0

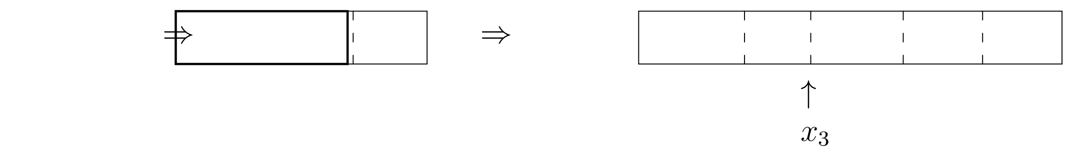
Then fold from the left and unfold:



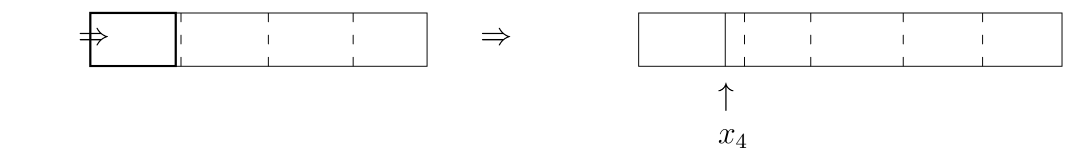
Again:



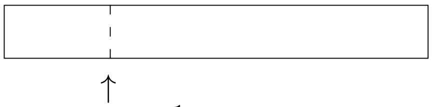
Then fold from the right and unfold:



And one last time:



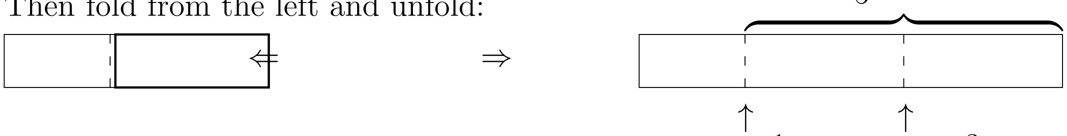
Why does it work?



Guess $x_0 = \frac{1}{5} + \epsilon$

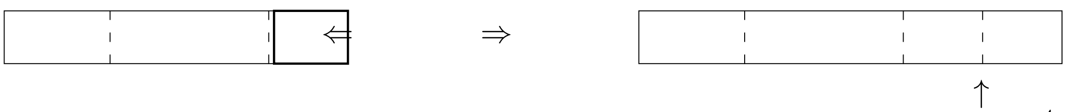
Then fold from the left and unfold:

$\frac{4}{5} - \epsilon$



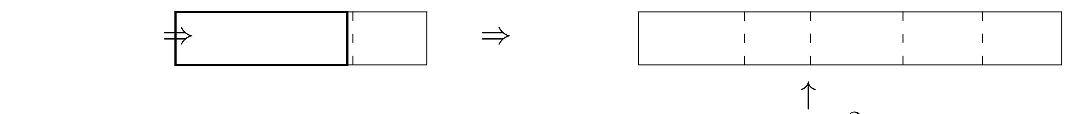
$x_0 = \frac{1}{5} + \epsilon$ $\quad x_1 = \frac{3}{5} + \frac{\epsilon}{2}$

Again:



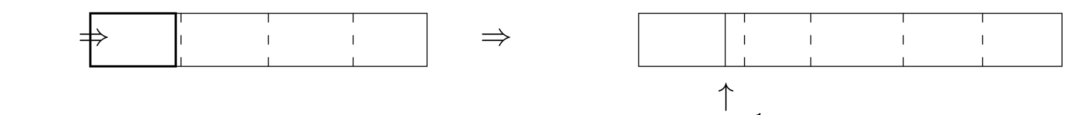
$x_2 = \frac{4}{5} + \frac{\epsilon}{4}$

Then fold from the right and unfold:



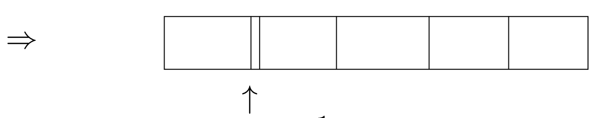
$x_3 = \frac{2}{5} + \frac{\epsilon}{8}$

And one last time:



$x_4 = \frac{1}{5} + \frac{\epsilon}{16}$

Repeating the whole process gives an even better estimate:



$x_8 = \frac{1}{5} + \frac{\epsilon}{256}$

The strategy is based on the side that has an **even number** of fifths.

