Voting Theory: Methods (Donald Saari, UC Irvine)

- Plurality
- Majority (by exhaustion)
- Run-off elections
- Approval voting (vote for one or two)
- Preferential Methods
  - Instant Run-off
  - Single transferable vote
  - Positional weighting

    e.g. da Borda (1770) Count:
    last = 0, next = 1 point. . . first = $n - 1$
  - Pair-wise matching (Condorcet 1780s)

Countries with Preferencial Systems:

1. Instant Run-off: Fiji, Hong Kong, Ireland, Papua New Guinea

2. Single Transferable Vote: Australia, Estonia, Malta, New Zealand, Northern Ireland

3. Contingent vote: Czech Republic, Sri Lanka

4. Borda Count: Nauru, Slovenia

The Trouble with Democracy

Preferential Ballot:

Milk $>$ Juice $>$ Soda: 5
Soda $>$ Juice $>$ Milk: 4
Juice $>$ Soda $>$ Milk: 3

- Plurality: Milk

- Instant Run-off: Soda

- Anti-Plurality: Juice

Positional weighting for three candidates:

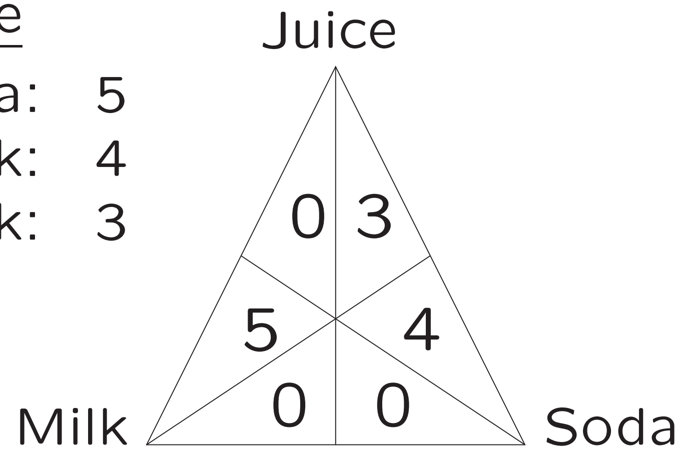
first = 1 point

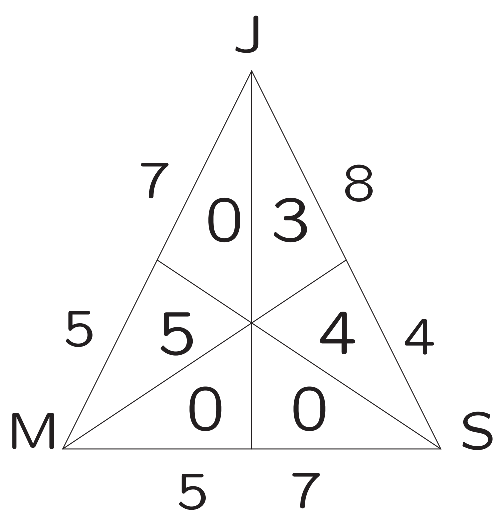
second = s points (where $0 \leq s \leq 1$)

third = 0 points

Voter Profile Triangle

Milk $>$ Juice $>$ Soda: 5
Soda $>$ Juice $>$ Milk: 4
Juice $>$ Soda $>$ Milk: 3





Juice is the Condorcet Winner

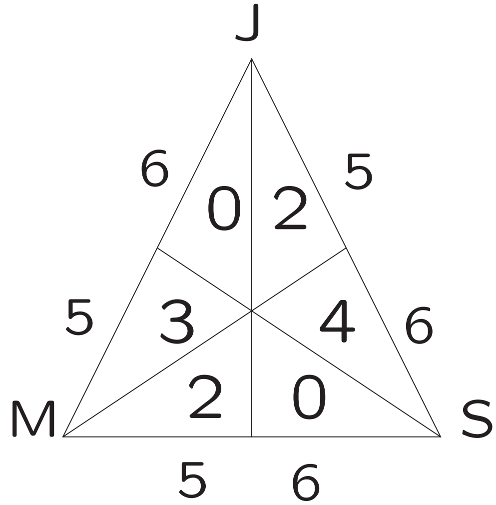
Positional weighting:
$$\begin{aligned} M &= 5 \\ S &= 4 + 3s \\ J &= 3 + 9s \end{aligned}$$

If $s < \frac{2}{9}$ then Milk is the positional winner.

