Contents

1 Statements and Sets

1.1 What is Discrete Math? What Is This Course About?

In this chapter and the next we will study sets, which lie at the very foundation of mathematics. We will say that a **set** is a collection of objects; we will call the objects **elements** of the set. Of course with a little thought one finds this definition to be rather circular, but I will assume that we all have some idea of what a set is. The elements of the sets we will be interested in will *usually* be numbers, but they can be other objects as well, like letters of the alphabet, geometric shapes, people, and so on.

I will use sets to try to explain what the term *discrete* means, from a mathematical point of view. Let us consider two sets, which we will call A and B. Set A is all of the numbers between 1 and 5, including both 1 and 5; technically speaking, I mean all the *real* numbers between 1 and 5, inclusive. So this includes numbers like 4, $2\frac{1}{3}$ and 1.875, but it also includes some oddballs, like $\sqrt{3}$, π and e. (Recall that set A can be concisely described using the **interval notation** $[1, 5]$.) Set B will be the numbers 2, 4, 6, 8, ..., the positive even numbers. (The numbers $0, -2, -4, ...$ are considered to be even also.)

Sets A and B are the same in one way: both sets are **infinite sets**, meaning they contain infinitely many elements. However, the two sets are different in the following way. The elements in set B can be "counted off" in such a way that we know that a given number in the set will be "counted" at some point. This is not possible with set A; there is no way to order the numbers in that set so that they can be counted off. A set like set B is called a **countable set**, and set A is an **uncountable set**. A major part of this course will focus on sets that are either finite or countable, and the mathematics associated with them - discrete mathematics - but we will also do some things with uncountable sets as well.

NOTE: Even though both A and B are infinite sets, you will eventually see that set A contains more elements than set B!

In this course we will first learn how to state mathematical facts precisely and explore the concept of a set. After that we come to the other major concept of mathematics besides sets, functions. Once we understand sets, functions we will go about seeing how to prove that mathematical facts are true. Then we will move on to various topics that use those ideas.

1.2 Simple and Conditional Statements

Performance Criteria:

1. (a) Determine whether a collection of words and/or symbols is a statement.
   (b) Determine whether a simple statement is unconditionally true. If it is not, give specific examples of situations for which it is true and it is not true.
   (c) Re-write a conditionally true statement as a true conditional statement.
   (d) Write true conditional statements.

For us a **statement** is a collection of words and/or symbols that has meaning to us and can, or could, be determined to be true or false. The "could" refers to situations in which one or more things are left unspecified, but the truth of the statement can be determined when those things *are* specified. Here are some examples of statements:

$$2+3=7, \qquad -7 \text{ is an integer}, \qquad x+y=7, \qquad x+3>5$$

The truth of the first two statements can be determined as they are, whereas the truth of the third statement depends on the values of x and y and the truth of the last statement depends on the value of x. Some examples of collections of symbols that are NOT statements are

$$x-y, \qquad x^2+3$$

None of the statements given above can be broken apart into smaller pieces that are themselves statements. We will call statements like this **simple statements** - this is not a commonly used term, but we will use it in this course.

NOTE: Throughout these notes, and in other texts, statements will be made about unspecified numbers, like x and y, or unspecified sets, like A and B. In these situations

- when the same letter occurs more than once in the same scenario, *it represents the same number or same set in all cases,*
- when two different letters are used, this indicates that the two things represented are possibly different from each other, but *they are allowed to be the same as well.*

1. Determine whether each of the following collections of symbols is a statement (S) or not a statement (NS).

   (a) $x^2+3x+2=0$ (b) $x+5>x$ (c) 2^x+3
   (d) $xy=x+y$ (e) $x-y$ (f) $x+2x=3x$

A statement is **unconditionally true** if it is true under all circumstances. For example, the statement $x+y=y+x$ is true for all real numbers x and y, so this statement is unconditionally true. On the other hand, the statement $x+3>5$ is **conditionally true**. It is true only when $x>2$.

2. Go back to each of the parts of Exercise 1 that ARE statements. Determine whether each of them is unconditionally true.

3. For each of the statements of Exercise 1 that are *conditionally* true, give a *specific* example of when it is true, and one when it it is not.

From this point on, if you are asked whether a statement is true, we mean UNCONDITIONALLY TRUE. The very essence of being a statement is then the fact that a statement can be determined to be true or false, so to each statement we can attach either the word "true" or the word "false". This is called the **truth value** of the statement. *Exercise 2 shows that it is sometimes possible to determine the truth value of a statement containing unspecified objects.*

There are many mathematical statements that most of us take to be true, like $1x = x$ and $x + x = 2x$. Such statements come in two forms:

- **Axioms:** These are statements that we take to be true, but which cannot be proven to be true. Two examples are $1x = x$ and $xz + yz = (x + y)z$.
- **Theorems:** These are statements that can actually be proven to be true, based on previously accepted facts and/or previously proven theorems. An example is the statement $x + x = 2x$.

Let's prove that $x+x = 2x$, based on the axioms given above and the fact that $1+1 = 2$:

$$x + x = 1x + 1x = (1 + 1)x = 2x.$$

We don't have time to work through discovering and proving all the theorems of algebra from the basic axioms, so we will not make much of the above distinction.

Later we will study how to prove some basic mathematical facts. For the time being we will generally not prove things that are "obvious" to be true. However, from this point on we are obligated to provide a **counterexample** for any statement we run across that is not unconditionally true, to show that it in fact is not unconditionally true. An example would be that if $x = 1$ the statement $x + 3 > 5$ is not true. *Note that a counterexample is not a long, drawn out explanation, it is simply one case that shows something is not unconditionally true.*

4. Provide counterexamples to show that each of the following statements is not unconditionally true.

   (a) $-x \leq x$ (b) $\dfrac{x^2}{x} = x$ (c) $\sqrt{x^2} = x$

As we proceed, it will at times be convenient to discuss simple statements in an abstract sense, without specifying actual statements. We will use the upper case letters P, Q, and occasionally R to denote general simple statements. Now consider the statement $x+3 > 5$, which we noted before is not unconditionally true. We will sometimes wish to create a more complicated statement containing this statement that *IS* unconditionally true; we use a statement of the form "If P, then Q" to do this. Such a statement is called a **conditional statement**. This means that whenever P is true, Q will "automatically" be true as well. In the case of $x + 3 > 5$, we could write

$$\text{if } x = 4, \text{ then } x+3 > 5 \qquad \text{OR} \qquad \text{if } x > 2, \text{ then } x+3 > 5.$$

We usually wish to make the statement P be the "broadest" conditions under which Q must be true, so the second statement here is preferred. *From this point on, when asked to write a conditional statement you should always follow this preference.*

5. Write a true conditional statement with Q being the statement from Exercise 1(a).

6. Write true conditional statements with Q being each of the statements from Exercise 4.

7. Write as many conditional statements about even and odd numbers, like "If m and n are even, then $m+n$ is even", as you can, *considering only the operations of addition and multiplication.* There are five more such statements.

A given conditional statement can be written a variety of ways. For example the statement "If m and n are even, then $m+n$ is even" can also be written

- ◇ If m and n are even, $m+n$ is even.
- ◇ $m+n$ is even if m and n are even.
- ◇ m and n even implies $m+n$ is even.

As you probably somehow understand already, the statement "If P, then Q" is only true if the truth of P "causes" or implies the truth of Q. *A conditional statement is false if we can find a situation where P is true AND Q is not.*

8. Determine whether each of the conditional statements is true or false. For each that is false, provide an example for which P is true and Q is not.

   (a) If m is divisible by 3, them m is divisible by 6. (Here we mean "evenly" divisible - we will define the term "divisible" more precisely later.)
   (b) If m is divisible by 6, them m is divisible by 3.
   (c) If m is even, then $m+1$ is odd.
   (d) If $x < y$, then $-3x < -3y$.

1.3 Sets

Performance Criteria:

1. (e) Determine whether a number is a natural number, integer and/or real number. Recognize or be able to give the correct notation for these sets of numbers.
   (f) Describe a set (i) in words, (ii) by listing (when appropriate), (iii) using set builder notation, or (iv) using interval notation (when appropriate).
   (g) Determine whether an object is an element of a given set.
   (h) Use correct notation for sets and "is an element of".
   (i) Describe a set of real numbers graphically.
   (j) Determine whether a set is continuous or discrete, finite or infinite.

In the Section 1.1 we "defined" a **set** to be a collection of objects. Because we cannot really define a set without using other terms whose meanings are unclear, a set is what is called an *undefined term.* You know from the discussion in the previous section that we call the objects in a set **elements**. Here are two important conditions that sets *MUST* meet:

(1) No element can occur more than once in a set.

(2) There is no order to elements in a set. Thus the sets $\{1, 2, 3\}$ and $\{3, 2, 1\}$ are the same set. (However, it is common practice when listing elements of a set to list them from smallest to largest if they are numbers.)

Suppose that we have a set in mind. We have two immediate concerns. First, we need to determine a symbol used to represent the set (its name). Then we must give a written description of the set that is clear, concise and unambiguous, so that a reader can determine exactly all the elements of the set. *We will usually represent a set with a single upper case (capital) letter.* There may be occasional exceptions to this, but they will be rare. A few sets that we will refer to regularly have permanently designated symbols. Perhaps the three most important sets in mathematics are the natural numbers, the integers and the real numbers. The **natural numbers** are the numbers 1,2,3,4,..., and we will denote them by $\mathbb{N}$. The **integers** are the natural numbers and their negatives, and zero. That is, they are the numbers $..., -3, -2, -1, 0, 1, 2, 3, ...$ We denote the integers by $\mathbb{Z}$.

The **real numbers** are difficult to describe without getting painfully technical, but basically they are all of the numbers you ever came across before you tried to take the square root of a negative number. They include all of the natural numbers, fractions, decimals, even-numbered roots of non-negative numbers, all odd-numbered roots, π, e, and of course negatives of any of these. The real numbers will be denoted by $\mathbb{R}$.

NOTE: A major emphasis of this course will be the clear written and verbal communication of ideas, and it is extremely important that correct notation and terminology are used by all of us. Start now in making an effort to use correct notation!

As stated before, the objects in a set are called elements of the set. We use the symbol $\in$ to indicate that something is an element of a particular set. For example, $5 \in \mathbb{N}$ means "five is an element of the natural numbers". (Often we will say instead "five is *in* $\mathbb{N}$".) Similarly, we could write $-\sqrt{3} \in \mathbb{R}$. At times we will also wish to indicate symbolically that something is *NOT* and element of a set. For this we use the symbol $\notin$; for example, $-7 \notin \mathbb{N}$.

Describing Sets

We will have a variety of ways of describing a set in writing. Let us consider two examples. Suppose that set A is all the natural numbers between 3 and 7, including both of those numbers, and set B is all the real numbers between 3 and 7, including 3 but NOT including 7. The first way we can describe a set is with a **written verbal description**, which is what I have just done for A and B. If a set is finite, we can describe it by simply **listing** the elements of the set. For example, we write

$$A = \{3, 4, 5, 6, 7\}.$$

The symbols $\{\ \}$ indicate that we are dealing with a set. It is also acceptable to describe infinite sets by listing as well, *as long as they are discrete sets.* For example, the odd natural numbers could be denoted by O and described by

$$O = \{1, 3, 5, 7, ...\}.$$

One problem with listing to describe an infinite set is that there is no reason to expect that a pattern shown will continue to hold. However, *we will use listing to describe an infinite set when we think it clearly describes the set in question and, in those cases, we will assume that any pattern we see in the list continues.*

A "fancier" (but not necessarily better!) way to describe a set is what is referred to as **set builder notation**. In one form this consists of giving a statement telling the set out of which we will select the elements of our set, then giving another statement that prescribes the criteria used for selecting the elements of that set. We separate these two things with a vertical bar, taken to mean "such that". For our set A we have

$$A = \{x \in \mathbb{N} \mid 3 \le x \le 7\}.$$

This is read as "A is the set of natural numbers x such that x is greater than or equal to three and less than or equal to seven." *Note that when using either listing or set builder notation to describe a set, the symbols* $\{$ *and* $\}$ *are used to indicate that we are dealing with a set.*

1. (a) Describe the set B defined above, using set builder notation.

   (b) Recall from your algebra courses that the set of all real numbers between two given numbers, possibly including either or both of those numbers, can be described using something called **interval notation**. This consists of

   - listing the two numbers, smallest first then largest, separated by a comma,

- enclosing the pair of numbers with square brackets [ ] and/or parentheses ( ), using [or] if the number is to be included, (or) if it is not. Of course, as you probably already know, we can also have the combinations [ ) and ( ].

Give the interval notation for set B.

2. Describe the natural numbers greater than 7 by

   (a) listing, (b) set builder notation.

3. Give a verbal description and set builder notation for the interval $(-\infty, 2]$.

NOTES:

(1) Interval notation is only appropriate for continuous sets of real numbers, and listing is only appropriate for finite or countable sets.

(2) An interval of the real numbers with endpoints a and b that are both finite numbers is called a **finite interval**; the interval $B = [3, 7)$ is such an interval. Any other interval, like $(-\infty, 2]$, is an **infinite interval**. *Note that a finite interval is NOT a finite set.*

(3) When denoting an interval for which the two endpoints are numbers, *the smaller number is always listed first.* If the interval involves either (or both) of the symbols $-\infty$ or ∞, $-\infty$ is always written first and ∞ is always written second.

(4) The symbols $-\infty$ and ∞ do not represent numbers, so they cannot be included in sets of numbers. Therefore, $-\infty$ is *always* preceded by (and ∞ is always followed by).

When describing continuous sets of real numbers, interval notation is almost always more efficient than set builder notation. One exception is for a set like all real numbers except 3. Using interval notation, this set is $(-\infty, 3) \cup (3, \infty)$. Set builder notation for the same set is $\{x \in \mathbb{R} \mid x \neq 3\}$.

Sometimes we use set builder notation in a manner different than that described previously. Suppose that we wish to describe the set of even numbers. (We will always take the even numbers to be both positive and negative, along with zero. Similarly for the odds, but zero is not odd.) In this case, we first give a general form of an even integer, involving an unknown quantity. We then give a *statement* of the condition or conditions that the unknown quantity must meet. Using set builder notation, the set of even integers is then given by

$$\{2n \mid n \in \mathbb{Z}\}.$$

If we wanted just the positive even numbers we would write

$$\{2n \mid n \in \mathbb{N}\}.$$

4. Describe each of the following sets by listing.

   (a) $\left\{ \dfrac{n}{n+1} \;\middle|\; n \in \mathbb{N} \right\}$ (b) $\{4n + 1 \mid n \in \mathbb{Z}\}$

5. Describe each of the following sets using set builder notation in the way just described.

   (a) $\{1, 4, 9, 16, 25, ...\}$
   (b) The set of all (including negative) multiples of seven.
   (c) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...\}$
   (d) $\{..., \frac{1}{27}, \frac{1}{9}, \frac{1}{3}, 1, 3, 9, 27, ...\}$
   (e) The set of all odd numbers.

