2 Quadratic Forms and Diagonalization

Consider the quadratic polynomial in two variables:

$$p(x, y) = 3x^2 + 2xy + 3y^2$$

We can see that the level sets of p are ellipses by making a linear change of coordinates that eliminates the xy term. To see how to do this we must write p as a **quadratic form**.

$$\begin{aligned} p(x, y) &= 3x^2 + xy + yx + 3y^2 \\ &= x(3x + 1y) + y(1x + 3y) \\ &= [x, y] \begin{bmatrix} 3x + 1y \\ 1x + 3y \end{bmatrix} \\ &= [x, y] \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

Recall from your linear algebra class that a **symmetric matrix** has a number of very nice properties. In particular, they always have real eigenvalues, as well as a complete set of eigenvectors. This means that symmetric matrices are always **diagonalizable**. We review the diagonalization process for a 2×2 matrix below.

We first need to find the **eigenvalues** of the matrix.

$$\begin{aligned} \det\left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) &= \det\left(\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} \right) \\ &= (3-\lambda)^2 - 1 \\ &= \lambda^2 - 6\lambda + 8 \\ &= (\lambda - 4)(\lambda - 2) \end{aligned}$$

This is the characteristic polynomial of the matrix, whose zeros are the eigenvalues, 4 and 2. To find the corresponding eigenspaces,

$$\begin{aligned} \mathrm{eig}_4\left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \right) &= \mathrm{null}\left(\begin{bmatrix} 3-4 & 1 \\ 1 & 3-4 \end{bmatrix} \right) \\ &= \mathrm{null}\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \right) \\ &= \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \end{aligned}$$

Similarly,

$$\mathrm{eig}_2\left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$$

With eigenvectors in hand, we may construct a change of basis matrix, P, and its inverse.

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Under this change of basis, the matrix is diagonalized. (Of course you may check this by multiplying out the matrices.)

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Returning to our example quadratic form, we can use the diagonalization...

$$\begin{aligned} p(x,y) &= [x,y] \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= [x,y] \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= [x+y,\ x-y] \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{x+y}{2} \\ \frac{x-y}{2} \end{bmatrix} \end{aligned}$$

..to help us make a linear change of coordinates. One choice is to let

$$s = \frac{x+y}{2},\ t = \frac{x-y}{2} \quad \Leftrightarrow \quad x = s+t,\ y = s-t$$

Then,

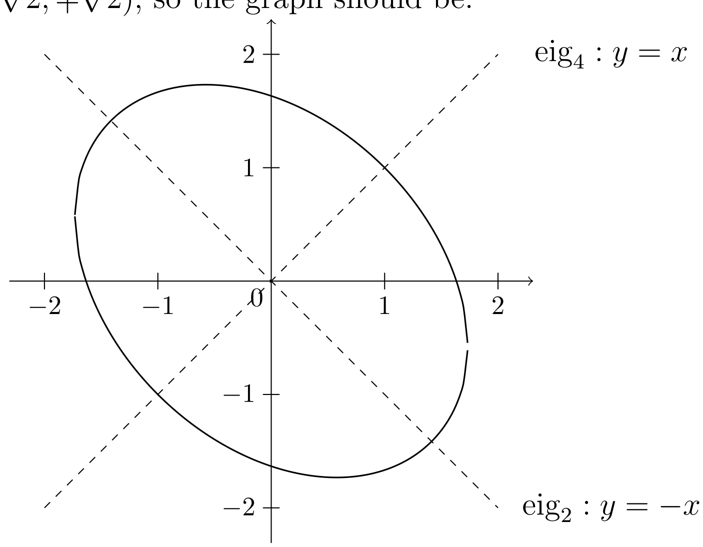
$$p(x,y) = p(s+t, s-t) = [2s,\ 2t] \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = 8s^2 + 4t^2$$

Of course we may check that our linear change of coordinates really does what we say it does by simply substituting into p:

$$p(s+t, s-t) = 3(s+t)^2 + 2(s+t)(s-t) + 3(s-t)^2 = 8s^2 + 4t^2 \checkmark$$

It's easy to see now that the level sets of p are ellipses, but just to beat this example entirely to death, let's plot one of the level sets, say $p(x,y) = 8$.

Note that if $t = 0$, then $8s^2 + 4t^2 = 8 \Rightarrow s = \pm 1$, while if $s = 0$ then $t = \pm\sqrt{2}$. These are the maximum and minimum values of s and t, corresponding to the points $(\pm 1, \pm 1)$ and $(\pm\sqrt{2}, \mp\sqrt{2})$, so the graph should be:



The same process can be applied to any quadratic form.

$$\begin{aligned} p(x,y) &= Ax^2 + Bxy + Cy^2 \\ &= x(Ax + \tfrac{1}{2}By) + y(\tfrac{1}{2}Bx + Cy) \\ &= [x,y]\begin{bmatrix} A & \tfrac{1}{2}B \\ \tfrac{1}{2}B & C \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= [x,y]P\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}P^{-1}\begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

The nature of the resulting (non-trivial) level sets depends entirely on the eigenvalues, λ_1 and λ_2. If they have the **same sign**, then the level sets will be **ellipses**. If they have **opposite signs**, then the level sets will be **hyperbolas**. If one of the eigenvalues **is zero**, then the level sets for a pure quadratic form will correspond to lines. However, if there are lower order terms, the level sets will generally be **parabolas**.