If $s > \frac{2}{9}$ then Juice is the positional winner.

Exercise: Fill in a voter profile triangle, and find the Condorcet winner (if there is one), and the positional winner(s).

Milk $>$ Juice $>$ Soda: 3
Milk $>$ Soda $>$ Juice: 2
Soda $>$ Juice $>$ Milk: 4
Juice $>$ Soda $>$ Milk: 2



Soda is the Condorcet Winner

$$\text{Positional weighting:} \quad \begin{aligned} \text{M} &= 5 \\ \text{S} &= 4 + 4s \\ \text{J} &= 2 + 7s \end{aligned}$$

If $s < \frac{1}{4}$ then Milk is the positional winner.

If $\frac{1}{4} < s < \frac{2}{3}$ then Soda is the positional winner.

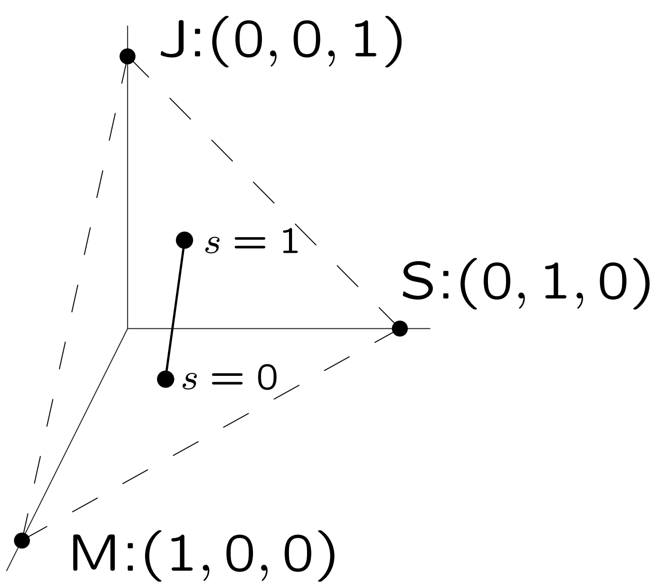
If $s > \frac{2}{3}$ then Juice is the positional winner.

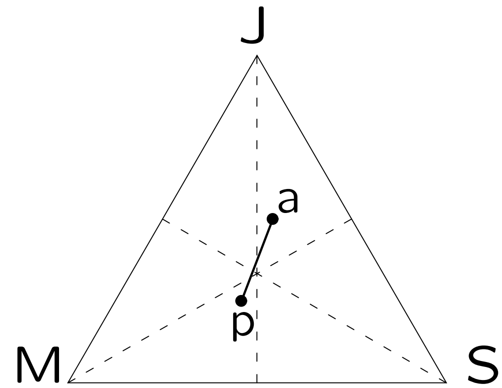
Procedure line

We can make a normalized vector from the weighted positional tallies.

$$\begin{array}{rl} \text{M} & = 5 \\ \text{S} & = 4 + 3s \\ \text{J} & = 3 + 9s \\ \hline \text{Total} & = 12 + 12s \end{array}$$

$$\begin{aligned} \vec{w}_s &= \left\langle \frac{5}{12(1+s)}, \frac{4+3s}{12(1+s)}, \frac{3+9s}{12(1+s)} \right\rangle \\ &= \left(\frac{1}{1+s}\right) \left\langle \frac{5}{12}, \frac{1}{12}, \frac{-6}{12} \right\rangle + \left\langle 0, \frac{3}{12}, \frac{9}{12} \right\rangle \end{aligned}$$

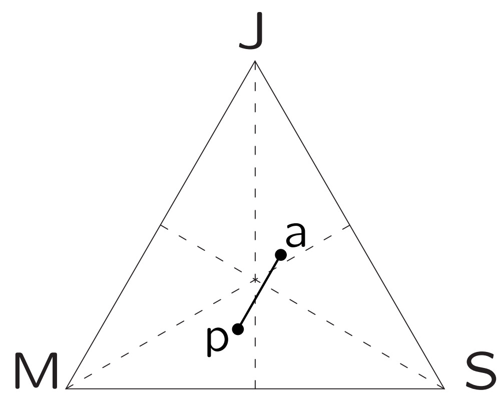




$$\begin{aligned} \mathsf{M} &= 5 \\ \mathsf{S} &= 4 + 4s \\ \mathsf{J} &= 2 + 7s \end{aligned}$$