Note that it does not make sense to describe a set by $\{n \in \mathbb{Z} \mid 2n\}$, since this translates verbally to "the set of all numbers n in the integers such that two times n". The notation $\{2n \mid n \in \mathbb{Z}\}$ *IS* correct, because it translates as "the set of all numbers two times n such that n is an integer". *In general, the expression after the "such that" bar MUST be a statement, whereas the expression before the bar need not be.*

6. A final way that we will sometimes represent a set is **graphically**. You should have already done this for intervals in some earlier classes. This exercise will remind you of that, if you have forgotten. Before you head into it, let me tell you that what you have always called "the number line" will be referred to now as the **real line**, since it represents the set of all real numbers.

   (a) The interval $[-3, 1)$ is a subset of $\mathbb{R}$. To represent it graphically, we draw the real line and put a dot at -3 and an open circle at 1. We then shade the part of the number line between the dot and the circle. Do this.
   (b) Repeat part (a) for the interval $[-1, \infty)$.

7. Of course discrete sets can be represented graphically as well. Represent the set $\{1, 2, 3\}$ graphically.

8. Go back to Section 1.1 and read about continuous and discrete sets again. Then classify each of the following sets as continuous (C) or discrete (D), *AND* finite (F) or infinite (I). *Each set will have TWO letters describing it.*

   (a) $[-5, 1)$ (b) $\{1, 3, 5, 7, ...\}$ (c) $\{x \in \mathbb{R} \mid x > 8\}$ (c) $\{3, 6, 9\}$

1.4 Subsets and Cardinality

Performance Criteria:

1. (k) Determine whether a set is a subset of a given set; if it is not, tell why.
   (l) Use correct notation for "is a subset of".
   (m) Give the largest and smallest subset of a set.
   (n) Tell what is meant by the universal set and the empty set, and recognize or give the notation for the empty set.
   (o) Determine the cardinality of a given set.

We are very interested in situations where one set is "contained" in another. Suppose that the conditional statement "If $x \in A$, then $x \in B$ is true. Then we say that A is a **subset** of B. Note that this is equivalent to saying that every element of A is also an element of a set B. Symbolically we denote A is a subset of B by $A \subseteq B$. Another way of stating this in words is that "A is contained in B". *A set A is NOT a subset of a set B if there is an element in A (one is enough!) that is not in B.* When we want to indicate symbolically that a set A is *NOT* a subset of a set B we write $A \nsubseteq B$. Thus if $A = \{2, 4, 6, 8\}$ and $B = \{4, 5, 6, 7, 8, \}$, we have $A \nsubseteq B$ because $2 \in A$ and $2 \notin B$.

1. In each case, determine whether C is a subset of D. If C is not a subset of D, write a brief statement (like the one just given above for A and B) explaining why it is not.

   (a) $C = \{1, 3\}, \quad D = \{1, 2, 3, 4\}$
   (b) $C = \{1, 3, 5\}, \quad D = \{1, 2, 3, 4\}$
   (c) $C = \{1, 3\}, \quad D = [1, 3]$
   (d) $C = \{1, 3\}, \quad D = (1, 3)$

   In (d) we determine from context that $(1, 3)$ is an interval, not an ordered pair!

   (e) $C = \{1, 2, 3\}, \quad D = \{1, 2, 3\}$
   (f) $C = [1, 3], \quad D = \{1, 2, 3\}$

2. Can a set be a subset of itself? If so, must it be?

3. There is no reason that a set cannot contain just one element; such sets are sometimes referred to as "singletons", but you don't need to know this terminology. Two of the statements

   $$5 \in A\ , \qquad\qquad 5 \subseteq A\ , \qquad\qquad \{5\} \subseteq A$$

   use proper notation and one does not. Which does not, and why? This exercise is meant to point out the difference between an element of a set and a singleton set as a subset of a set.

4. Suppose that A is *any* set. What is the largest possible subset of A?

Exercises 1(e), 2 and 4 illustrate that *every set is a subset of itself.* If A is a subset of B but is not equal to B itself, we say that A is a **proper subset** of B. We are not usually interested in making a big deal out of whether or not a subset is proper.

NOTE: *We define two sets to be equal if each is a subset of the other.* This seems fairly trivial, but we will make use of this definition later.

The Universal Set and the Empty Set

Whenever we are working with sets, there is usually some large set "in the background" that the sets under consideration are subsets of. For example, the background set for the set $A = \{3, 4, 5, 6, 7\}$ that we have been working with is $\mathbb{N}$ and the background set for $B = [3, 7)$ is $\mathbb{R}$. The relevant background set is called the **universal set**. If there is no designated universal set, we just denote the universal set by U and assume it is some set containing all of the sets under consideration. The universal set is not something that we will always need to be concerned with, but it will arise occasionally.

We also have a need for a set that has no elements in it, which we call **empty set**, or **null set**. The need for the empty set is analogous to the fact that when counting things, we need the number zero to indicate when there is nothing to count. The symbol for the empty set is $\emptyset$. (From here on, you need to take care to use the symbol 0 for zero and $\emptyset$ for the empty set!) Note that the symbol $\emptyset$ is taken to include the set brackets $\{$ and $\}$, so $\{\emptyset\}$ *is incorrect notation for the empty set.* (This notation does have meaning in certain situations, but it is rarely needed.) Because of a rule of logic that I don't want to go into here, *the empty set is a subset of every set.*

5. Give all subsets of the set $\{a, b\}$.

Cardinality of a Set

The **cardinality** of a set A is the number of elements in the set. For example, the cardinality of $\{2, 4, 6, 8, 10\}$ is five, and the cardinality of the empty set is zero. We use the notation $|A|$ for the cardinality of set A.

For now we will say that a set like the natural numbers or real numbers has infinite cardinality. Later we will see that there are in fact different sizes of infinity, and the natural numbers and real numbers will have different cardinalities!

6. Finish the conditional statement "If $A \subseteq B$, then ..." with a statement about the cardinalities of A and B. Use correct notation and remember that $|A|$ and $|B|$ are *numbers*, so whatever you say about them must make sense for numbers.

1.5 Binary Operations, Conjunctions and Disjunctions

Performance Criteria:

1. (p) Give some examples of binary operations on numbers, and on other objects.
   (q) Determine the result of a binary operation on two numbers.
   (r) Determine whether a binary operation is commutative and/or associative.
   (s) Determine the truth value of a conjunction or disjunction.

Binary Operations

In Section 2.4 we will see why we make a distinction between various sets of numbers like the natural numbers, integers and real numbers. Before doing that, we need to develop the ideas of a binary operation on a set and closure under a binary operation. In this section we develop the idea of a binary operation, saving closure for later.

A **binary operation** is a way of taking two mathematical objects and obtaining from them another mathematical object. This definition is not very precise, but it should do for now. After we have developed the idea of a function we can come back to this idea and make it more precise. For now we need to note several things:

- A binary operation takes *two* objects and returns *one* object.
- The single object that is returned must be *unique*. What we mean by this is that when two given objects are operated on, there must be only one result possible.
- We really need to specify the set of objects that a given binary operation acts on. That is, the set from which the two objects to be operated on are chosen.

You are of course very familiar with four binary operations on the set of real numbers, $+$, $-$, $\cdot$ and $\div$ (with the understanding that division by zero is not allowed). When we want to denote a general binary operation we will use $\star$ to signify the operation.

There are some other binary operations on numbers that we would like to consider; you should be familiar with some of them:

- $\gcd(a,b)$ is the **greatest common divisor** of a and b, defined on the non-zero integers. It is the largest integer that divides into both a and b. (Again, we mean divides "evenly". Note that by being the largest such integer, it will be positive.)
- $\text{lcm}(a,b)$, also defined on the non-zero integers, is the **least common multiple** of a and b. It is the smallest natural number (so it is positive) that both a and b divide into.
- $\min(a,b)$ is the smaller of a and b and $\max(a,b)$ is the larger of a and b. If $a=b$, then $\min(a,b)=\max(a,b)=a=b$. Both are defined on the set of real numbers.

1. Find each of the following, if possible.

(a) $\gcd(6, 9)$	(b) $\text{lcm}(-6, 9)$	(c) $\min(6, -9)$	(d) $\max(6, 9)$
(e) $\gcd(17, 3)$	(f) $\text{lcm}(4.1, 7)$	(g) $\min(4.1, 7)$	(h) $\max(9, 6)$

2. Find $\gcd(5, 5)$ and $\max(5, 5)$.

As this last exercise (and your experience with $+$, $-$, $\cdot$ and $\div$) shows, a binary operation can operate on two numbers that are the same.

Binary operations themselves have certain properties. In particular, binary operations can be commutative or associative. A binary operation $\star$ on a set A is **commutative** if $a \star b = b \star a$ for every pair (a, b) for which $\star$ is defined.

3. (a) Addition is clearly commutative. Which of multiplication, subtraction and division are commutative? For any that are not, provide counterexamples.
   (b) Which of gcd, lcm, min and max are commutative?

The binary operation $\star$ on the set A is **associative** if $a \star (b \star c) = (a \star b) \star c$ whenever the two sides of this equation are defined. *Note that the associate property involves THREE elements of* A *at a time*, whereas the commutative property involves just two at a time.

4. (a) Which of addition, multiplication, subtraction and division are associative? Don't forget to provide counterexamples for those that are not!
   (b) Which of gcd, lcm, min and max are associative? Again, provide counterexamples...

5. Consider the binary operation $\star$ defined on $\mathbb{R}$ by $a \star b = a^b$ whenever the operation can be carried out.
   (a) Show that the operation is *not* commutative by providing a counterexample.
   (b) Are there *any* values of a and b, $a \neq b$, for which $a^b = b^a$?
   (c) For which, if any, natural numbers is $\star$ *not* defined?
   (d) For which, if any, integers is $\star$ not defined?

Conjunctions and Disjunctions

In Section 1.2 we saw how to combine two simple statements to create a conditional statement. We will call any statement made up of two or more simple statements a **compound statement**. (Again, this is not commonly used terminology.) As mentioned before, when studying compound statements it is convenient to talk about simple statements without giving them specifically. To this end, in this section we will again take $P$and Q to represent arbitrary simple statements.

The compound statements we will consider are "P and Q", and "P or Q". The first of these is called a **conjunction** and the second is a **disjunction**. A specific example of a conjunction is

$$9 \text{ is a perfect square and } 10 \text{ is odd,}$$

and an example of a disjunction is

$$\text{for all real numbers } x, \; x^2 \geq 0 \text{ or } x + x = 2x.$$

Note that one or both of these statements might not be true - they are simply examples of a conjunction and a disjunction.

Given a conjunction or disjunction, we need to have consistent rules for determining its truth value, based on the truth values of P and Q individually. In the following exercises you will see that you already know these rules.

6. Determine the truth value of each of the following statements.

   (a) For all real numbers x, $|x| \geq x$ and $x^2 \geq 0$.
   (b) For all real numbers x, $x^2 x^3 = x^5$ and $7x - 2x = 9x$.
   (c) Six is odd and 18 is divisible by 5.

7. Based on the previous exercise, a conjunction "P and Q" is true if ...?

8. Determine the truth value of each of the following statements.

   (a) For all real numbers x, $|x| \geq x$ or $x^2 \geq 0$.
   (b) For all real numbers x, $x^2 x^3 = x^5$ or $7x - 2x = 9x$.
   (c) Six is odd or 18 is divisible by 5.

9. Based on the previous exercise, a disjunction "P or Q" is true if ...? (Throughout these notes, the word "or" will always allow the possibility of "both" as well. This is the common interpretation of "or" in logic and mathematics.)

1.6 Binary Operations on Sets

Performance Criteria:

1. (t) Find the union, intersection or difference of two sets; recognize or use correct notation for any of these.
   (u) Find the complement of a set; recognize or use correct notation for this.
   (v) Determine whether a statement of equality of sets is true. Give a counterexample if not.
   (w) Give conditions under which a statement of equality of sets is true.
   (x) Determine commutativity or associativity of binary operations on set.
   (y) Give the relationships between the cardinalities of A, B and the cardinalities of the results when they are operated on by any of the operations $\cup$, $\cap$ and $-$.
   (z) Know the distributive properties for binary set operations.

In the previous section you were introduced to the idea of a binary operation; our focus there was on binary operations on numbers. In this section we see some binary operations on sets. Note that each of these *is a method for combining two sets to get a SINGLE new set.* Given any two sets A and B, the **union** of them is the set of elements that are in either A or B, or both. The notation for the union of A and B is $A \cup B$. So symbolically,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

As emphasized above, $A \cup B$ is a *single* set!

The set of all elements that are in *both* A and B is called the **intersection** of the two sets, denoted by $A \cap B$. That is,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

If the sets A and B have no elements in common, then their intersection is the empty set; in that case we say that the two sets are **disjoint**.

1. Let $A = \{2, 4, 6, 8, 10\}$ and $B = \{3, 4, 5, 6\}$. Find $A \cap B$ and $A \cup B$.

Given the *single set* A with universal set U, the set of all elements not in A is called the **complement** of A. It is denoted by A'. (Some books will use the notation $\overline{A}$ for the complement of A.) Complementation is a **unary operation** - it is an operation that needs only one "input" (assuming that the universal set is known). Given two sets A and B, the set of all elements of A that are not in B is called the **difference of** A **and** B; sometimes this is called the relative complement of B with respect to A. We will denote this set by $A - B$, and we will often say "A minus B" instead of "the difference of A and B." Symbolically we now have

$$A' = \{x \mid x \notin A\} \qquad \text{and} \qquad A - B = \{x \mid x \in A \text{ and } x \notin B\}.$$

(Here it is understood in the definition of A' that the elements x must come from the universal set U.)

2. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{2, 4, 6, 8, 10\}$ and $B = \{3, 4, 5, 6\}$.
   (a) Find A' and B', $A - B$.
   (b) Find $A' \cap B'$ and $(A \cap B)'$. Are they the same?
   (c) Give a set $C \subseteq U$ such that A and C are disjoint.

3. (a) Is it possible to have $A \cap B = A$? If so, give an example, and give the most general conditions under which it can happen *in the form of a conditional statement*.
   (b) Is it possible to have $A \cup B = A$? If so, give an example, and give the most general conditions under which it can happen *in the form of a conditional statement*.
   (c) Is it possible to have $A \cap B = A \cup B$? Again, give an example if it is possible and give the most general conditions under which it can happen.

4. (a) Draw graphical representations of $[-3, 2)$ and $[-1, \infty)$ on separate real lines, one directly over the other. Then draw a graphical representation of $[-3, 2) \cap [-1, \infty)$ on another real line directly below those two. Give the interval notation for that set. Describe how to use the first two graphs to obtain the third.
   (b) Repeat the above for $[-3, 2) \cup [-1, \infty)$.

