4 Fredholm Alternative

Consider the Neumann boundary value problem, similar to the type we have been solving:

$$ux = f(x), \quad \frac{du}{dx}(0) = 0 = \frac{du}{dx}(L) \tag{3}$$

It is not immediately obvious that, in general, this problem has **no solution**. In fact, we'll show that this problem has a solution if and only if:

$$\int_0^L f(x)\,dx = 0$$

The first way to understand this is to just try to solve it by integration.

$$u(x) = \int_0^x \int_0^{\bar{x}} f(t)\,dt\,d\bar{x} + c_1 x + c_2$$

It is clear that for this definition, $u''(x) = f(x)$. The boundary conditions, however, become problematic.

$$\frac{du}{dx}(x) = \int_0^x f(t)\,dt + c_1$$

$$\frac{du}{dx}(0) = \int_0^0 f(t)\,dt + c_1 = c_1, \text{ so } \frac{du}{dx}(0) = 0 \;\Rightarrow\; c_1 = 0$$

Thus,

$$\frac{du}{dx}(L) = \int_0^L f(t)\,dt$$

Once c_1 is fixed at zero by the left endpoint, the only way the right endpoint condition can be satisfied is if $\int_0^L f(t)\,dx = 0$.

We can understand this more intuitively by remembering that this problem mirrors the problem of finding a steady-state heat distribution on a bar with internal heat sources/sinks and **insulated ends**. There would only be such a steady state solution if the **net heat flow** into the bar is zero. That is,

$$\int_0^L Q(x)\,dx = 0 \;\Rightarrow\; \int_0^L f(x)\,dx = 0$$

Finally we can see that there is no Green's Function for this problem since

$$\frac{d^2G}{dx^2}(x, x_0) = \delta(x - x_0), \quad \frac{dG}{dx}(0, x_0) = 0 = \frac{dG}{dx}(L, x_0)$$

would have a solution of the form:

$$G(x, x_0) = \begin{cases} ax + b & \text{if } \quad x < x_0 \\ cx + d & \text{if } \quad x > x_0 \end{cases}$$

But the boundary conditions would require that $a = c = 0$. The resulting G would have the derivative,

$$G'(x, x_0) = (d - b)\delta(x - x_0)$$

Then

$$G''(x, x_0) = (d - b)\delta'(x - x_0) \neq \delta(x - x_0)$$

for any value of b or d. So there is no such function, G.

This is consistent with the other statements since

$$\int_0^L \delta(x - x_0)\, dx = 1 \neq 0$$

Which is to say there is a net flow **out** of the bar from the isolated "cold point" at x_0, so there will never be a steady-state.

We can generalize this idea as to when a particular non-homogeneous BVP has a solution with a peculiarly named theorem.

Theorem 4.1: (Fredholm Alternative)
Let $\mathcal{L}$ be a Sturm-Liouville differential operator, and consider solutions to $\mathcal{L}[u] = f(x)$ with boundary conditions such that $\mathcal{L}$ is self-adjoint.

1. If the only solution to $\mathcal{L}[u] = 0$ satisfying the boundary conditions is $u = 0$, (that is, if $\lambda = 0$ is **not** an eigenvalue of $\mathcal{L}$),
   then there is a **unique** solution to the BVP.

2. If there are non-trivial solutions to $\mathcal{L}[u] = 0$ satisfying the boundary conditions, (this if $\lambda = 0$ **is** an eigenvalue of $\mathcal{L}$),
   then there are either **no solutions** or **infinitely many solutions**, and **there is no Green's Function**.

Example 4.1: Show that the BVP described in 3 has either no solution or infinitely many solutions.

It's easily seen that $\phi_0(x) = 1$ is an eigenfunction with eigenvalue zero to Equation 3.

$$\frac{d^2\phi_0}{dx^2} = 0, \quad \frac{d\phi_0}{dx}(x) = 0 \;\Rightarrow\; \frac{d\phi_0}{dx}(0) = 0 = \frac{d\phi_0}{dx}(L)$$

Thus, by Theorem 4.1, since $\lambda = 0$ is an eigenvalue, there are either no solutions or infinitely many solutions to Equation 3.

Proof sketch of Theorem 4.1:

Consider the potential solution u written as a generalized Fourier series in the eigenfunctions of $\mathcal{L}$, ϕ_n.

$$u(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

Now we multiply $\mathcal{L}[u] = f(x)$ by $\phi_N \sigma$ and integrate.

$$\begin{aligned}
\int_0^L \phi_N(x) f(x)\, dx &= \int_0^L \phi_N(x) \mathcal{L}[u]\, dx \\
&= \int_0^L \mathcal{L}[\phi_N] u(x)\, dx \\
&= \int_0^L -\lambda_N \phi_N(x) u(x) \sigma(x)\, dx \\
&= -\lambda_N a_N \int_0^L (\phi_N(x))^2 \sigma(x)\, dx
\end{aligned}$$

If $\lambda_n \neq 0$ for all n, then we have a unique definition of all the a_ns.

$$\Rightarrow \quad a_n = \frac{\int_0^L \phi_n(x) f(x)\, dx}{-\lambda_N \int_0^L (\phi_n(x))^2 \sigma(x)\, dx}$$

And so u exists and is unique.

If, on the other hand, one of the eigenvalues is zero (say $\lambda_m = 0$), then for u to be a solution it must be that:

$$\int_0^L \phi_m(x) f(x)\, dx = 0$$

Thus if $\int_0^L \phi_m(x) f(x)\, dx \neq 0$ we know that there is no solution.

Further if $f(x) = \delta(x - x_0)$, then $u(x) = G(x, x_0)$ and

$$\int_0^L \phi_m(x_0) \delta(x_0 - x)\, dx_0 = \phi_m(x_0)$$

Then for G to exist we would need $\phi_m(x_0) = 0$ for every x_0. That would imply that ϕ_m is identically zero, which is not a valid eigenfunction.

Finally, if $\int_0^L \phi_m(x) f(x)\, dx = 0$ then

$$\tilde{u}(x) = \sum_{n \neq m} a_n \phi_n(x)$$

is a solution, because we have a formula for all the other a_ns. However then,

$$u_c(x) = c\phi_m(x) + \tilde{u}(x)$$

is **also** a solution since:

$$\mathcal{L}[u_c] = c\mathcal{L}[\phi_m] + \mathcal{L}[\tilde{u}] = 0 + f(x) = f(x)$$

So we now have infinitely many solutions u_c since c can be any real number.