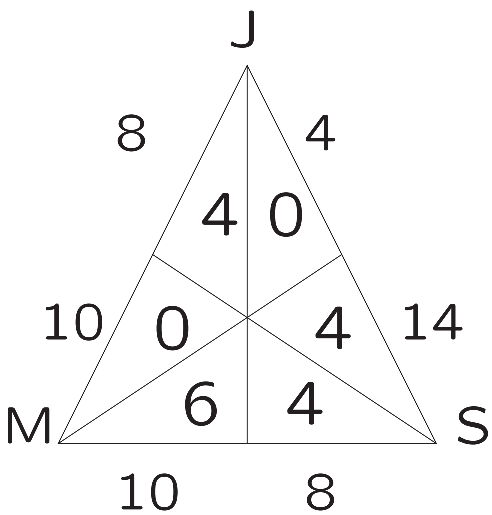
$$\mathsf{p} = \left\langle \frac{5}{11}, \frac{4}{11}, \frac{2}{11} \right\rangle$$

$$\mathsf{a} = \left\langle \frac{5}{22}, \frac{8}{22}, \frac{9}{22} \right\rangle$$



Comparing Methods

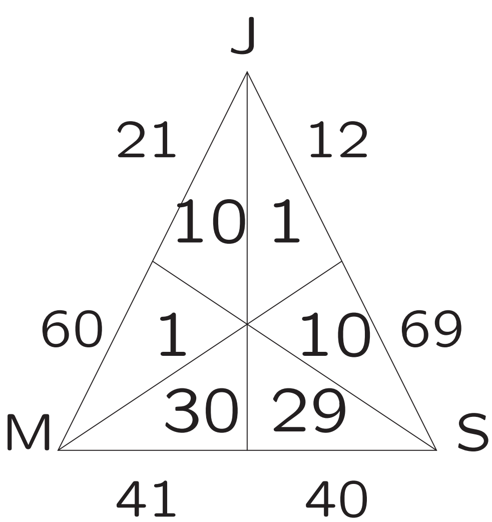
Condorcet vs. da Borda



M is the Condorcet winner

$$
\begin{aligned}
M:&\ 6+8(.5) = 10\\
S:&\ 8+6(.5) = 11\\
J:&\ 4+4(.5) = 6
\end{aligned}
$$

S is the da Borda winner



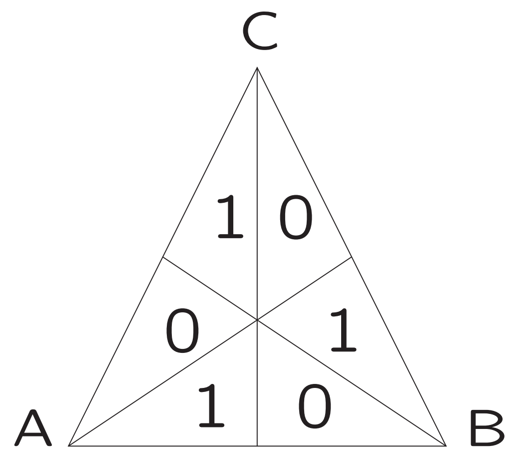
M is the Condorcet winner

$$
\begin{aligned}
M:&\ 31+39(.5) = 50.5\\
S:&\ 39+31(.5) = 54.5\\
J:&\ 11+11(.5) = 16.5
\end{aligned}
$$