Left of Fold	Right of Fold	Fold from the
$\frac{1}{5}$	$\frac{4}{5}$	Right
$\frac{3}{5}$	$\frac{2}{5}$	Right
$\frac{4}{5}$	$\frac{1}{5}$	Left
$\frac{2}{5}$	$\frac{3}{5}$	Left
$\frac{1}{5}$		

What would be the folding strategy for one **seventh**? Or **four ninths?**

Left of Fold	Right of Fold	Fold from the
$\frac{1}{7}$	?	?

The folding strategy for one **seventh**

Left of Fold	Right of Fold	Fold from the
$\frac{1}{7}$	$\frac{6}{7}$	Right
$\frac{4}{7}$	$\frac{3}{7}$	Left
$\frac{2}{7}$	$\frac{5}{7}$	Left
$\frac{1}{7}$		

$$x = \frac{1}{7} + \epsilon \;\Rightarrow\; x = \frac{1}{7} + \frac{\epsilon}{8}$$

The folding strategy for four **ninths**

Left of Fold	Right of Fold	Fold from the
$\frac{4}{9}$	$\frac{5}{9}$	Left
$\frac{2}{9}$	$\frac{7}{9}$	Left
$\frac{1}{9}$	$\frac{8}{9}$	Right
$\frac{5}{9}$	$\frac{4}{9}$	Right
$\frac{7}{9}$	$\frac{2}{9}$	Right
$\frac{8}{9}$	$\frac{1}{9}$	Left
$\frac{4}{9}$		

$$x = \frac{4}{9} + \epsilon \;\Rightarrow\; x = \frac{4}{9} + \frac{\epsilon}{64}$$

Fractions written as decimals

Base 10

$$\frac{1}{5} = 0.2 = 2\left(\frac{1}{10}\right)$$

$$\frac{1}{7} = 0.\overline{142857} = 1\left(\frac{1}{10}\right) + 4\left(\frac{1}{10}\right)^2 + 2\left(\frac{1}{10}\right)^3 + 8\left(\frac{1}{10}\right)^4 + \dots$$

Base 2

$$\frac{1}{5} = 0\left(\frac{1}{2}\right) + 0\left(\frac{1}{2}\right)^2 + 1\left(\frac{1}{2}\right)^3 + \dots ?$$

Writing $\frac{1}{5}$ as a binary decimal:

$$\frac{1}{5} - \frac{1}{8} = \frac{3}{40} \quad \text{and} \quad \frac{3}{40} - \frac{1}{16} = \frac{1}{80}$$

$$\begin{aligned}
\Rightarrow \frac{1}{5} &= \frac{1}{8} + \frac{1}{16} + \frac{1}{80} \\
\frac{1}{5} &= \frac{1}{8} + \frac{1}{16} + \frac{1}{16}\left(\frac{1}{5}\right) \\
&= (.0011)_2 + \frac{1}{16}\left(\frac{1}{5}\right) \\
&= (.0011)_2 + \frac{1}{16}\left((.0011)_2 + \frac{1}{16}\left(\frac{1}{5}\right)\right) \\
&= (.00110011)_2 + \frac{1}{16}^2\left(\frac{1}{5}\right) \\
&= (0.\overline{0011})_2
\end{aligned}$$

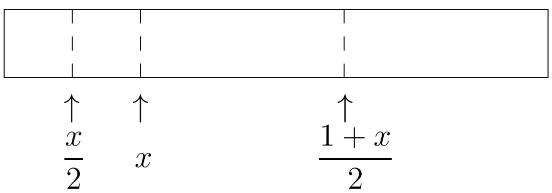
Notice the similarity:
0 0 1 1 (repeating) ??? Fold: Right-Right-Left-Left (repeating)

Symbolic Dynamics

For $0 < x < 1$ consider the functions:

$$\text{Fold from Left} \quad T_0(x) = \frac{x}{2}$$

$$\text{Fold from Right} \quad T_1(x) = x + \frac{1-x}{2} = \frac{1}{2} + \frac{x}{2}$$



$$\begin{aligned}
x &= \tfrac{x_1}{2} + \tfrac{x_2}{4} + \tfrac{x_3}{8} + \ldots &&= (.x_1x_2\ldots)_2 \\
T_0(x) &= \tfrac{x_1}{4} + \tfrac{x_2}{8} + \tfrac{x_3}{16} + \ldots &&= (.0x_1x_2\ldots)_2 \\
T_1(x) &= \tfrac{1}{2} + \tfrac{x_1}{4} + \tfrac{x_2}{8} + \tfrac{x_3}{16} + \ldots &&= (.1x_1x_2\ldots)_2
\end{aligned}$$

$$\begin{aligned}
(T_0 \circ T_0 \circ T_1 \circ T_1)(x) &= (.0011x_1x_2\ldots)_2 \\
(T_0 \circ T_0 \circ T_1 \circ T_1)^2(x) &= (.00110011x_1x_2\ldots)_2 \\
\lim_{n\to\infty}(T_0 \circ T_0 \circ T_1 \circ T_1)^n(x) &= (\overline{.0011})_2 = \frac{1}{5}
\end{aligned}$$