5. Repeat Exercises 1 and 2 for $U = \mathbb{R}$, $A = (-2, 4.1]$, $B = [1, 8]$.

6. (a) Sketch the graph of the set

$$\{x \in \mathbb{R} \mid x \geq -2\} \cap \{x \in \mathbb{R} \mid x \neq 3\}.$$

   (b) *Carefully* write your answer to part (a) using interval notation. *Note that this sort of says that (in this case) an intersection is a union!*

7. Let $A = (3, 5]$ and $B = \{3, 4, 5, 6, 7\}$. Find $A \cup B$, $A \cap B$, $A - B$ and $B - A$.

8. Describe the set $\mathbb{R} - \{2\}$
   (a) with interval notation (b) with set builder notation (c) in words

9. (a) Which of union, intersection and difference of sets are commutative?
   (b) Which of union, intersection and difference of sets are associative?

10. (a) Fill the first blank of the following statement with $<$, $>$, $\leq$, $\geq$ or $=$ and the second blank with $+$, $-$, $\cdot$ or $\div$:

$$|A \cup B| \ \underline{\qquad} \ |A| \ \underline{\qquad} \ |B|$$

   Your answer should be valid whether or not A and B are disjoint, or regardless of whether one is a subset of the other.

(b) You should not have used $=$ in the first blank of part (a). Under what circumstances would $=$ be correct there? Write a conditional statement that summarizes this.

(c) Try to finish the statement $|A \cup B| =$ in such a way that it holds unconditionally. Your answer should contain the quantities $|A|$, $|B|$ and $|A \cap B|$.

1.7 Chapter 1 Exercises

> **Performance Criteria:**
>
> 1. (aa) Apply concepts and techniques of the chapter to new situations.

1. Consider again the sets

$$A = \{3, 4, 5, 6, 7\} \qquad \text{and} \qquad B = [3, 7).$$

   (a) Why is it not true that $A \subseteq B$?
   (b) Why is it not true that $B \subseteq A$?

2. (a) Give a set A, containing more than one element, for which $\{5\} \subseteq A$ makes sense.
   (b) Give a set B, containing more than one element, for which $\{5\} \in B$ makes sense.

3. A student uses the notation $[3, 7]$ to describe the set $\{3, 4, 5, 6, 7\}$. What is wrong with this?

4. In a previous exercise you should have used the notation $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ to describe the set $\{1, \frac{1}{2}, \frac{1}{3}, ...\}$. What set is described by $\{\frac{1}{x} \mid x \in \mathbb{R},\ x \neq 0\}$? Give your answer three ways:

   (a) In words.
   (b) As the union of two intervals.
   (c) As the difference of two sets.

5. In plane geometry, four-sided polygons are called *quadrilaterals*. For this exercise we will take the set of quadrilaterals to be the universal set, and we will denote that set by Q (not to be confused with the set $\mathbb{Q}$ of rational numbers). There are a number of specific types of quadrilaterals, some familiar, some not so familiar:

   ⋄ *Rectangles* are quadrilaterals with four right angles.

   ⋄ *Squares* are quadrilaterals with four right angles and four sides of equal length.

   ⋄ *Parallelograms* are quadrilaterals with opposite sides parallel.

   ⋄ *Rhombuses* are quadrilaterals with opposite sides parallel and all four sides the same length.

   ⋄ *Trapezoids* are quadrilaterals with *exactly* one (no more, no less) pair of opposite sides parallel.

   We will denote the sets of these various types of quadrilaterals by Re, S, P, Rh and T, respectively.

   (a) Give all subset relations within these sets, ignoring Q. There are five of them.
   (b) For each pair of sets that are not disjoint and for which one is not a subset of the other, tell what the intersection of the two sets is. (In each case the answer will be one of the given sets.)

2 More on Sets and Statements

2.1 Quantifiers and Negations

Performance Criteria:

2. (a) Determine the truth value of a statement containing a quantifier.
   (b) Negate a simple statement or a compound statement.

Quantifiers

In the previous chapter we considered questions like "In general, can we say that $A \cap B = A$?" We interpret such a question to mean "Is $A \cap B = A$ *for all* sets A and B?" The words "for all" are what we call a **quantifier**. We will have just two quantifiers, "for all" (also stated as "for every", "for each") and "there exists" (or "there is").

1. (a) The statement "For all sets A and B, $A \cap B = A$" is false. Give a *specific* counterexample.
   (b) Consider the statement "There exist sets A and B for which $A \cap B = A$." This is taken to be true if there is at least one pair of sets for which it is true. What is the truth value of this statement? Explain.

2. Determine the truth value of each of the following statements.
   (a) There exist $x, y \in \mathbb{R}$ such that $x - y > x + y$.
   (b) For all $x, y \in \mathbb{R}$, $x - y > x + y$.
   (c) For all sets A and B, $A' \subseteq B$.
   (d) There exist $m, n \in \mathbb{N}$ such that $m + n \in \mathbb{N}$.
   (e) For all $m, n \in \mathbb{N}$, $m + n \in \mathbb{N}$.
   (f) There exist sets A and B such that $A \cup B \subseteq A \cap B$.
   (g) For all sets A and B, $A \cap B \subseteq A \cup B$.

3. Most statements that we will encounter will be of the "for all" variety, which are easier to show false than to prove true. Give counterexamples for any false "for all" statements in the previous exercise.

4. On the other hand, "there exist" statements are easier to show true than false, since truth can be validated by *one* example. Provide such an example for each "there exist" statement from Exercise 2 that you said was true.

5. Find a value of x for which $x + 2x = 5x$ is true. Is the statement true for any other values?

From this point on we will assume, unless told otherwise, that the quantifier "for all sets" is implied for any statement about sets. That is, we take a statement like $A \cap B = A$ to mean "For all sets A and B, $A \cap B = A$". The statement is then false, as shown in

Exercise 1(a). We will also assume that statements about numbers include the quantifier "for all real numbers" unless told otherwise. Thus $x + 2x = 5x$ is false, since it is only true when $x = 0$. On the other hand, $3(x + 1) = 3x + 3$ is true, since it holds for all real numbers.

Negations

For reasons which may not be clear to you for a while, it is very important that we be able to determine "opposites" of statements, which we call **negations**. For example, the negation of "today is Friday" is "today is not Friday". Clearly, when forming the negation (we will call this process *negating*) of a simple statement consisting of words, we just insert the word "not" in the correct place. As you may know, the symbols $=$, $\in$ and $\subseteq$ are negated by placing a slash through them: $\neq$, $\notin$ and $\not\subseteq$. Negations of simple statements are usually easy to form, at least if quantifiers are not involved.

6. Negate each of the following statements.

   (a) $A \subseteq C$ (b) x is a real number (c) x is divisible by 4

7. Negate $x > 7$, without using the symbol $\not>$. *This is the correct way to negate an inequality, and you should always negate one in this way.*

To guide us in determining the negations of more complicated statements we make the following obvious, but very important, observation. *If a statement is true, its negation is false, and vice-versa.* Thus, negating a statement consists of creating another statement whose truth value is the opposite as that of the original statement.

8. (a) Explain why the statement "For all $x \in \mathbb{R}$, $x + 3 = 7$" is false.
   (b) Is the statement "For all $x \in \mathbb{R}$, $x + 3 \neq 7$" true?
   (c) Is the statement in (b) the negation of the statement in (a)? Explain.

The above exercise shows that a statement containing a quantifier cannot be negated by simply negating the "mathematical part" of the statement. We must not only do that, but we must also change the quantifier itself; if it was "for all" we change it to "there exists - such that", and vice versa. The negation of the statement in Exercise 8(a) is then "There exists $x \in \mathbb{R}$ such that $x + 3 \neq 7$".

9. Show that "There exists $x \in \mathbb{R}$ such that $x + 3 \neq 7$" is true.

NOTE: When negating a statement, the new statement must, of course, make sense grammatically!

10. Negate each of the following statements. (Some may be false statements - simply negate them without concerning yourself with their truth value.)

    (a) For all sets A and B, $A \cap B \subseteq A$.
    (b) There exist sets A and B for which $A \cap B = A$.
    (c) There exist $x, y \in \mathbb{R}$ such that $x - y > x + y$.
    (d) For all sets A and B, $A' \subseteq B$.

(e) For all $m, n \in \mathbb{N}$, $m + n \in \mathbb{N}$.
(f) There exist sets A and B such that $A \cup B \subseteq A \cap B$.

Next we look at negating conjunctions and disjunctions. (Recall that a conjunction is a statement of the form "P and Q" and a disjunction has the form "P or Q".)

11. Consider the statement "$x > 7$ or $x = -3$", and *do NOT assume that the quantifier "for all real numbers" holds.* Then the statement can be either true or false, depending on the value of x.

    (a) Give two values of x for which the statement is true.
    (b) Give two values of x for which the statement is false.
    (c) What must one insure about x in order to make the statement false?

Recall now that we said the negation of any statement is another statement whose truth value is the opposite of the original. Your answer to Exercise 11(c) should have indicated that the negation of the statement of Exercise 11 is "$x \leq 7$ and $x \neq -3$". Thus we can see that *the negation of "P or Q" is "not P* <u>*and*</u> *not Q".* This should make sense: for "P or Q" to be false, it must be the case that *both* P and Q must be false. That is, both "not P" and "not Q" must be true.

12. (a) What do you suppose the negation of "P and Q" is? Explain.
    (b) Give a value of n for which "n is even and $n > 10$" is true.
    (c) Give the negation of "n is even and $n > 10$".
    (d) Does your answer to (b) make your answer to (c) true, or false?
    (e) Now find a value that makes the statement in (b) false. Does this value make your answer to (c) true, or false?

13. Negate each of the following.

    (a) $x \in A$ or $x \in B$ (b) $x \in A$ and $x \notin B$
    (c) n is divisible by two and n is not divisible by three

The ability to negate statements is needed in order to translate statements into other forms. For example, consider the statement $x \in (A \cap B)'$. By definition of complement, this translates to

$$x \text{ is not in } A \cap B.$$

But this is the same as saying "not $x \in A \cap B$. Recall that $x \in A \cap B$ means

$$x \in A \text{ and } x \in B.$$

As we now know, the negation of this is

$$x \notin A \text{ or } x \notin B.$$

Thus $x \in (A \cap B)'$ is equivalent to $x \notin A$ or $x \notin B$.

14. (a) Translate $x \in (A \cup B)'$ into an equivalent statement, in the manner just shown.
    (b) Translate $x \in (A - B)'$ into an equivalent statement.

We will finish this section by determining how to negate a conditional statement. Remember again that to negate any statement we must find a statement whose truth value is the opposite of the original statement.

15. Consider the conditional statement "If m is divisible by 2, then m is divisible by 4." Explain why this statement is false.

Your answer should have been based on a the fact that you can find a value of m (many values, actually) for which "m is divisible by 2 and m is not divisible by 4" is true. Since the truth value of this statement is the opposite of that of the original conditional statement, it is the negation of the conditional statement. This illustrates that *the negation of "If P, then Q" is "P and not Q".*

16. Give the negation of each of the following conditional statements. (Note that the statements given may not necessarily be true.)

    (a) If $A \subseteq B$, then $A \cap B = B$.
    (b) If $x < y$, then $x + 3 < y + 3$.
    (c) If $A \subseteq B$, then $B' \subseteq A'$.
    (d) If m is an integer, then m is even or m is odd.
    (e) If $x + y = 7$, then $x = 4$ and $y = 3$.

Most of the statements in this last exercise contain implied quantifiers. For example, (a) is taken to mean "For all sets A and B, if $A \subseteq B$, then $A \cap B = B$." Negating the statement then means not only negating the conditional statement, but changing the quantifier as well:

"There exist sets A and B such that $A \subseteq B$ and $A \cap B \neq B$."

17. Rewrite the statement from Exercise 16(b) to include the implied quantifier, then give its negation.

The ability to negate statements is important. One reason is that if we want to show that a statement is false, it is sufficient to show its negation is true, and vice-versa.

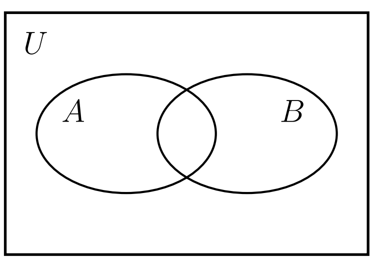
18. Assume the quantifier "for all" is implied for Exercises 16(c),(e). Determine the truth value of each of those statements. If your answer is false, explain.

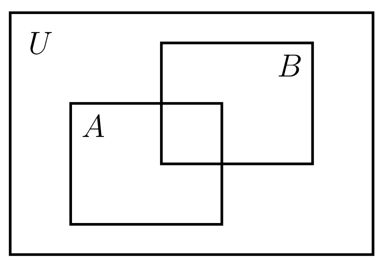
2.2 Venn Diagrams

Performance Criteria:

2. (c) Give the set represented by any region of a Venn diagram.
   (d) Create a Venn diagram for some given sets.
   (e) Draw a Venn diagram with a given set shaded.
   (f) Draw all possible Venn diagrams for two sets in a universal set.
   (g) Use Venn diagrams to determine the truth of a given set relation.
   (h) Use Venn diagrams to discover set relations.
   (i) Draw a Venn diagram describing the relationships between sets of mathematical objects.

There is a very useful tool for visualizing the relationships between sets, and operations on sets. It is a type of diagram called a **Venn diagram**. The picture below and to the left is an example of a Venn diagram for the two sets $A = \{3, 4, 5\}$ and $B = \{5, 6, 7\}$, with the universal set $U = \{1, 2, 3, ..., 10\}$. Basically a Venn diagram consists of a large rectangle that represents the universal set, and the interior of the rectangle contains ovals or circles representing sets. If two sets have a non-empty intersection, their ovals overlap to indicate this. If a set A is a subset of a set B, its oval is inside the oval for B. Even though most Venn diagrams you will see in books use rectangles for universal sets and ovals for other sets, there is no reason that one can't use rectangles (or other shapes) for all the sets, like the picture below and to the right.





1. Each region of a Venn diagram represents certain subsets of the universal set. For example, the part of the oval for B that does not overlap the oval for A is the set $B - A$.

   (a) The region where the ovals for A and B overlap represents the elements that are in both sets. What subset of U is this?
   (b) What subset of U does the part of the oval for A that does not overlap B's oval represent?
   (c) What subset of U does the region outside the two ovals represent?

2. For the following, let $U = \mathbb{R}$.

   (a) Draw a Venn diagram for the sets $A = \{3n \mid n \in \mathbb{N}\}$ and $B = \{3n \mid n \in \mathbb{Z}\}$.