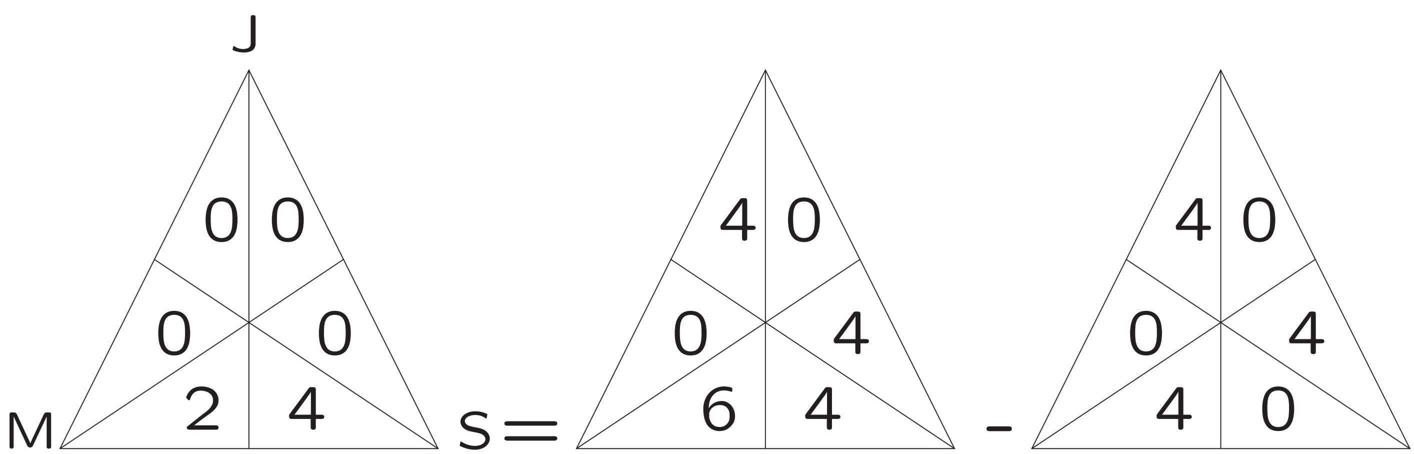
S is the da Borda winner

Cycle Profile

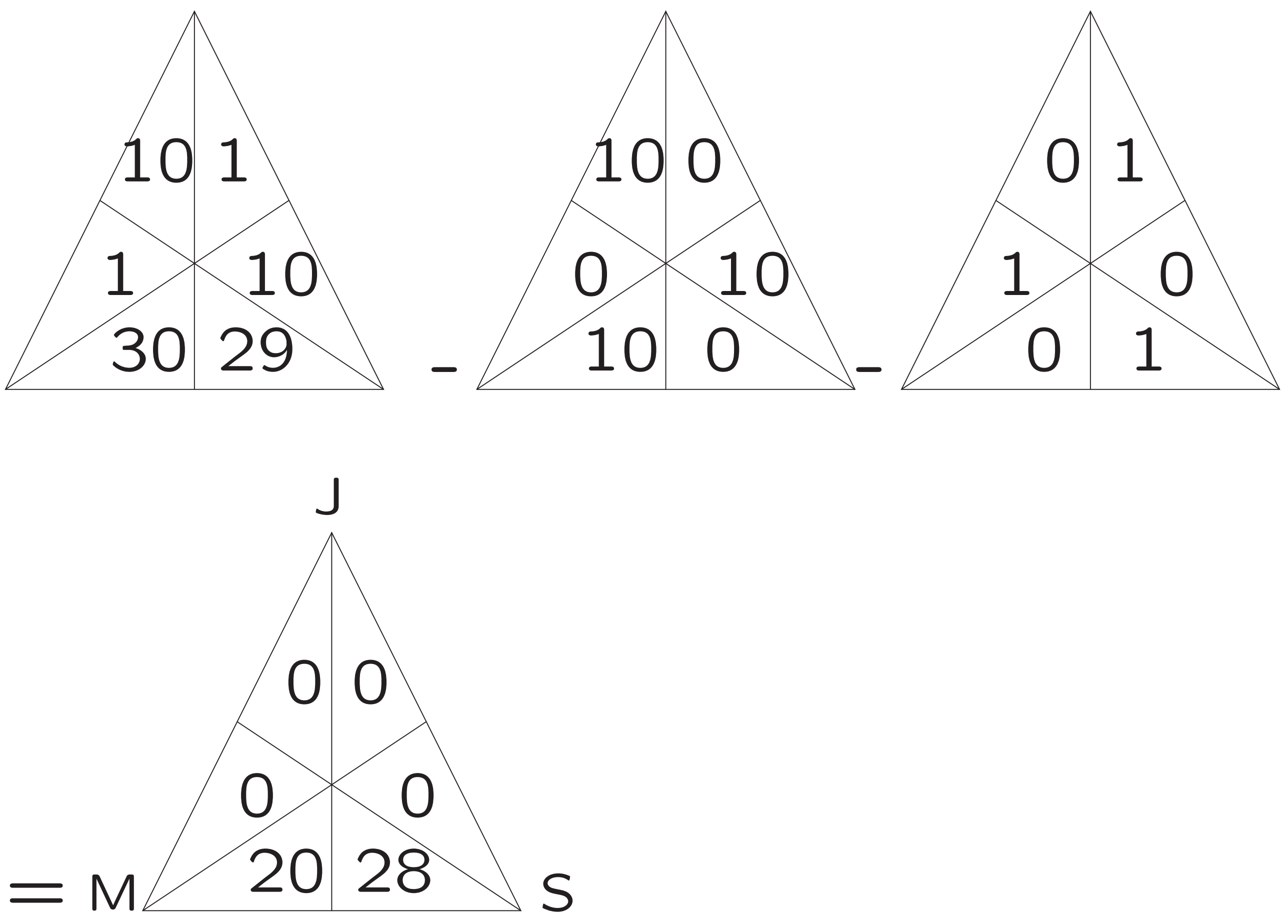
A > B > C: 1
B > C > A: 1
C > A > B: 1



Surely this is a tie!



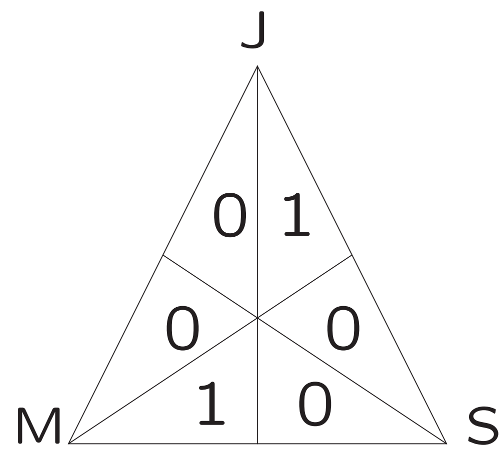
S is the clear winner by any method
(positional, pair-wise)



S is the clear winner by any method (positional, pair-wise)

Reversal Profile

M	$>$	S	$>$	J:	1
J	$>$	S	$>$	M:	1



Three-way Condorcet tie. . . positional?

M = J = 1,   S = $2s$

Borda count $\Rightarrow s = \frac{1}{2} \Rightarrow$ Tie!

Edwin Edwards, David Duke, Buddy Roemer