Now, since the determinant of a matrix is just the product of its eigenvalues, we can determine the nature of the level sets without going through the whole diagonalization process.

$$\lambda_1\lambda_2 = \det\left(\begin{bmatrix} A & \tfrac{1}{2}B \\ \tfrac{1}{2}B & C \end{bmatrix}\right) = AC - \frac{1}{4}B^2 = -\frac{1}{4}(B^2 - 4AC)$$

So the sign of the **discriminant** (yes, that thing from Ye Old Quadratic Formula!) tells us the nature of the level sets:

$B^2 - 4AC < 0 \quad \Rightarrow \quad$ ellipses
$B^2 - 4AC = 0 \quad \Rightarrow \quad$ parabolas
$B^2 - 4AC > 0 \quad \Rightarrow \quad$ hyperbolas

The same classification scheme applies to second order, linear, partial differential operators.

$$\begin{aligned} \mathcal{L}u &= A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} \\ &= \tfrac{\partial}{\partial x}\left(A\tfrac{\partial u}{\partial x} + \tfrac{1}{2}B\tfrac{\partial u}{\partial y}\right) + \tfrac{\partial}{\partial y}\left(\tfrac{1}{2}B\tfrac{\partial u}{\partial x} + C\tfrac{\partial u}{\partial y}\right) \\ &= \left[\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}\right]\begin{bmatrix} A & \tfrac{1}{2}B \\ \tfrac{1}{2}B & C \end{bmatrix}\begin{bmatrix} \tfrac{\partial}{\partial x} \\ \tfrac{\partial}{\partial y} \end{bmatrix}u \\ &= \left[\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}\right]P\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}P^{-1}\begin{bmatrix} \tfrac{\partial}{\partial x} \\ \tfrac{\partial}{\partial y} \end{bmatrix}u \end{aligned}$$

As we will see, the behavior of solutions to partial differential equations involving these types of linear operators will vary radically depending on the respective signs of the eigenvalues. Thus, though there isn't really a geometric interpretation that I'm aware of, PDEs involving this sort of linear operator are classified similarly to quadratic forms:

$$\begin{array}{lcl} B^2-4AC<0 & \Rightarrow & \text{elliptic equation} \\ B^2-4AC=0 & \Rightarrow & \text{parabolic equation} \\ B^2-4AC>0 & \Rightarrow & \text{hyperbolic equation} \end{array}$$

Let's consider one final example,

$$\mathcal{L}u = 3\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2}$$

This operator can be diagonalized in exactly the same way as our early quadratic form. That is,

$$\begin{aligned}
\mathcal{L}u &= \tfrac{\partial}{\partial x}\left(3\tfrac{\partial u}{\partial x} + \tfrac{\partial u}{\partial y}\right) + \tfrac{\partial}{\partial y}\left(\tfrac{\partial u}{\partial x} + 3\tfrac{\partial u}{\partial y}\right) \\
&= \left[\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}\right] \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} u \\
&= \left[\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}\right] \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} u \\
&= \left[\tfrac{\partial}{\partial x} + \tfrac{\partial}{\partial y},\ \tfrac{\partial}{\partial x} - \tfrac{\partial}{\partial y}\right] \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) \end{bmatrix} u
\end{aligned}$$

The exact same linear transform we used for the quadratic form can be applied,

$$s = \frac{x+y}{2},\ t = \frac{x-y}{2} \quad \Leftrightarrow \quad x = s+t,\ y = s-t$$

The derivatives transform in a similar—but not precisely the same—way as the variables. We can see this by just applying a derivative and the chain rule to a function u. Thus,

$$\frac{\partial}{\partial s}u = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial u}{\partial x}1 + \frac{\partial u}{\partial y}1 = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)u$$

So, $\frac{\partial}{\partial s} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. A similar calculation shows $\frac{\partial}{\partial t} = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$.

In the new variables:

$$\begin{aligned}
\mathcal{L}u &= \left[\tfrac{\partial}{\partial s},\ \tfrac{\partial}{\partial t}\right] \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\frac{\partial}{\partial s} \\ \frac{1}{2}\frac{\partial}{\partial t} \end{bmatrix} u \\
&= 2\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}
\end{aligned}$$

We can see that this is an **elliptic** operator, and PDEs involving it will be **elliptic equations**. However, if that was all we needed to know, then we could have found that out by simply calculating the discriminant.

$$B^2 - 4AC = 2^2 - 4(3)(3) = -32 < 0 \ \Rightarrow\ \text{elliptic}$$

Practice:

1. Consider the quadratic polynomial: $p(x, y) = 2x^2 + 6xy + 2y^2$.

   (a) Will the level sets of p be ellipses, hyperbolas, parabolas, or lines?

   (b) Write p as a quadratic form with a symmetric matrix.

   (c) Diagonalize the matrix.

   (d) Find a linear change of coordinates that removes the xy "cross term" from p.

   (e) Sketch a non-trivial level set of p.

   (f) Use the same linear transformation you found above to remove the mixed partial "cross term" from the linear operator:

$$\mathcal{L}u = 2\frac{\partial^2 u}{\partial x^2} + 6\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2}$$

2. Consider the quadratic polynomial: $p(x, y) = 9x^2 + 12xy + 4y^2 + 10x - 2y$.

   (a) Will the level sets of p be ellipses, hyperbolas, or parabolas?

   (b) Write p as a quadratic form with a symmetric matrix.

   (c) Diagonalize the matrix.

   (d) Find a linear change of coordinates that removes the xy "cross term" from p.

   (e) Sketch a non-trivial level set of p.

   (f) Use the same linear transformation you found above to remove the mixed partial "cross term" from the linear operator:

$$\mathcal{L}u = 9\frac{\partial^2 u}{\partial x^2} + 12\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} + 10\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y}$$