(b) Draw a Venn diagram for the sets $B = \{3n \mid n \in \mathbb{Z}\}$ and $C = \{3^n \mid n \in \mathbb{Z}\}$.

(c) Draw a Venn diagram for all three sets A, B and C.

3. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{2, 4, 6, 8, 10\}$ and $B = \{3, 4, 5, 6\}$. Consider the sets $A' \cap B'$ and $(A \cap B)'$. The first is obtained by taking the complements of A and B, then intersecting those. The second is obtained by first intersecting A and B, then taking the complement of the result. Draw Venn diagrams of each of these two sets. Are they the same?

Venn diagrams are very useful for suggesting things about sets, but *they cannot be trusted to tell the truth all of the time.* You will see this in one of the exercises below.

4. The diagram on the previous page does not show the only possible relationship between two sets A and B within a universal set U; it only shows the case where A and B are not disjoint. There are four others. Try to find and draw each of them.

5. Let $U = \{1, 2, 3, ..., 9, 10\}$ and give specific sets A and B illustrating the situation shown in each of your Venn diagrams from Exercise 3.

6. To study the relationships between two sets A and B, the most general arrangement is the one for which the two sets intersect, but neither is a subset of the other, shown above. For each of the following, draw a Venn diagram with A and B arranged in that manner and shade the indicated set.

   (a) $A \cap B$ (b) $A \cup B$ (c) $A - B$ (d) B' (e) $A \cup B'$

7. (a) Use the Venn diagram from 6(a) to deduce a subset relationship between A and $A \cap B$.

   (b) What would you have deduced about $A \cap B$ if you would have used a Venn diagram in which A is a subset of B?

   (c) Is your conclusion from (a) correct for all possible arrangements of A and B, as given in Exercise 4?

8. What subset relationship holds in general (meaning for all possible relationships between A and B, as found in Exercise 4) for A and $A \cup B$?

9. Suppose that $A \subseteq B$. In Exercise 7(b) you should have deduced that in this case $A \cap B = A$.

   (a) What can we deduce about $A \cup B$ if $A \subseteq B$? Write your answer as a conditional statement.

   (b) What can we deduce about $A - B$ if $A \subseteq B$? Write your answer as a conditional statement.

10. Draw separate Venn diagrams (using the arrangement shown on the previous page) with $(A \cap B)'$ and $A' \cap B'$ shaded. Clearly the two sets are not equal. Then draw two more Venn diagrams with $(A \cup B)'$ and $A' \cup B'$ shaded. Use your four diagrams to make two conjectures. These relationships are special enough to have their own names - they are called **DeMorgan's Laws**. Later you will prove that they always hold.

11. Suppose that $A \subseteq B$. What is the relationship between A' and B'? Write your answer as a conditional statement.

12. (a) Express $A - B$ as the intersection of two sets. Give the most general answer possible. This means to tell what holds when no particular assumptions are made about how the sets A and B are related.

    (b) Express $(A - B)'$ as the union of two sets. Again, give the most general answer possible.

13. Draw a Venn diagram for the sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ with universal set $\mathbb{R}$.

14. Letting $U = \mathbb{R}$, sketch a Venn diagram for the sets $A = (3, 5]$, $B = \{3, 4, 5, 6, 7\}$ and $C = \{9, 10, 11\}$.

15. There are some special types of triangles:

    ◇ *Right triangles* are triangles with one right angle.

    ◇ *Isosceles triangles* are triangles with at least two sides of equal length.

    ◇ *Equilateral triangles* are triangles with three sides of equal length.

    (a) Let T, R, I and E represent the sets of all triangles, right triangles, isosceles triangles and equilateral triangles, respectively. Draw a Venn diagram showing the relationship between the various types of triangles, taking T to be the universal set.

    (b) Give a subset relationship between two of the types of triangles.

    (c) What can you say about the intersection of the set of right triangles and the set of equilateral triangles?

    (d) What can you say about the intersection of the set of right triangles and the set of isosceles triangles?

There is a powerful method of thinking about concepts that can often prove useful when learning new things. It is called *analogy*, and it involves seeing similarities in things that one might think at first have little to do with each other. The power of analogy lies in the fact that once some similarities are found, others are likely to follow. Sets and and some of the above operations on them are similar to numbers, with the operations of addition and multiplication. (In a sense, there is no need for subtraction and division, since subtraction is just addition of the opposite and division is multiplication by the reciprocal.) Union of sets is like addition of numbers and intersection of sets is like multiplication of numbers, with the empty set playing the role of zero. You will make use of this analogy in the next exercise.

16. One of the basic axioms (unproven rules) of the real numbers is the *distributive property of multiplication over addition*, which says that $a(b + c) = ab + ac$. Does there seem to be a similar property for unions and intersections of sets? If so, state the property and give a *simple* example which supports your conjecture. *Note that the distributive property contains two operations, multiplication and addition. Your statement needs to include both union and intersection.* (Realize that one example does not PROVE the property; it simply suggests that it MAY be true in general.)

2.3 Closure of Sets Under Binary Operations

> **Performance Criteria:**
>
> 2. (j) Determine whether a set is closed under a binary operation; give a set under which a binary operation is closed.

1. Consider the set $A = \{-1, 0, 1\}$.

   (a) Perform all possible multiplications of two numbers from this set. (Don't forget that this includes multiplying elements by themselves, as discussed after Exercise 1.3.2.) What results are possible?

   (b) Perform all possible additions of two numbers from this set. What results are possible?

In this last exercise you should have seen that if your whole numerical world consisted of the set A and all you wanted to do was multiply numbers, life could proceed just fine. On the other hand, the set A cannot handle addition without needing to expand itself. We say that A is **closed** *under the operation of multiplication*, and *it is NOT closed under the operation of addition*.

In general, we say that a set A is closed under a binary operation $\star$ if the following holds: For all $a, b \in A$ for which $a \star b$ is defined, $a \star b \in A$ as well. Thus the set A from Exercise 1 is not closed under addition because $1 \in A$ but $1 + 1 = 2 \notin A$. (Remember that a binary operation can work on two objects that are really the same.) We should note a couple important things at this point:

- It is the *set* that is closed or not closed, but
- the *operation* is what is used to determine whether or not the set is closed.

Thus one must always have a set *AND* an operation to consider the concept of closure. (The type of closure we are talking about is technically *algebraic closure*. An interval of the form $[1, 5]$ is called a closed interval, but this means something called *topological closure*, which some of you may encounter someday.)

2. In each of the following, a set and an operation are given. Determine whether the set is closed under the operation; if not, provide a counterexample. (Remember that such a counterexample consists of giving two things that are in the set, and showing that the result of operating on them is not in the set. See the explanation above as to why A is not closed under addition.)

   (a) Is $A = [-1, 1]$ closed under multiplication?

   (b) Is $A = [-1, 1]$ closed under subtraction?

   (c) Is $B = [-1, 0) \cup (0, 1]$ closed under division?

   (d) Define the binary operation $\text{diff}(a, b) = |a - b|$ on the set of all real numbers. Is $A = [-1, 1]$ closed under diff?

(e) Define the binary operation $a \star b = a^b$. Are the natural numbers closed under $\star$? (Consider only pairs for which $\star$ is defined.)

(f) Are the integers closed under $\star$ from (e)? (Again, consider only pairs for which $\star$ is defined.)

(g) Is the set $B = \{-1, -2, -3, -4, ...\}$ closed under addition?

(h) Is the set $[0, \infty)$ closed under multiplication?

(i) Is the set $[0, \infty)$ closed under subtraction?

(j) Is the set $C = \{1, 3, 9, 27, ...\}$ closed under multiplication?

(k) Is the set $C = \{1, 3, 9, 27, ...\}$ closed under division?

3. Let $E = \{2, 4, 6, 8, ...\}$ and $O = \{1, 3, 5, 7, ...\}$

   (a) Is E closed under addition? Multiplication?

   (b) Is O closed under addition? Multiplication?

4. (a) Is $\{1, 2, 3, 6\}$ closed under gcd? Under lcm?

   (b) Is $\{1, 3, 4, 6\}$ closed under gcd? Under lcm?

   (c) Can you find a set that is closed under gcd but not under lcm? Vice-versa?

5. Consider the binary operation diff from Exercise 2(d). Try to find sets of each of the following kinds that are closed under diff.

   (a) finite

   (b) infinite countable

   (c) finite interval

   (d) infinite interval

6. Try to find sets of each of the types given in Exercise 5 that *are not* closed under diff.

2.4 Origins of Sets of Numbers

Performance Criteria:

2. (k) Determine whether a number is an element of the natural numbers, whole numbers, integers, and/or the rational numbers.
   (l) Tell what gave rise to the development of the integers from the whole numbers, and the rational numbers from the integers.
   (m) Give some examples of unary operations on numbers.

Armed with the concept of closure under a binary operation, we are now ready to look at the sets of numbers that we will be working with throughout this course. We will develop them in the order that they (probably) arose historically. The very first need for numbers was in the process of counting; the numbers used for this purpose are the natural numbers, which we have already defined.

One of the more interesting things about the development of the number systems we use is that the natural numbers were used for a long time (about 1000 years?) before it occurred to anyone to invent a number to use for the number of "things" if there were no "things" - zero! When we include zero with the natural numbers, we have the set

$$\mathbb{W} = \{0, 1, 2, 3, 4, ...\},$$

called the **whole numbers**.

You will now see how the idea of closure comes into play when considering sets of numbers.

1. (a) Are the whole numbers closed under addition? If not, what additional numbers must we include to get the smallest set containing the whole numbers that is closed under addition?
   (b) Repeat (a) for multiplication.
   (c) Repeat (a) for subtraction.

You can see that once the binary operation of subtraction is introduced, the set of whole numbers is no longer "big enough" to account for the results of subtractions in which the first number is smaller than the second. (Historically, this may have been discovered at the same time as the concept of debt came about!) To accommodate the results of subtracting *any* two whole numbers, we need the new set that is formed by including the negatives of all the natural numbers with the whole numbers. This set is the **integers**, $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$, denoted by $\mathbb{Z}$. Note that the integers ARE closed under subtraction.

2. Are the integers closed under addition? Multiplication? Division?

Take note of your first two answers to this exercise. *Later we will make regular use of the fact that the integers are closed under both addition and multiplication.* We will take this as fact throughout the rest of these notes.

When we divide two integers, the result may not be an integer, so we now need the numbers formed by dividing one integer by another (not allowing division by zero). The next important set of numbers larger than the integers is the **rational numbers**, which are all quotients of two integers, not allowing division by zero. Of course any integer x is also a rational number, since it can be expressed as $\frac{x}{1}$, so *the rational numbers contain the integers*. The symbol for the rational numbers is $\mathbb{Q}$. Note that the rational numbers are not easily described by listing, but we *can* describe them with set builder notation:

$$\mathbb{Q} = \left\{ x \;\middle|\; x = \frac{m}{n} \,,\ m, n \in \mathbb{Z} \text{ and } n \neq 0 \right\}$$

The rational numbers are closed under all four basic operations. *Therefore, if all the mathematics that we wished to do used only the operations of addition, subtraction, multiplication and division, the rational numbers would be all that we needed.*

3. An operation that takes one "input" and gives one "output" is called a **unary operation**. For example, consider the operation of rounding a number to the nearest integer. It takes in any number (real number, technically) and gives out a number. The "square" operation is a unary operation as well. The set operation of complementation is also a unary operation. Give a few more examples of unary operations; there are a number of them that you have been exposed to in previous courses.

A common unary operation is the square root. If we allow the square root to act on any non-negative rational number, the result may not necessarily be a rational number. In order to have a set of numbers that is closed under the operation of square root we must include numbers like $\sqrt{3}$ with the rational numbers. All the oddballs of mathematics like roots, π and e are called the **irrational numbers**; they are just the numbers that are not rational. *The rational numbers and irrational numbers together make up the real numbers.* There is no commonly accepted notation for the irrational numbers; since they are all the real numbers that are not rational, we can describe them symbolically by $\mathbb{R} - \mathbb{Q}$.

Note that if one wishes to be able to take square roots of negative numbers, the complex numbers must be considered. (That is, the real numbers are not closed under square root.) *We will not work with complex numbers in this course.*

2.5 More On Sets

Performance Criteria:

2. (n) Determine whether an object is a an element and/or a subset of a set of sets.
   (o) Determine whether a given set of sets is closed under union, intersection and/or difference.
   (p) Give the power set of a set. Give the cardinality of the power set of a set.
   (q) Determine whether a set of sets is a partition of a given set. If not, tell why.
   (r) Give the Cartesian product of two sets; give the cardinality of the Cartesian product of two sets.
   (s) Graph a given subset of $\mathbb{R} \times \mathbb{R}$.

Sets of Sets

It is not usual to be interested in a set whose elements are themselves sets. Consider the set

$$A = \{\emptyset, \{1\}, \{2\}, \{2,3\}, \{1,2,3\}\}.$$

1. What is the cardinality of A?

2. Determine whether each of the following is (i) an element of A, (ii) a subset of A, (iii) neither an element nor a subset of A or (iv) both an element and a subset of A.

   (a) 2 (b) $\{2\}$ (c) $\{\{2\}\}$ (d) $\emptyset$ (e) $\{1,2,3\}$
   (f) $\{\{1\},\{2,3\},\{1,2,3\}\}$

3. When considering a set of numbers, we could ask whether the set was closed under a binary operation like addition or multiplication. When considering a set of sets, the binary operations we might consider are union, intersection and difference. Is A closed under union? Intersection? Difference?

4. Consider the set $B = \{\emptyset, [0,1),\ [0,2),\ [1,2)\}$.

   (a) What is the cardinality of B?
   (b) Is B closed under union? Intersection? Difference?
   (c) Is there any change to your answer to (b) if the empty set is removed from B?

I just made up the two sets used in the above exercises. However, there are certain situations in which sets of sets arise very naturally. You will see these in the remaining exercises.

5. For a given set A, the **power set** of A is the set of all possible subsets of A. The power set of A will be denoted $P(A)$; note that *it is a set of sets.* Thus $|P(A)|$ makes sense, and represents the number of sets in the power set.

   (a) Let $A = \{1, 2, 3\}$. Write out $P(A)$, *indicating symbolically that it is a set of sets.* The power set should contain eight sets; that is, $|P(A)| = 8$.
   (b) For $B = \{a, b, c, d\}$, what is $P(B)$? What is $|P(B)|$?
   (c) Remember that for a set C, $|C|$ is called the cardinality of C, and it is the number of elements in C. If $|C| = 1$, what is $|P(C)|$?
   (d) If $|D| = 2$, what is $|P(D)|$?
   (e) In general, if $|A| = n$, what do you think $|P(A)|$ is?
   (f) What is $P(\emptyset)$? Does this cause a problem with your answer to (e)?