“Krook vs. Klan”

Profile Subspaces

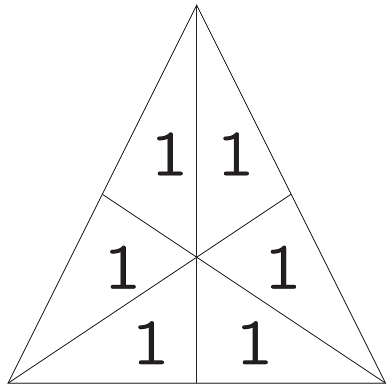
Consider the vector space of all three candidate voter profiles:

$$\mathbf{V} \cong \mathbb{R}^6$$

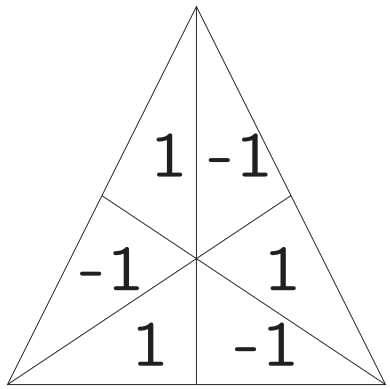
Decompose $\mathbf{V}$ into four orthogonal subspaces:

- $\mathbf{K}$: line along which no outcomes change
- $\mathbf{C}$: line along which pair-wise tallies change, but positional tallies do not
- $\mathbf{R}$: plane along which positional tallies change, but pair-wise tallies do not
- $\mathbf{B}$: plane along which all positional and pair-wise outcomes agree