Note, by the way,

$$(T_0 \circ T_0 \circ T_1 \circ T_1)\left(\frac{1}{5}\right) = (T_0 \circ T_0 \circ T_1 \circ T_1)(\overline{.0011}) = .0011\overline{0011} = \frac{1}{5}$$

So $\dfrac{1}{5}$ is a fixed point of the map $(T_0 \circ T_0 \circ T_1 \circ T_1)$.

How about the expansion for $\frac{1}{7}$?

Let $\frac{1}{7} = (.y_1y_2\ldots)_2$

$$\begin{aligned}
(T_0 \circ T_0 \circ T_1)\left(\frac{1}{7}\right) &= \frac{1}{7} \\
(.001y_1y_2y_3\ldots)_2 &= (.y_1y_2y_3\ldots)_2 \\
\Rightarrow & \;\; y_1 = 0,\; y_2 = 0,\; y_3 = 1 \\
\Rightarrow & \;\; y_4 = y_1 = 0,\; y_5 = y_2 = 0,\; y_6 = y_3 = 1 \\
\Rightarrow \frac{1}{7} &= (.\overline{001})_2
\end{aligned}$$

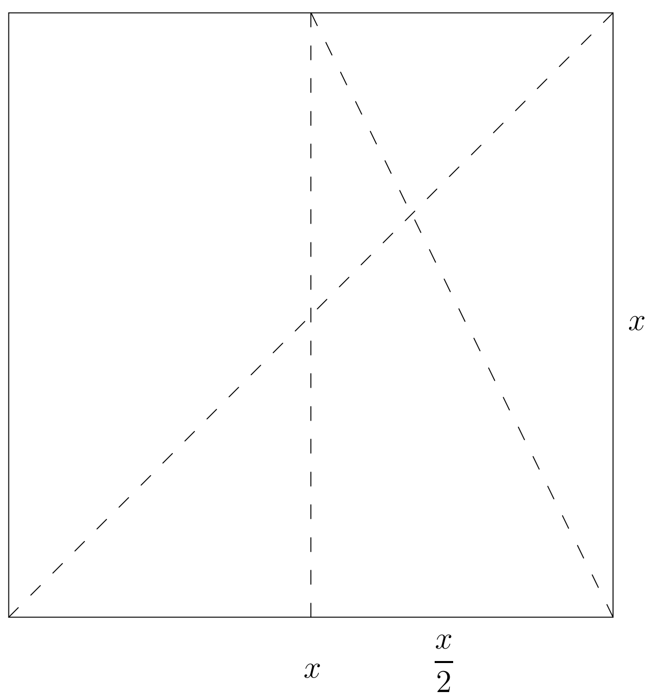
What about $\dfrac{1}{19}$?

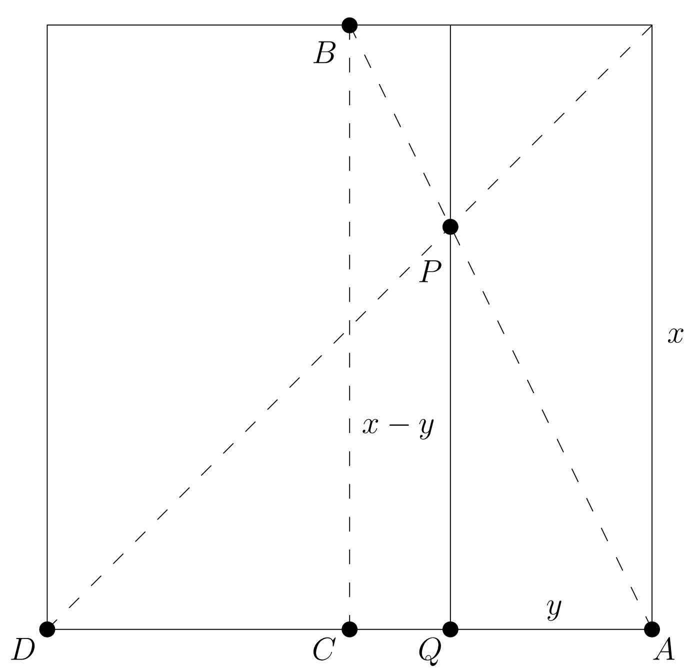
19-ths Left	19-ths Right	Fold from the
1	18	R
10	9	L
5	14	R
12	7	L
6	13	L
3	16	R
11	8	R
15	4	R
17	2	R
18	1	L
9	10	R
14	5	L
7	12	R
13	6	R
16	3	L
8	11	L
4	15	L
2	17	L
1		

$$\frac{1}{19} = (.\overline{000011010111100101})_2$$

Folding into Exact Thirds

Fold an x by x square paper along the dotted lines.
(In fact it works even if the paper isn't square.)