Mathematical concepts (as well as concepts in other technical disciplines) are often presented in a very terse form that is at first difficult to understand. Once such a concept is understood, we find the idea to be very simple and we wonder why its definition seems to be so confusing or unilluminating. That is because the definition is constructed to be as concise as possible, with no "loopholes". Historically, the idea came first, followed by a definition that may or may not have been correct. As time went on the definition was modified to be correct and as precise as possible. The process is usually reversed when the rest of us come to the idea later - the definition is given and we must sometimes struggle to see the idea that it describes. In the next exercise you will grapple with the definition of a concept that is really quite simple, a partition of a set. (Don't neglect to think about what the word partition means to you in your own field, or just in general!) Here is the definition:

A **partition** of a set A is a set $\{A_1,\ A_2, ...A_n\}$ of subsets of A for which

- none of $A_1, A_2, ..., A_n$ are empty,
- $A_1 \cup A_2 \cup \cdots \cup A_n = A$ and
- $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

6. Determine whether each of the following is a partition of the set $A = \{1, 2, 3, 4\}$. For any that are not, explain why.

   (a) $\{\{1\},\ \{1, 2\},\ \{1, 2, 3\},\ \{1, 2, 3, 4\}\}$.
   (b) $\{\{1\},\ \{2\},\ \{3, 4\}\}$.
   (c) $\{\{1, 2, 3, 4\},\ \{1, 2\},\ \{3, 4\}, \emptyset\}$.
   (d) $\{\{1\},\ \{2\},\ \{3\}\}$.

7. What set or sets could be included in

$$\{[0, 1), [1, 3), (3, 4)\}$$

   to obtain a partition of the interval $[0, 6]$? If there is more than one answer possible, give two.

If you haven't recognized it by now, a partition of a set is just a set of its subsets that don't "overlap" (they're disjoint), and which "combine" to give the whole set (their union is the whole set). You probably think of a number of partitions already:

- The integers partition into evens and odds.
- The real numbers partition into positives, negatives and zero.
- The real numbers partition into rationals and irrationals.

Cartesian Products of Sets

Given two sets A and B we can create a new set consisting of all possible ordered pairs made up of an element of A and an element of B, *in that order.* This new set is called the **Cartesian product** of A and B, denoted by $A \times B$. Each ordered pair is enclosed in parentheses, with its two elements separated by a comma, like $(2, 5)$. Of course this notation also represents the interval of real numbers between 2 and 5, not including either. *One must determine from the context what the meaning of* $(2, 5)$ *is.*

8. If $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$, write out $A \times B$. Indicate symbolically that your answer is a *set.*

9. Discuss the validity of the statement $A \times B = B \times A$.

10. If A has m elements and B has n elements, how many elements does $A \times B$ have?

A Cartesian product that you are all familiar with is $\mathbb{R} \times \mathbb{R}$, the set of all ordered pairs of real numbers. This is xy-plane from algebra and calculus. When you graph an equation like $y = x^2$, you are really graphing a subset of $\mathbb{R} \times \mathbb{R}$; in particular, you are graphing the set

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2\}.$$

Another subset of $\mathbb{R} \times \mathbb{R}$ is $\mathbb{Z} \times \{2\}$, whose graph is shown to the right.

11. Take $\mathbb{R} \times \mathbb{R}$ to be the universal set. For each of the following, sketch and label xy-axes, then indicate the given set in your diagram.

    (a) $\mathbb{Z} \times \mathbb{Z}$. This set is sometimes referred to as the lattice of integer points.

    (b) $\mathbb{R} \times \{0\}$. What is this set?

    (c) $\{3\} \times \mathbb{R}$

    (d) $[-1, 3] \times [2, 4]$

    (e) $(-1, 3] \times [2, 4)$ Note that you must somehow distinguish some sides of the resulting rectangle from others! Use solid lines when pairs on the boundary are included in the set, dashed when they are not. Put open or closed circles on the corners, indicating whether or not those points are included.

    (f) $[2, \infty) \times \mathbb{R}$

    (g) $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = -y\}$

    (h) $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x + y \in \mathbb{Z}\}$

12. Using the set definition for part (g) of the previous exercise as an example, describe the sets from parts (a), (c), (e) and (f) of that exercise using set builder notation.

2.6 Converses, Biconditional Statements and Contrapositives

Performance Criteria:

2. (t) Form the converse or contrapositive of a conditional statement.
   (u) Understand the relationships between the truth of a statement and the truth of its converse and contrapositive.

1. The **converse** of a conditional statement "If P then Q" is the statement "If Q then P." Give the converse of each of the following conditional statements.
   (a) If m is divisible by 2 and by 3, then m is divisible by 6.
   (b) If m and n are odd, then $m + n$ is even.
   (c) If $A \subseteq B$, then $A \cap B = A$.
2. (a) Which of the original statements from Exercise 1 are true?
   (b) Determine the truth values of the converses of the statements from Exercise 1.
   (c) Suppose that a conditional statement is true. Is its converse necessarily true?

You should have observed that *the converse of a true conditional statement is not necessarily true*. By now you have seen a number of conditional statements that are false. Our interest in conditional statements, however, is that we want to use them to state certain mathematical facts (*true* facts, that is). From this point on our only concern with false statements is that we would like to recognize them as false.

Sometimes there will be true conditional statements whose converses are true as well. For example, the converse of the true conditional statement given in Exercise 1(a),

If m is divisible by 6, then m is divisible by 2 and by 3,

is true as well. In such cases, rather than giving two conditional statements we write one statement that combines both. To do this, we use the phrase "if, and only if":

m is divisible by 2 and by 3 if, and only if, m is divisible by 6.

Such a statement is called a **biconditional statement**. The biconditional statement is true only if both conditional statements are true, and vice-versa.

3. Write the following pair of conditional statements as a biconditional statement:

   If $A = B$, then $A \cap B = A \cup B$.

   If $A \cap B = A \cup B$, then $A = B$.

4. Write each of the following biconditional statements as two conditional statements. Then determine whether the biconditional statement is true; if it is not, explain why.

(a) $A \subseteq B$ if, and only if, $A \cap B = B$.
(b) $A \subseteq B$ if, and only if, $B' \subseteq A'$.
(c) $x + y = 7$ if, and only if, $x = 4$ and $y = 3$.

We have seen how to take a conditional statement and obtain from it another conditional statement, its converse. There is another way to take a conditional statement "If P then Q" and get another conditional statement from it. This is done by "switching" P and Q *and* negating both of them: "If not Q then not P." This new conditional statement is called the **contrapositive** of the original statement.

5. Form the contrapositive of each of the conditional statements below.

   (a) If m is even, then $m + 1$ is odd.
   (b) If $A \subseteq B$, then $A \cap B = B$.
   (c) If $x < y$, then $x + 3 < y + 3$.
   (d) If $A \subseteq B$, then $A \cap B = A$.

6. (a) Which of the original statements from Exercise 1 are true?
   (b) Determine the truth values of the contrapositives of the statements from Exercise 1.
   (c) Suppose that a conditional statement is true. Do you think its contrapositive is necessarily true?

This last exercise points out the importance of contrapositive: *the truth value of a conditional statement is the same as that of its contrapositive.* Thus, if we want to show a conditional statement is true but we're having trouble doing it, it might be easier to show the contrapositive is true. That then means the original conditional statement is true as well.

7. Form the contrapositive of each statement below; each involves a conjunction or disjunction.

   (a) If m and n are odd, then $m + n$ is even.
   (b) If m is an integer, then m is even or m is odd.
   (c) If $x + y = 7$, then $x = 4$ and $y = 3$.

2.7 Chapter 2 Exercises

Performance Criteria:

2. (v) Apply concepts and techniques of the chapter to new situations.
   (w) Find the closure of a set under an operation.
   (x) Give a partition of a set that meets certain conditions.
   (y) Give the symmetric difference of two sets.
   (z) Give two definitions of the symmetric difference in terms of unions, intersections and complements.

1. Consider again the sets

$$A = \{3, 4, 5, 6, 7\} \qquad \text{and} \qquad B = [3, 7).$$

   (a) Why is it not true that $A \subseteq B$?
   (b) Why is it not true that $B \subseteq A$?

2. (a) Give a set A, containing more than one element, for which $\{5\} \subseteq A$ makes sense.
   (b) Give a set B, containing more than one element, for which $\{5\} \in B$ makes sense.

3. A student uses the notation $[3, 7]$ to describe the set $\{3, 4, 5, 6, 7\}$. What is wrong with this?

4. In a previous exercise you should have used the notation $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ to describe the set $\{1, \frac{1}{2}, \frac{1}{3}, ...\}$. What set is described by $\{\frac{1}{x} \mid x \in \mathbb{R},\ x \neq 0\}$? Give your answer three ways:

   (a) In words.
   (b) As the union of two intervals.
   (c) As the difference of two sets.

5. In Exercise 16 of Section 2.2 you found that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. For each of the following,

   - write what a "distributive property" would look like,
   - draw a Venn diagram for each side of your equation, shading the described set, and state that the distributive property seems to hold if it does,
   - give *SPECIFIC* counterexamples for any of the "distributive properties" that do not hold.

   (a) $A \cup (B - C)$ (b) $A \cap (B - C)$ (c) $A - (B \cup C)$ (d) $A - (B \cap C)$

6. In plane geometry, four-sided polygons are called *quadrilaterals*. For this exercise we will take the set of quadrilaterals to be the universal set, and we will denote that set by Q (not to be confused with the set $\mathbb{Q}$ of rational numbers). There are a number of specific types of quadrilaterals, some familiar, some not so familiar:

   ◇ *Rectangles* are quadrilaterals with four right angles.

   ◇ *Squares* are quadrilaterals with four right angles and four sides of equal length.

   ◇ *Parallelograms* are quadrilaterals with opposite sides parallel.

   ◇ *Rhombuses* are quadrilaterals with opposite sides parallel and all four sides the same length.

   ◇ *Trapezoids* are quadrilaterals with *exactly* one (no more, no less) pair of opposite sides parallel.

   We will denote the sets of these various types of quadrilaterals by Re, S, P, Rh and T, respectively.

   (a) Give all subset relations within these sets, ignoring Q. There are five of them.
   (b) For each pair of sets that are not disjoint and for which one is not a subset of the other, tell what the intersection of the two sets is. (In each case the answer will be one of the given sets.)
   (c) Draw a Venn diagram that shows the relationships between the various types of quadrilaterals. Don't forget to include the universal set!

7. Refer to Exercise 14 of Section 2.2. There are some more obscure types of triangles as well:

   ◇ *Scalene triangles* are triangles in which no two sides are of equal length.

   ◇ *Acute triangles* are triangles with no angle greater than *or equal to* $90°$.

   ◇ *Obtuse triangles* are triangles with one angle greater than $90°$.

   Using the letters S, A, O for these new types of triangles, draw a Venn diagram showing the relationship between all types of triangles described in this exercise *and* Exercise 14 of Section 2.2.

8. (a) Find a *proper* subset A of
   $$\mathbb{Z}$$
   with $|A| \geq 2$ that is closed under addition. Then find another such set that is of a different nature.
   (b) Repeat (a) for subtraction, if possible.
   (c) Repeat (a) for multiplication, if possible.
   (d) Repeat (a) for division, if possible.

9. Repeat the previous exercise, but requiring $|A| = 1$.

10. If possible, find a proper subset of

$$\mathbb{Z}$$

other than $\mathbb{N}$ or $\mathbb{W}$ that is closed under

(a) addition
(b) subtraction
(c) multiplication
(d) division

11. In exercise 2(a) of Section 2.3 you found that the set $[-1, 1]$ is closed under multiplication. Discuss the closure of sets of the form $[-a, a]$ (where $a \in \mathbb{R}$) under multiplication.

12. Consider the sets $A = \{2n \mid n \in \mathbb{N}\}$ and $B = \{3n \mid n \in \mathbb{N}\}$.

(a) Are A and/or B closed under addition? Multiplication?
(b) Is $A \cup B$ closed under addition? Multiplication?
(c) Is $A \cap B$ closed under addition? Multiplication?
(d) Suppose that A and B are two sets that are both closed under a binary operation $\star$. Is it necessarily true that $A \cup B$ is closed under $\star$? How about $A \cap B$?

13. In this exercise you will find a *discrete* proper subset A of $\mathbb{R}$ that is closed under division and contains the number two.

(a) Begin with $A_0 = \{2\}$. Do all possible divisions using two elements from A_0, and add any new numbers that you get to the set A_0 to form a new set, A_1.
(b) Now do all possible divisions with two elements A_1 and form a new set A_2 by adding any new elements obtained to A_1.
(c) Repeat the process with A_2 to get A_3. This is the critical step - you should find that $|A_3| = 5$. That is, the cardinality of A_3 is five.
(d) Continue the process until you can determine the infinite, but discrete, set that you are building, call it A. This set is called the **closure** of the set $\{2\}$ under division; it is the smallest set containing $\{2\}$ that is closed under division. Describe A by listing.
(e) Describe A using set builder notation.

14. (a) Find the closure of the set $\{12, 18\}$ under gcd.
(b) Find the closure of the set $\{12, 15, 20\}$ under gcd.

15. In Exercise 3 of Section 2.5 you found that the set

$$A = \{\emptyset, \{1\}, \{2\}, \{2, 3\}, \{1, 2, 3\}\}$$

is closed under intersection but not under union or difference.

(a) Find the closure of A under union. Is the result closed under intersection?
(b) Find the closure of A under difference. Is it closed under intersection?

16. Suppose that $m \in \mathbb{N}$. Is $2m + 1 \in \mathbb{N}$? Explain, using the term "closed".

17. What is the name of the set $\mathbb{R} - \mathbb{Q}$?

18. There is no reason that the set A or any of the subsets $A_1, ..., A_n$ in a partition need to be finite. Additionally, the number of sets in the partition need not be finite either.

    (a) Give a finite partition $A_1, ..., A_n$ of $\mathbb{N}$ for which at least one of $A_1, ..., A_n$ is infinite. *Include at least two sets.*

    (b) Give an infinite partition $A_1, A_2, A_3, ...$ of $\mathbb{N}$ for which at least one of $A_1, ..., A_n$ is infinite.

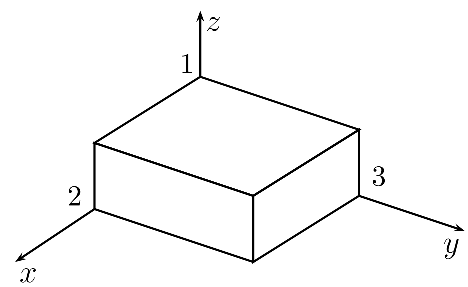
    (c) Give an infinite partition $A_1, A_2, A_3, ...$ of $\mathbb{N}$ for which all of $A_1, A_2, A_3, ...$ are finite.

    (d) Give a partition of $\mathbb{N}$ consisting of three infinite sets.

