9 Separating Variables in the Spherical Wave Equation

There are a number of important application of PDEs in spherical coordinates. One can discuss the Heat Equation or LaPlace's Equation in (or outside) of a sphere. We will work through the Wave equation in a sphere, and leave the other (generally simpler) applications for homework exercises.

We want to consider the equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

where $u = u(\rho, \phi, \theta, t)$ and ∇^2 is the LaPlacian operator, which in Cartesian coordinates is simply:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

(As before, c is the wave speed.)

The model for this might be something like seismic waves bouncing around inside the Earth. (However our "simple" Wave equation assumes the medium is homogeneous (thus c is a constant)—which the interior of the Earth certainly is not!)

We'll impose the Dirichlet boundary condition that on the surface of a sphere of radius R, while also implicitly assume that u is bounded in the center of the sphere and along the polar axis. In sum, the boundary conditions will be:

$$u(R, \phi, \theta, t) = 0, \quad u(\rho, \pi, \theta, t) = u(\rho, -\pi, \theta, t), \quad \frac{\partial u}{\partial \theta}(\rho, \pi, \theta, t) = \frac{\partial u}{\partial \theta}(\rho, -\pi, \theta, t)$$

as well as require u to be bounded at $\rho = 0$, $\phi = 0$, and $\phi = \pi$.

To get a unique solution we'll also require the Initial conditions:

$$u(\rho, \phi, \theta, 0) = F(\rho, \phi, \theta) \quad \text{and} \quad \frac{\partial u}{\partial t}(\rho, \phi, \theta, 0) = G(\rho, \phi, \theta)$$

We begin by separating the time dependence from the spacial dependence: $u = h(t)w(\rho, \phi, \theta)$.

$$h''w = c^2 h \nabla^2 w \;\Rightarrow\; \frac{h''}{c^2 h} = \frac{\nabla^2 w}{w} = -\lambda$$

Thus we have:

$$\boxed{h'' + c^2 \lambda h = 0} \tag{8}$$

and

$$\nabla^2 w = -\lambda w$$

The time-dependent equation will be straight forward once we know the eigenvalues λ. To determine that, however, we have to treat the quite intimidating spacial equation. In spherical coordinates (where the LaPlacian much more complicated):

$$\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left[\rho^2 \frac{\partial w}{\partial \rho}\right] + \frac{1}{\rho^2 \sin\phi}\frac{\partial}{\partial \phi}\left[\sin\phi \frac{\partial w}{\partial \phi}\right] + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 w}{\partial \theta^2} + \lambda w = 0$$

Continuing the separation by substituting: $w = f(\rho)g(\phi)q(\theta)$,

$$\frac{gq}{\rho^2}\frac{d}{d\rho}\left[\rho^2\frac{df}{d\rho}\right] + \frac{fq}{\rho^2\sin\phi}\frac{d}{d\phi}\left[\sin\phi\frac{dg}{d\phi}\right] + \frac{fg}{\rho^2\sin^2\phi}\frac{d^2q}{d\theta^2} + \lambda fgq = 0$$

After multiplying through by $\left(\frac{\rho^2\sin^2\phi}{fgq}\right)$ we can separate off the θ dependent terms,

$$\frac{\sin^2\phi}{f}\frac{d}{d\rho}\left[\rho^2\frac{df}{d\rho}\right] + \frac{\sin\phi}{g}\frac{d}{d\phi}\left[\sin\phi\frac{dg}{d\phi}\right] + \lambda\rho^2\sin^2\phi = -\frac{1}{q}\frac{d^2q}{d\theta^2} = \mu$$

This gives a familiar eigenvalue problem for the θ dependence:

$$\boxed{\frac{d^2q}{d\theta^2} + \mu q = 0}, \quad q(\pi) = q(-\pi), \quad q'(\pi) = q'(-\pi) \tag{9}$$

As we saw before $\mu = m^2$ for some non-negative integer m, while the eigenfunctions are $\cos(m\theta)$ and $\sin(m\theta)$.

So that's two variables down with two to go... Unfortunately it starts to get pretty hairy from this point on. Looking at the ρ and ϕ equation with $\mu = m^2$,

$$\frac{\sin^2\phi}{f}\frac{d}{d\rho}\left[\rho^2\frac{df}{d\rho}\right] + \frac{\sin\phi}{g}\frac{d}{d\phi}\left[\sin\phi\frac{dg}{d\phi}\right] + \lambda\rho^2\sin^2\phi = m^2$$

Dividing through by $\sin^2\phi$, we can separate the last two variables:

$$\frac{1}{f}\frac{d}{d\rho}\left[\rho^2\frac{df}{d\rho}\right] + \lambda\rho^2 = -\frac{1}{\sin\phi\, g}\frac{d}{d\phi}\left[\sin\phi\frac{dg}{d\phi}\right] + \frac{m^2}{\sin^2\phi} = \xi$$

This gives us two unfamiliar Sturm-Liouville equations:

$$\boxed{\frac{d}{d\rho}\left[\rho^2\frac{df}{d\rho}\right] - \xi f = -\lambda\rho^2 f} \tag{10}$$

$(p = \rho^2,\ q = -\xi,\ \sigma = \rho^2)$

$$\boxed{\frac{d}{d\phi}\left[\sin\phi\frac{dg}{d\phi}\right] - \left(\frac{m^2}{\sin\phi}\right) g = -\xi\sin\phi\, g} \tag{11}$$

$(p = \sin\phi,\ q = -\left(\frac{m^2}{\sin\phi}\right),\ \sigma = \sin\phi)$

Our approach will be to analyze equation (11) first, determine the eigenvalues ξ, then substitute those eigenvalues into equation (10). The eigenvalues of equation (10) are our first separation constant λ, which we will use to solve the time dependent equation (8).