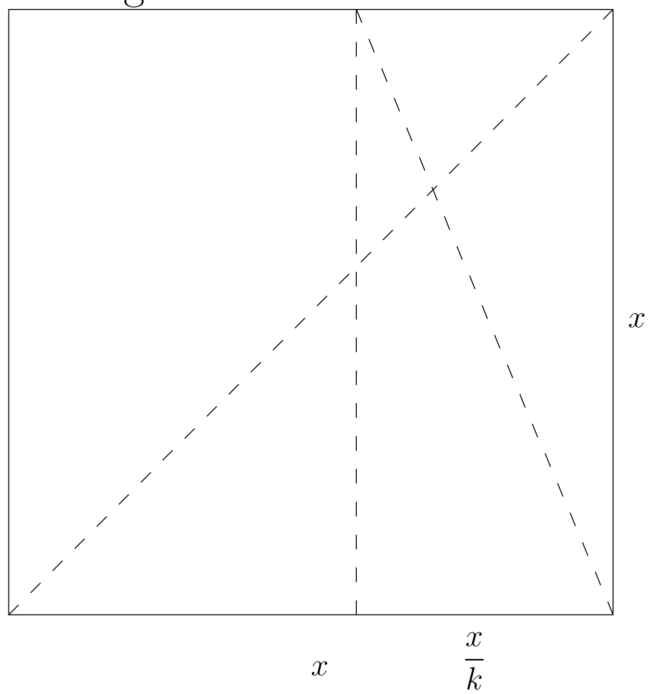


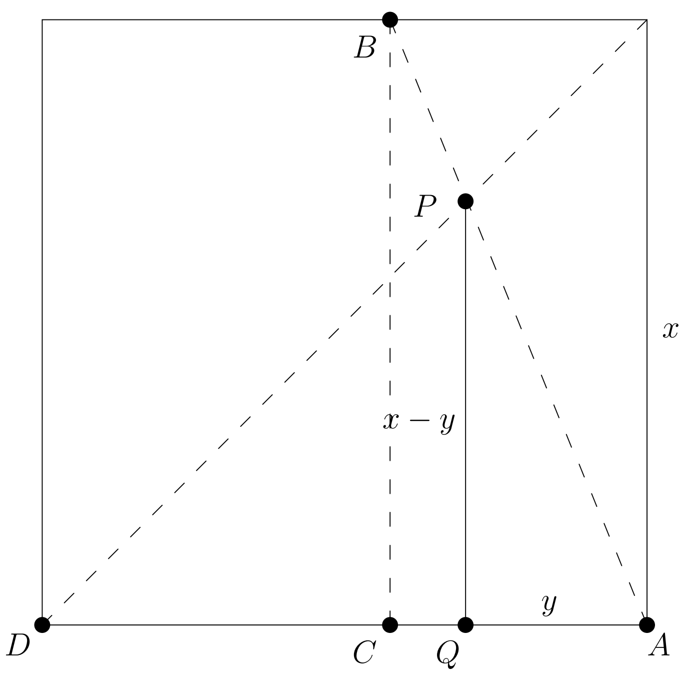
$\overline{DQ} = \overline{PQ} = x - y$

By similar triangles APQ and ABC,

$$\begin{aligned} \frac{x-y}{y} &= \frac{x}{x/2} = 2 \\ \Rightarrow x - y &= 2y \\ \Rightarrow x &= 2y + y = 3y \\ \Rightarrow y &= \frac{x}{3} \end{aligned}$$

Folding into Exact N-ths





$\overline{DQ} = \overline{PQ} = x - y$

By similar triangles APQ and ABC,

$$\begin{aligned} \frac{x-y}{y} &= \frac{x}{x/k} = k \\ \Rightarrow x - y &= ky \\ \Rightarrow x &= ky + y = (k+1)y \\ \Rightarrow y &= \frac{x}{k+1} \end{aligned}$$