    (e) Give an infinite partition $A_1, A_2, A_3, ...$ of $\mathbb{N}$ for which all of $A_1, A_2, A_3, ...$ are infinite.

19. (a) Describe the solid box (which includes both its interior and its faces and edges) in $\mathbb{R}^3$ shown to the right as a Cartesian product of intervals.

    (b) Describe the same box, but without the points on the right hand face.



20. Given two sets A and B, the **symmetric difference** of A and B is the set consisting of all elements that are in A or in B, *but not in both.* We denote the symmetric difference of A and B by $A \triangle B$. Note that this is yet another binary operation on sets.

    (a) Find $A \triangle B$ for the sets A and B of Exercise 1 of Section 1.6.

    (b) Find $[1, 3) \triangle (2, 4)$.

    (c) Find $[1, 3) \triangle (5, 6)$.

    (d) For the rest of this exercise, take the sets A and B to be *any* sets, not necessarily the ones you used for parts (a), (b) or (c). Describe $A \triangle B$ two different ways using union, intersection and difference. (**Hint:** Draw a Venn diagram of A and B and shade $A \triangle B$.)

    (e) What are $A \triangle A$, $A \triangle \emptyset$ and $A \triangle U$?

    (f) Sketch a Venn diagram for three sets A, B and C, in which each set intersects each of the other two. Shade $(A \triangle B) \triangle C$, *being careful about the part of the diagram where all three sets intersect.* Draw the same Venn diagram again and shade $A \triangle (B \triangle C)$ on it. Is symmetric difference associative?

    (g) What set is $A \triangle B$ if A and B are disjoint?

    (h) What set is $A \triangle B$ if $A \subseteq B$?

    (i) This operation should look familiar to you if you are a computers major. What operation is it, in "computerspeak"?

3 Functions

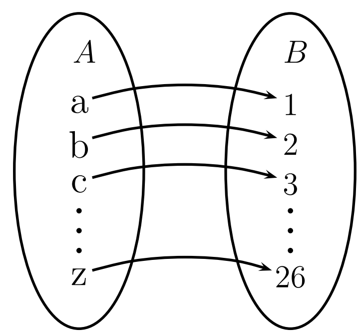
3.1 Introduction

Performance Criteria:

3. (a) Determine whether an assignment rule is a function; if not, tell why.
   (b) Determine the domain, range and target set of a function.
   (c) Give several examples of functions whose domains are not sets of numbers.

Those of you who have taken a physics class, or perhaps even a high school physical science class, know that light is a rather elusive concept. It has been found that for some purposes light can be thought as rays emanating from a source. This works well when thinking about things like reflection or refraction of light. However, when examining the fact that light is actually made up of parts (spectra, the wavelengths making up light), it is more productive to think of light as being made up of waves.

Similarly, we will find that there is more than one way to think of the concept of a function. In this chapter we will look at functions in the way that is probably familiar to all of you. That is, we think of a **function** as a rule for assigning to each element of a set A an element of a second set B. Let's look at a concrete example. Suppose that $A = \{a, b, c, d, ..., z\}$ and $B = \mathbb{R}$. A simple rule would be to assign to a the number 1, to b the number 2, and so on. The diagram to the right illustrates this.



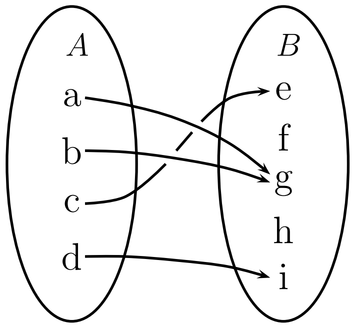
The function is *the rule AND the set* A. (We must specify those two - when the second set B is not specified, we can take it to be any set that contains *at least* all elements that the function assigns to elements of A, maybe more.) There are two conditions that must be met here for our rule to be a function:

(i) The rule must assign something in B to *every* element of A.

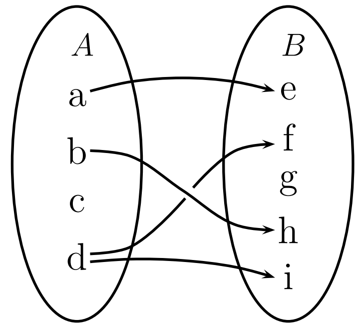
(ii) The rule cannot assign to an element of A more than one element in B.

1. On the next page are diagrams showing ways to assign to each element of $A = \{a, b, c, d\}$ an element of a set B whose elements are shown in each case. (Assume that the elements shown are *all* the elements of that set B.) For each diagram, determine whether the assignment rule shown is a function. That is, does it satisfy (i) and (ii) above? If not, tell which is (are) violated, and how.

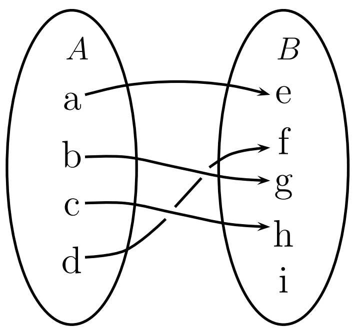
(a)



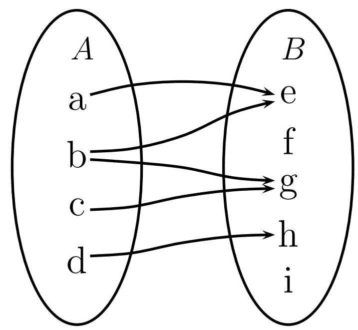
(b)



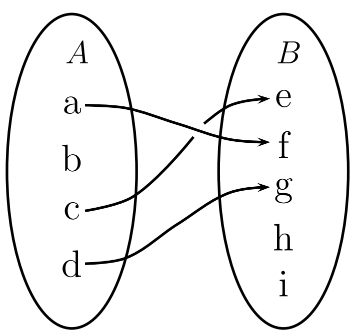
(c)



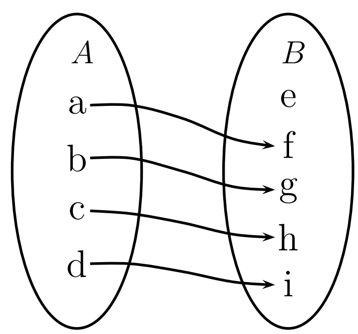
(d)



(e)



(f)



Notice here that we have to be careful not to read into (i) and (ii) anything that is not there. As you should now see, it is possible that

- different elements of A can be assigned the same element of B (Exercise 1(a)), and
- not every element of B has to be assigned to an element of A (Exercises 1(a),(f)).

We will always name our functions, usually (but not necessarily always) with the lower case letters f, g and h. Suppose that the function described before Exercise 1 is f. Then we indicate that f assigns 3 to c by writing $f(c) = 3$. If we use a different rule, we use different letter. For example we could again let $A = \{a, b, c, d, ..., z\}$ and $B = \mathbb{R}$, and assign to each vowel the number 0 and to each consonant the number 1. (Remember vowels and consonants?) This is a function because every letter will be assigned a value (every letter is either a vowel or a consonant) and no letter can be assigned more than one value (no letters are both vowels *and* consonants). If we call this function g we then have $g(m) = 1$ and $g(u) = 0$. (Of course there could be a little confusion concerning $g(g) = 1$, but we will rarely, if ever, see such a problem again.) Notice that f and g both assign real numbers to letters of the alphabet, but the assignment rules used are different.

Now for a little terminology. The set A is called the **domain** of the function. We will call the B the **target set** of the function. The term "domain" is used universally, but different people have different names for the set B. (I looked in three different books, and each author had a different name for this set. I like the name "target set" because it seems natural when we think of the function as sort of "shooting things from A into B", as illustrated in the diagrams we have been looking at.) Using this language, (i) and (ii) say that every element in the domain must be assigned an element in the target set, and a domain element can only be assigned one element of the target set.

On the other hand, not every element in the target set needs to be "hit" by the function when assigning values to the elements of the domain. The subset of the target set consisting of all elements that *are* assigned to elements of the domain is called the **range** of the function.

For the sake of brevity, we have some notation for the domain and range of a function; the domain of a function f is denoted by $\text{Dom}(f)$ and the range is denoted by $\text{Ran}(f)$. When referring to arbitrary elements of the domain and range of a function we generally use x for domain elements and y for range elements. Thus the domain of a function might be $\{x \in \mathbb{R} \mid \text{blah, blah}\}$ and the range might be $\{y \in \mathbb{R} \mid \text{yada, yada}\}$. As an example, consider the function $h(x) = x^2 + 3$, where x is only allowed to have *integer* values. Then

$$\text{Dom}(h) = \mathbb{Z} \qquad \text{and} \qquad \text{Ran}(h) = \{3, 4, 7, 12, 19, ...\}$$

2. Consider the two functions f and g described previously. The domain of both is the letters of the alphabet, and the target set for both is the real numbers. Give the range of each, using listing or set builder notation, whichever seems most appropriate.

Prior to this course, you have likely thought of a function as being just a rule telling how the function assigns values, like $f(x) = x^2$. *Whenever we are given a function as just a rule, we will assume that the domain of the function is the largest subset of the real numbers for which the function is defined, and the target set is all of the real numbers.* The range may or may not be all real numbers.

3. Give the domain and range for each of the following functions *using interval notation* and the Dom and Ran notations.

   (a) $f(x) = -x^2$ (b) $g(x) = 2x - 1$

   (c) $h(x) = \sqrt{5 - x}$ (d) $f(x) = \dfrac{1}{x+3}$

4. Restate the domain and range for each of the functions in Exercise 3, using set builder notation. (Just write $\mathbb{R}$ in the case that the set being described is all real numbers. Note that the set consisting of all real numbers except 2 can be clearly and concisely described by $\{x \in \mathbb{R} \mid x \neq 2\}$.

Suppose that we are considering the function f, whose domain is all real numbers, that assigns to each real number its square. We will sometimes show the domain and target set of the function by writing $f : \mathbb{R} \to \mathbb{R}$. So we might write something like

$$\text{Define the function } f : \mathbb{R} \to \mathbb{R} \text{ by } f(x) = x^2.$$

When the same rule is used with different domain sets, *the resulting functions are not the same*. This is because *a function consists of both a rule and a domain.* For example, the function defined by $g : \mathbb{N} \to \mathbb{R}$, $g(x) = x^2$ is not the same as the function f that was just defined. When we use a rule on a set smaller than the largest set that it is mathematically feasible on, we say that we have **restricted** the domain of the function.

5. Restricting the domain of a function will sometimes change the range of the function.
   (a) What are the ranges of the functions f and g just given?
   (b) Is it possible to restrict the domain of f without changing the range? (Of course you should give an example if you answer yes!)

6. Suppose that the domain of each of the functions in Exercise 3 is restricted to the largest subset of $\mathbb{Z}$ that is allowable. Give the domain and range of each function, using any correct notation.

The letter x used in the definition $g(x) = x^2$ is what is called a "dummy variable"; it is simply a letter used for the purpose of showing how the function g assigns values to domain elements. We could just as well use any other letter for the dummy variable. We will usually use x for the dummy variable when the domain is some continuous subset of the real numbers, and n when the domain is $\mathbb{N}$ or $\mathbb{Z}$. Here we would use the dummy variable x with f, and the variable n with g.

We should note at this point that neither the domain nor the target set of a function need be sets of numbers. For example, we could consider the function that assigns to every triangle the perimeter of the triangle. For the purpose of these notes, *after the following exercise we will assume that the target set of every function is some set of numbers.* This will keep us from having to belabor certain points that are not essential to your understanding of functions. The domains of our functions will generally be sets of numbers as well.

7. Describe some more functions whose domains are all triangles. There are at least six or seven that come to mind for me right now!

8. Try to determine another function whose domain is not a set of numbers - the target set may or may not be a set of numbers. Be sure to specify the domain and the rule for assigning a value to each element of the domain. Be creative - try not to just give some minor variation on the example that I just gave!

9. Give a function whose domain *and* target set are not sets of numbers.

3.2 The Floor and Ceiling Functions

Performance Criteria:

3. (d) Evaluate the floor or ceiling function.
   (e) Determine whether a statement about the floor and/or ceiling function is true.
   (f) Create a function making given assignments using the floor or ceiling function.

Given that this course is called *Discrete* Math, we are primarily interested in discrete sets like $\mathbb{N}$, $\mathbb{W}$, and $\mathbb{Z}$. There are some functions that take the real numbers $\mathbb{R}$, which is the most important continuous set, to the integers, which are a classic example of a discrete set. We will see two such functions in this section.

1. (a) Give the set of all integers less than or equal to 2.74 by listing. (Of course you will need to use the ... notation.) What is the largest element of that set?
   (b) Repeat (a) for the number -2.74.
   (c) Repeat (a) for 7.
   (d) Repeat (a) for π.

2. We now define a function whose domain is all real numbers as follows: The function assigns to any real number x the greatest (largest) integer that is less than or equal to x. What will the function assign to

   (a) $3\frac{2}{3}$? (b) -6.1? (c) -11? (d) 1.99?

The function that you have been using is sometimes called the "greatest integer function" (by mathematicians, mostly) or the **floor function** (more common in computer science). We will stick with the latter, so we would say "the floor of -6.1 is -7." Symbolically we will use the symbols $\lfloor\ \rfloor$ to denote the floor of a number, in the same way that we use $|\ |$ to denote the absolute value function. Thus $\lfloor -6.1 \rfloor = -7$. We will see that the floor function can be used to build many other useful functions.

3. Is it possible for $\lfloor x \rfloor = x$? Elaborate.

4. (a) What is the range of the floor function?
   (b) What are the ranges of the functions $f(x) = \lfloor 3x \rfloor$ and $g(x) = 3\lfloor x \rfloor$?
   (c) What is the range of the function $h(x) = x - \lfloor x \rfloor$?

5. (a) Define $f : (0, 5) \to \mathbb{R}$ by $f(x) = \lfloor x \rfloor$. What is $\text{Ran}(f)$?
   (b) Does your answer change if the domain of f is changed to $[0, 5]$? If so, what is the new range?
   (c) Repeat (b) for $[0, 5)$.
   (d) Repeat (b) for $(0, 5]$.

The **ceiling function** is just like the floor function, except that it assigns to any real number x the smallest integer that is *greater than or equal to* x. The notation for the ceiling function is $\lceil x \rceil$.

6. Find each of the following:

   (a) $\lceil 8.7426 \rceil$ (b) $\lceil -5\frac{1}{2} \rceil$ (c) $\lceil 7 \rceil$ (d) $\lceil \pi \rceil$

7. Try finding both $\lfloor x+1 \rfloor$ and $\lceil x \rceil$ for some real numbers. What do you conclude?

8. (a) What is the relationship between $\lfloor -x \rfloor$ and either $\lfloor x \rfloor$ or $\lceil x \rceil$?

   (b) Give a similar relationship for $\lceil -x \rceil$.

9. Give a formula defining the ceiling function in terms of the floor function. **Hint:** Use the results of the previous exercise.