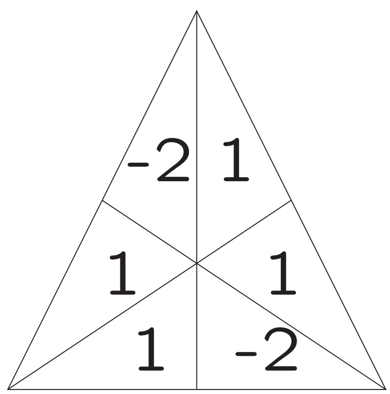
$\mathbf{K} =$ span of



$\mathbf{C} =$ span of



$\mathbf{R} =$ span of



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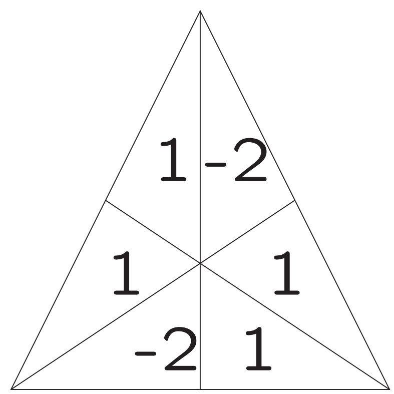
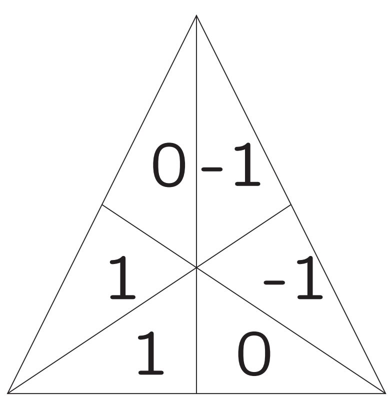


$\mathbf{B} =$ span of



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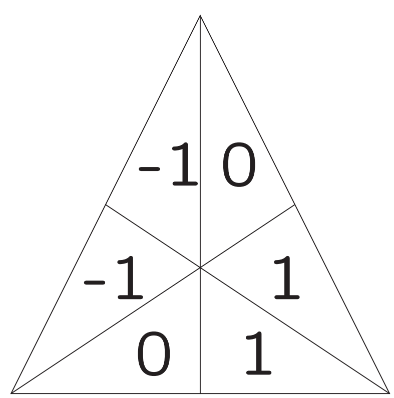
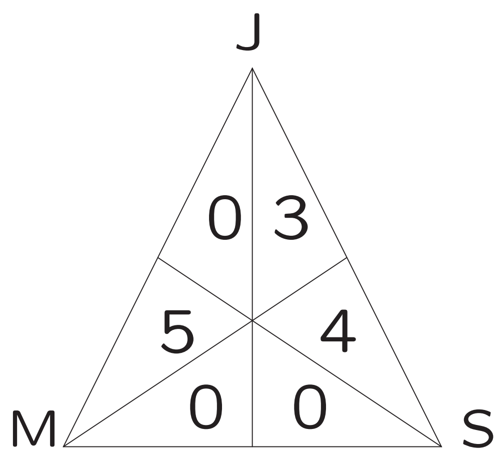


Writing our first voter profile with respect to this basis:

$\vec{P} =$



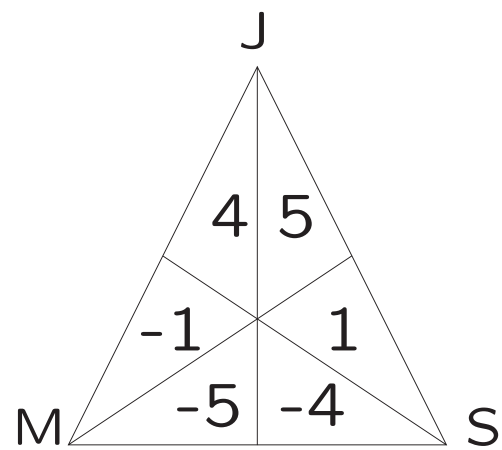
$$= 2\vec{K} - \tfrac{2}{3}\vec{C} + \tfrac{3}{2}\vec{R}_M + \vec{R}_S - \tfrac{5}{6}\vec{B}_M - \tfrac{2}{3}\vec{B}_S$$

To find the coefficent of $\vec{K}$, for example:

$$\langle \vec{K}, \vec{P} \rangle = k\langle \vec{K}, \vec{K} \rangle \;\Rightarrow\; 12 = k6 \;\Rightarrow\; k = 2$$

Considering only the **Basic** terms:

$-\frac{5}{6}\vec{B}_M - \frac{2}{3}\vec{B}_S =$ $\frac{1}{6}$



J is the Condorcet winner:

$\text{J} = 10 > -10 = \text{M}, \;\; \text{J} = 8 > -8 = \text{S}$

J is the positional winner:

$\text{J} = 9, \text{S} = 3, \text{M} = -6$

(independent of s)