10. For $m, x \in \mathbb{N}$, define a function $f(x) = m\lfloor \frac{x}{m} \rfloor$.

    (a) Find the range of f.

    (b) What does the function f do?

11. Define functions q, $r : \mathbb{W} \to \mathbb{W}$ by $q(x) = \lfloor \frac{x}{5} \rfloor$ and $r(x) = x - 5\lfloor \frac{x}{5} \rfloor$. Describe what each of these two functions does.

12. Use the idea of the previous exercise to get a function that gives the digit in the tens place for any natural number greater than or equal to ten.

13. Use the floor and/or ceiling functions to define a function, whose domain is the set $\{n \in \mathbb{N} \mid 100 \le n \le 999\}$, that reverses the order of the digits of the number.

3.3 Images and Preimages of Sets

Performance Criteria:

3. (g) Determine the image of a set under a function.
   (h) Determine the inverse image of a set under a function.
   (i) Determine whether a statement about unions or intersections of images or inverse images of sets is true. If not, give appropriate counterexamples.

We now introduce a new concept relating to functions. Suppose that $f : A \to B$ is a function, and $S \subseteq A$. If we restrict f to S and look at its range, it is a subset of B that we call the **image** of S under f. We denote the image of S under f by $f(S)$. Note that $f(x)$ is an *element* of the target set of f, whereas $f(S)$ is a *subset* of the target set. Formally,

$$f(S) = \{y \in B \mid \text{there exists an } x \in S \text{ for which } f(x) = y\}.$$

1. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x - 5$. Find $f([3, 7])$ and $f(\{1, 2, 3\})$.
   (b) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2$. Find $g([3, 7])$ and $g([-1, 5))$.
   (c) Let $f : \mathbb{R} \to \mathbb{Z}$ be defined by $h(x) = \lfloor x \rfloor$. Find $h([3, 7])$.

2. Draw a diagram something like those in Section 3.1 that illustrates the sets S and $f(S)$.

3. Suppose that $f : A \to B$ is a function and S and T are subsets of A. Which of the following seem to be true? For any that aren't, give appropriate counterexamples and try to determine a similar statement that *IS* true by replacing $=$ with $\subseteq$ or $\supseteq$.

   (a) $f(S \cup T) = f(S) \cup f(T)$ (b) $f(S \cap T) = f(S) \cap f(T)$

Suppose that we have a function $f : A \to B$ and $T \subseteq B$. We define the **preimage** of T under f to be the set of all x in A such that $f(x)$ is in T. It is denoted by $f^{-1}(T)$; a more concise definition can be given symbolically:

$$f^{-1}(T) = \{x \in A \mid f(x) \in T\}.$$

4. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x - 5$. Find $f^{-1}(\{1, 2, 3, 4\})$ and $f^{-1}([4, 7])$.
   (b) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2$. Find $g^{-1}(\{7\})$ and $g^{-1}([4, 49])$.

5. Let $g : \mathbb{R} \to \mathbb{Z}$ be the floor function $g(x) = \lfloor x \rfloor$. Find $g^{-1}(\{3\})$ and $g^{-1}(\{-5, 9, 10\})$.

6. Draw a diagram something like those in Section 5.1 that illustrates the sets T and $f^{-1}(T)$.

7. Suppose that $f : A \to B$ is a function, $S \subseteq A$ and $T \subseteq \text{Ran}(f)$. Which of the following seem to be true? For any that aren't, give appropriate counterexamples and try to determine a similar statement that *IS* true by replacing $=$ with $\subseteq$ or $\supseteq$.

   (a) $f^{-1}[f(S)] = S$ (b) $f[f^{-1}(T)] = T$

8. Suppose that $f : A \to B$ is a function and S and T are subsets of B. Which of the following seem to be true? For any that aren't, give appropriate counterexamples and try to determine a similar statement that *IS* true.

   (a) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$

   (b) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

3.4 Functions From Functions

Performance Criteria:

3. (j) For given functions f and g and values of x, evaluate $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$.
   (k) For given functions f and g, give the domains of $f+g$, $f-g$, fg and $\frac{f}{g}$.
   (l) For given functions f and g and values of x, evaluate $(f \circ g)(x)$ and $(g \circ f)(x)$.
   (m) For given functions f and g, give the domains of $f \circ g$ and $g \circ f$.

Remember that a function (at least the way that we are thinking of functions right now) consists of

- ◇ a "rule" for obtaining an "output" from a given "input",
- ◇ a set of objects that constitute the allowable "inputs" of the function.

We give the above information as two statements, for example

$$\text{define } f : [0, \infty) \to \mathbb{R} \text{ by } f(x) = x^2.$$

The first statement tells the objects that the function will accept as inputs, the domain. (In this case the domain is $[0, \infty)$, by my choice.) The second statement is the rule that tells us how to get outputs from inputs.

Any time that we are considering a set of mathematical objects, one of the first considerations is whether there are ways to combine two of those objects to get a new object of the same kind. That is, can we define some *binary operations* on the object? If we were to define binary operations on functions, they would necessarily have to attend to how we combine the rules for two function *AND* how to combine the domains.

In this section we will see five (or seven, depending on how you count) binary operations for combining two functions to get a new function. They should all be familiar to you.

1. Give as many binary operations on two sets that you can think of.

The first binary operations on two functions f and g that we will consider are the functions $f+g$, $f-g$, fg and $\frac{f}{g}$. Since these are to be obtained from f and g we must tell how each of four functions works, *in terms of how f and g work.* To this end, we define

$$(f+g)(x) = f(x) + g(x)\ , \qquad (f-g)(x) = f(x) - g(x)\ ,$$

$$(fg)(x) = f(x)g(x)\ , \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

We will ignore for now the issue of how to combine the domains of f and g to obtain the domains of these functions.

2. Technically there are two more functions here, $g - f$ and $\frac{g}{f}$. What is it about the operations of subtraction and division that cause the need to say this?

3. Let $A = \{a, b, c, ..., z\}$ and $f : A \to \mathbb{R}$ again be the function defined by

$$f(a) = 1, \quad f(b) = 2, \quad f(c) = 3, \quad ..., \quad f(z) = 26.$$

Let $g : A \to \mathbb{R}$ be defined by assigning 0 to vowels and 1 to consonants. Find each of the following.

(a) $(f + g)(j)$ (b) $(fg)(e)$ (c) $\left(\frac{g}{f}\right)(c)$

(d) $\left(\frac{g}{f}\right)(i)$ (e) $\left(\frac{f}{g}\right)(i)$

4. When combining two functions with algebraic equations, we define the new functions $f+g$, $f-g$, fg and $\frac{f}{g}$ by finding algebraic equations that define each of these. This is done by simply combining the equations of f and g as directed by the definitions of $f+g$, $f-g$, fg and $\frac{f}{g}$. Consider the functions

$$f(x) = x^2, \qquad g(x) = \frac{x+3}{x-2},$$

with no regard for their domains for the time being. Find and simplify algebraic equations for each of the following, using the definitions given above.

(a) $f + g$ (b) $f - g$ (c) fg (d) $\frac{f}{g}$

We now consider the issue of how to combine $\text{Dom}(f)$ and $\text{Dom}(g)$ to get the domains of $f+g$, $f-g$, fg and $\frac{f}{g}$. We will assume that we want to find the least restrictive domain for each of these, given the domains of f and g.

5. Go back to Exercise 3 and note that both f and g have the same domain, A. Part (e) illustrates that A is not necessarily the domain of all four functions $f+g$, $f-g$, fg and $\frac{f}{g}$.

(a) Which of the functions $f+g$, $f-g$, fg and $\frac{f}{g}$ *CAN* be evaluated for all values in A?

(b) Which of the functions $f+g$, $f-g$, fg and $\dfrac{f}{g}$ *CANNOT* be evaluated for all values in A? What *ARE* the domains of those functions?

The two functions f and g from the previous exercise were a little unusual in that both have the same domain. The question you should now be asking is "what are the domains of $f+g$, $f-g$, fg and $\dfrac{f}{g}$ when the domains of f and g are not the same. The next exercise should help you answer this question.

6. Consider the functions

$$f(x) = \frac{3}{x-3}, \qquad g(x) = \sqrt{x+2}.$$

   (a) What are the domains of f and g?

   (b) What are the domains of $f+g$, $f-g$ and fg? How do we obtain the domains of those functions from the domains of f and g?

   (c) What is the domain of $\dfrac{f}{g}$? How do we obtain it from the domains of f and g?

7. Suppose now that we have two arbitrary functions $f : A \to C$ and $g : B \to D$. (A is no longer the set of the previous two exercises.) What are the domains of $f+g$, $f-g$, fg and $\dfrac{f}{g}$? Use the notations $\text{Dom}(f)$, $\text{Dom}(g)$, $\text{Dom}(f+g)$, etc. when stating your answers.

Recall that with sets we have methods for getting new sets by somehow "combining" two existing sets, but we can also get a new set from *one* existing set by taking the complement of the original set. There is a very important way to take one function and create from it a new function, called its inverse. We will get to that in Section 5.5. Let's finish this section with probably the most important way to combine two functions to get a new function.

8. Consider the functions $f(x) = 3x$ and $g(x) = x-2$. Find $g(7)$, then find f of that result; that is, find $f[g(7)]$.

In the above exercise you used two existing functions to create a new function by applying g, then applying f to the result. This new function is called the **composition** of f and g, or "f of g". It is denoted by $f \circ g$; you just determined $(f \circ g)(7)$.

9. (a) For the same f and g, find $(g \circ f)(-1)$. *Note the order in which the functions are applied!*

   (b) Find $(g \circ f)(7)$ and compare with your result from the previous exercise. Comment on what you see. What does this show in general?

   (c) You should recall from Math 111 that if we wish to compute $f \circ g$ for a number of values, it is most efficient to first determine a *simplified* formula for $(f \circ g)(x)$ algebraically, then apply it. For example, in this case

$$(f \circ g)(x) = f[g(x)] = f(x-2) = 3(x-2) = 3x-6.$$

   Do this for $g \circ f$.

10. Consider the functions $f(x) = \sqrt{x}$ and $g(x) = x + 1$

   (a) Find any of $(g \circ f)(9)$, $(f \circ g)(3)$ and $(f \circ g)(-5)$ that you can. What problem arises here?

   (b) What are the domain and range of f? Of g?

   (c) What conditions must be met on the domains and/or ranges of two functions f and g so that $f \circ g$ can be found for every value in the domain of g?

   (d) Give the least restrictive domain for g so that $f \circ g$ can be found for every element in that domain. *This is the domain of* $f \circ g$.

   (e) Why is it not necessary to restrict the domain of f for the composition $g \circ f$? What is the domain of $g \circ f$?

11. For $f(x) = x - 7$ and $g(x) = \frac{1}{x}$, find the domains of $f \circ g$ and $g \circ f$.

12. See if you can describe how to get $\text{Dom}(f \circ g)$ from $\text{Dom}(f)$ and/or $\text{Dom}(g)$.

3.5 One-To-One and Onto Functions, Inverses

Performance Criteria:

3. (n) Determine whether a function is one-to-one, onto, or both.
   (o) Modify the domain and/or target set of a function to make it one-to-one, onto, or both.
   (p) Determine whether a function has an inverse. If it does, find it and give its domain and range.

Some functions exhibit certain behaviors that make them very nice to work with in certain ways. In this section we will study those behaviors.

1. Consider the two functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$, $g(x) = 2x + 3$.

   (a) The target set of each function is $\mathbb{R}$. Find the range of each.

   (b) Choose a value y in the range of f. How many values of x are there such that $f(x) = y$? What are they? Is that the case for all values of y in the range of f?

   (c) Repeat (b) for g.

The above exercise illustrates the first two ideas in the title of this section. When the range of a function is the entire target set, we say that the function is **onto**. f is NOT onto, g is. A fancier term than onto is to say that g is a *surjection*, but we will stick with the term onto. If we think of a function as something that sort of "shoots" things from the domain into the target set, an onto function is one that "shoots" something onto every element of the target set. Thinking of a function as a rule that assigns elements of a target set to values in a domain set, an onto function is one for which each and every element of the target set is assigned to some domain value.

The exercise also showed that for each element y in the range of g there is only one domain element x such that $g(x) = y$. A function of this type is called **one-to-one**, meaning that each range element is assigned to only one domain element. (The fancy term for this is *injection*.) Although we are usually only concerned with whether or not a function is one-to-one, we could say in this case that f is two-to-one on all of its domain except zero. A function like the sine function from trigonometry is sometimes said to be many-to-one. (Infinitely many, in fact!)

2. You should notice that the function f from the previous exercise is neither one-to-one or onto, and g is both one-to-one *AND* onto. A natural question is whether a function can be one-to-one but not onto, or vice-versa.

   (a) Consider $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(x) = 2x$. Is f one-to-one? If f onto?

   (b) Consider $g : \mathbb{R} \rightarrow [-1, 1]$ defined by $g(x) = \sin x$. Is g one-to-one? If g onto?

(c) Consider $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = 2x$. Is h one-to-one? If h onto? *Compare this to part (a) - this illustrates the importance of the domain as a defining characteristic of a function!*

3. Determine whether each of the following functions is one-to-one and/or onto. *Make careful note of the domain and range of each.*

   (a) $f : \mathbb{R} \to \mathbb{Z}$ defined by $f(x) = 2\lfloor x \rfloor$

   (b) $g : \mathbb{R} \to \mathbb{Z}$ defined by $g(x) = \lfloor 2x \rfloor$

   (c) $h : \mathbb{Z} \to \mathbb{Z}$ defined by $h(x) = 3\lfloor \frac{x}{3} \rfloor$

4. Go back to Exercise 1 of Section 3.1. For each rule that *IS* a function, determine whether it is one-to-one, onto, or both.

5. What connection is there between the concepts of a one-to-one function and the preimage of a set under a function?

Even if a function is not one-to-one or onto, the domain and range of a function can usually (always?) be modified to make a function one-to-one and onto. This process is important, because functions that are both one-to-one and onto have a very nice property that you will see soon. The following exercises will describe how to modify the domain and/or range of a function to make it one-to-one or onto.

6. Often we can modify a function to make it one-to-one, onto, or both. A function can be made to be onto by simply "cutting down" the target set to the range. So if we say $f : \mathbb{R} \to [0, \infty)$ for the function f of Exercise 1 above, then f is an onto function. What should we make the target set of $h(x) = \sqrt{x}$ in order for it to be onto?

7. For each of the assignment rules of Exercise 1 of Section 3.1 that is a function, give a target set for which the function is onto.

8. We can also make the function f of Exercise 1 (of this section) one-to-one if we restrict its domain. Give two different restrictions of the domain of f that make it one-to-one, *without changing the range of* f.

9. For each of the following functions, determine first whether the function is onto the target set $\mathbb{R}$. If not, give the range of the function. Then determine whether the function is one-to-one and, if not, give a restricted domain (but restricted by the least amount necessary) for which the function is one-to-one. (Again, make it large enough that the range is not changed.)

   (a) $f(x) = -x^2$

   (b) $g(x) = 7x$

   (c) $h(x) = \sqrt[3]{x}$

   (d) $f(x) = \sqrt{5 - x}$

   (e) $g(x) = \dfrac{1}{x + 3}$

   (f) $h(x) = \dfrac{x^2}{x^2 + 1}$

10. Consider again the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 2x + 3$, which is both onto and one-to-one. Find a function $h : \mathbb{R} \to \mathbb{R}$ that takes every value in the range of g back to the domain value that it "came from". In other words, since $g(1) = 5$, $g(-7) = -11$ and so on, we must have $h(5) = 1$, $h(-11) = -7$, etc.

A function that is both one-to-one and onto is called a **bijection**. The important thing about bijections is this: Given a value in the target set of a bijection, there is *one, and only one* domain value that it came from. Again there is a domain value that it came from, and there is only one value that it came from. Now go back and look at the two conditions that are necessary for a function. What I essentially just said is that for a bijection there is a *function* that takes target set values back to domain values. This function is called the **inverse function** of the original function. For the exercise you just did, h is the inverse function of g. Rather than use two different letters, we write g^{-1} instead of h to indicate that it is the inverse function of g. Notice that g^{-1} DOES NOT mean $\dfrac{1}{g(x)}$!

11. Consider the functions g and h from the previous exercise. Find and simplify formulas for $(g \circ h)(x)$ and $(h \circ g)(x)$, in the manner of Exercise 9(c) of Section 3.4.

The previous exercise illustrates part of what it means for two functions to be inverses. Technically speaking, two functions f and g are inverses if

(i) The range of f is the domain of g and the range of g is the domain of f, and

(ii) $(f \circ g)(x) = x$ for all x in the domain of g and $(g \circ f)(x) = x$ for all x in the domain of f.

12. (a) The function $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = x^2$ is a bijection. What is f^{-1}?
    (b) The function $f : (-\infty, 0] \to [0, \infty)$ defined by $f(x) = x^2$ is also a bijection. (Technically this is a different function than the one from part (a), so I should really use a different letter for it but ...) What is f^{-1} in this case?

13. Give the domain and range for which each function of Exercise 9 is a bijection by writing $f : A \to B$, where A is the restricted domain and B is the range. Then give the inverse function.

3.6 Chapter 3 Exercises

Performance Criteria:

3. (q) Determine whether two integers are relatively prime.
   (r) Evaluate the Euler ϕ-function.
   (s) Use functions defined in this chapter, along with the sign function, to give function equations.
   (t) Evaluate a piecewise-defined function or create a piecewise-defined function making given assignments.
   (u) Evaluate a characteristic function or use a characteristic function to create a function making given assignments.

1. (a) The **greatest common divisor** (gcd) of two integers a and b is the largest integer that divides into both a and b. This concept is used when doing arithmetic operations like reducing fractions. We say that two integers are **relatively prime** if their gcd is 1. Give two integers that are relatively prime, and two that are not. Give the gcd for those that are not.

   (b) Which natural numbers less than 7 are relatively prime to 7?

   (c) Which natural numbers less 6 are relatively prime to 6?

   (d) We now define a function ϕ whose domain is the natural numbers as follows: for $n \in \mathbb{N}$, $\phi(n)$ is the number of natural numbers less than n that are relatively prime to n. Thus $\phi(7) = 6$ and $\phi(6) = 2$. This function is called the **Euler ϕ-function** (ϕ is pronounced "fee"), after the Swiss mathematician Leonhard Euler (pronounced "oiler"). What are $\phi(8)$, $\phi(12)$ and $\phi(17)$?

   (e) What is a natural question to ask about the function ϕ? *Think about the concepts discussed in this chapter.*

NOTE: The definition I just gave for the Euler ϕ-function is missing one important detail that I left out because it had no bearing on the exercise. ϕ is defined as described *for all natural numbers greater than or equal to two*, but $\phi(1) = 1$. I don't know why this is, but such things are not done unless there is a good reason for them!

2. Some functions are difficult or impossible to define with a single rule, but can easily be defined by giving two or more rules, each applying to different elements in the domain. Such functions are called **piecewise-defined** functions. An example is the function

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ x+3 & \text{if } x < 0. \end{cases}$$

Then $f(5) = 5^2 = 25$ since $5 \geq 0$ and $f(-2) = -2 + 3 = 1$ since $-2 < 0$.

(a) Consider the function $g : \mathbb{N} \to \mathbb{R}$ for which $g(1) = 0$, $g(2) = \frac{1}{2}$, $g(3) = 0$, $g(4) = \frac{1}{4}$, $g(5) = 0$, $g(6) = \frac{1}{6}$, etc. Give a piecewise definition of this function.

(b) Consider the piecewise defined function $h : \mathbb{R} \to \mathbb{R}$ given by

$$h(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

This is a function that you should know in some other form. Can you see what it is? If not, try putting in a few values for x to see what you get.

(c) Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

Give a single equation that defines this function. Do the same for $g : \mathbb{Z} \to \mathbb{Z}$ defined by

$$g(n) = \begin{cases} -1 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

These two simple functions can be very useful in building other functions!

3. The function $f : \mathbb{N} \to \mathbb{Z}$ assigns the following values:

$$1 \to 0\ , \qquad 2 \to 3\ , \qquad 3 \to -3\ , \qquad 4 \to 6\ , \qquad 5 \to -6\ , \ ...$$

Give a piecewise definition of this function.

4. Consider the function $f : \mathbb{N} \to \mathbb{W}$ that makes the following assignments:

$$1 \to 0\ , \quad 2 \to 0\ , \quad 3 \to 1\ , \quad 4 \to 1\ , \quad 5 \to 2\ , \quad 6 \to 2\ , \quad 7 \to 3\ , ...$$

(a) Is f onto? Is it one-to-one?

(b) Give a piecewise definition of the function, *without using the floor or ceiling functions.*

(c) If f is a bijection, give its inverse. If it is not, tell how to restrict its domain and/or range appropriately to obtain a function that is. Then give the inverse of that function.

(d) Give a *single* equation for (the original) f, using the floor and/or ceiling function.

5. Consider the function $h : \mathbb{Z} \to \mathbb{Z}$ defined by

$$h(n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

(a) Give a formula, using functions you saw in Exercise 2, for this function.

(b) Give two functions f and g for which $h = f \circ g$. Be sure to get the order correct!

6. We define the **signum function** on the real numbers by

$$\text{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Use this function and any others you learned in this chapter to give a *single* formula for the function $f : \mathbb{R} \to \mathbb{Z}$ that makes the following assignments:

$$43.7 \to 43\ , \qquad -18\frac{2}{3} \to -18\ , \qquad -2.94 \to -2\ , \qquad 1412.8333... \to 1412\ , \qquad 0 \to 0\ , ...$$

7. Give the domain and range for each of the following functions.

(a) $g(x) = \dfrac{\sqrt{x}}{x-5}$ (b) $h(x) = \dfrac{x^2}{x^2+1}$

8. Let A be a subset of some set B. We define the **characteristic function** of A, denoted χ_A and having domain B, by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

(a) Let $B = \mathbb{R}$ and $A = [2, 5]$. Find $\chi_{[2,5]}(7)$, $\chi_{[2,5]}(-4.5)$ and $\chi_{[2,5]}(\pi)$,

(b) Use the characteristic function to give a single equation defining the function from Exercise 2(a). (You will need to define a set A.)

(c) Use the characteristic function to give a single equation defining the first function from Exercise 2(c).

9. For a fixed value of $n \geq 0$, define $\text{rnd}[n](x) : \mathbb{R} \to \mathbb{R}$ for any x to be the number obtained by rounding x to n places past the decimal. (So, for example, rnd[3] and rnd[7] are two separate functions and $\text{rnd}[3](5.86572) = 5.866$.)

   (a) Find $\text{rnd}[2](3.7486)$ and $\text{rnd}[0](\pi)$.

   (b) Using the floor function, give an equation that defines rnd[0] for $x \geq 0$.

   (c) Give an example showing that your formula for (b) is not always valid for negative values.

   (d) Explain how the composition function $\text{rnd}[n] \circ \text{rnd}[n+1]$ works. (**Hint:** Pick a value of n, then try it on some numbers.

   (e) Is $(\text{rnd}[n] \circ \text{rnd}[n+1])(x) = \text{rnd}[n](x)$?

NOTE: The value N for the functions from the previous exercise can vary from function to function, but it is fixed for a given rnd function. Such a value is usually referred to as a **parameter** of the function.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be the absolute value function $f(x) = |x|$. Find $f([-8, 1])$ and $f((-8, -2])$.

11. Consider the function $f(x) = 3 - x^2$.

   (a) Find the ordered pairs $(x, f(x))$ for $x = -2$, -1, 0, 1, 2.

   (b) Plot those points on the Cartesian plane $\mathbb{R} \times \mathbb{R}$.

   (c) sketch what you think you would see if you graph the ordered pairs $(x, f(x))$ for all values of x on the portion of the x-axis you've drawn. The result is "the graph of f".

12. Sketch the graph of $g(x) = \lfloor 2x \rfloor$.

4 Proofs

4.1 The Real Numbers

Performance Criteria:

4. (a) Prove that two algebraic statements are equal, using the axioms of the real numbers.

In order to do any useful mathematics with real numbers we need to perform various operations on them. For example, we might decide that $5 + (x + 3) = x + 8$. When performing such operations, we follow various "rules"; for example, we all "know" that for any two real numbers a and b, $a + b = b + a$. Some of these rules are simply taken to be true, while others can be demonstrated to be true based on previously assumed rules. Rules that are simply assumed to be true are called **axioms** (or, sometimes, **postulates**). Rules that can be shown to be true based on previously assumed axioms or other previously demonstrated rules are called **theorems**. The word "property" is sometimes substituted for axiom or theorem (so we must know from experience which of the two it might be).

In mathematics we call the act of showing something is true based on previously known facts is called a **proof**. It is simply an *organized and logical* argument of why something is true. Written proofs can be presented in various ways. Most mathematicians use a format called a paragraph proof, but we will primarily use a method we'll call a "two-column" proof. The left column will contain a linear progression of statements, and the corresponding entry in the right hand column will be the justification for that particular entry. We will call the left column the "statement column", and the right column the "reason column".

OK, so how about an example? Before beginning, we need to decide what our basic axioms are. We will begin with the assumption that there is a binary operation that we can perform on two real numbers, called addition, that results in another real number. Of course we use the symbol $+$ for addition. The basic axioms of addition are as follows:

1) The **commutative property**: For any two real numbers x and y, $x + y = y + x$.

2) The **associative property**: For any three real numbers x, y and z, $(x + y) + z = x + (y + z)$.

3) **Additive identity**: For any real number x, $x + 0 = x$.

4) **Additive inverses**: Corresponding to any real number x there is a real number $-x$ such that $x + (-x) = 0$.

Let's note a few things before we proceed. First, addition is a binary operation, so it operates *two* real numbers to give a third. Therefore $w + x + y + z$ has no meaning, from a technical point of view. When we see such an expression, we will all agree that it really means

$$((w + x) + y) + z.$$

That is, we first add only w and x, resulting in a *single* real number. That number is then added to y, resulting in another single number. Finally, that number is added to z. This assumed order in which we perform each addition is part of a general set of rules we call **order of operations**.

The commutative law is only stated for two numbers, so we can only use it on two at a time. Thus we can't simply determine that $3x+5y+4 = 5y+4+3x$. We must really apply the previous discussion and the axioms as follows:

$$3x + 5y + 4 = (3x + 5y) + 4 = 3x + (5y + 4) = (5y + 4) + 3x = 5y + 4 + 3x. \qquad (1)$$

Finally, for a given number x, the number $-x$ is called the **negative** of x. Note that if $x = -3$, then the negative of x is 3; thus the negative of a given number is not necessarily a negative number!

Now we will see a two column proof of (1). In the left column we will write out the steps of (1) in a vertical arrangement that you should already be familiar with. The justification for each step will be given as the corresponding entry in the right hand column. We will number the steps, for reasons that will become more apparent in future sections.

	Statement	Reason	
1.	$3x + 5y + 4 = (3x + 5y) + 4$	order of operations	
2.	$= 3x + (5y + 4)$	associative property	
3.	$= (5y + 4) + 3x$	commutative property	
4.	$= 5y + 4 + 3x$	order of operations	$\square$

A big part of knowing how to do a proof is understanding how much detail your reader expects of you. For the algebraic proofs of this section, I will expect a distinct step any time that you apply any axiom. However, it will get a bit tedious to recognize the grouping dictated by order of operations every time that it occurs, so we will eliminate showing it when it is used to insert perentheses. So the proof above becomes

	Statement	Reason	
1.	$3x + 5y + 4 = 3x + (5y + 4)$	associative property	
2.	$= (5y + 4) + 3x$	commutative property	
3.	$= 5y + 4 + 3x$	order of operations	$\square$

1. Prove that $5 + x + 7 = x + 12$. Your proof should contain four steps, with the reason for the last step being "arithmetic" (adding 5 and 7 to get 12).

2. Prove that $x + 6 + x + 1 = 7 + x$.

You may or may not have noticed in the previous discussion that the expression $3x+5y+4$ contains two multiplications. Since I did not address the binary operation of multiplication there, let me do that now. We take multiplication to be a second *binary* operation, with x multiplied by y denoted by xy or $xcdoty$, having the following axioms:

5) The **commutative property**: For any two real numbers x and y, $xy = yx$.

6) The **associative property**: For any three real numbers x, y and z, $(xy)z = x(yz)$.

7) **Multiplicative identity**: For any real number x, $x \cdot 1 = x$.

8) **Multiplicative inverses**: Corresponding to any real number x there is a real number x^{-1} such that $x \cdot x^{-1} = 1$.

Note that each of these corresponds to a similar property for addition. Technically we should distinguish by writing "commutative law of addition" or "commutative law of multiplication", depending on which operation is being considered, but I will not require you to do that. In addition to all the axioms given so far, there is an additional axiom that tells us how addition and multiplication interact:

9) **Distributive property**: For any real numbers x, y and z, $x(y+z) = xy + xz$.

We often use the distributive property form "right to left." That is, $xy + xz = x(y+z)$.

There are some order of operations issues with multiplication as well. As with addition, we take xyz to mean $(xy)z$. Also, multiplication takes precedence over addition, in the sense that $x + yz$ means $x + (yz)$. You are already familiar with these order of operations rules.

We will now show how to use the axioms of addition and multiplication to show that $5x + 7 + 4x = 9x + 7$:

	Statement			Reason
1.	$5x + 7 + 4x$	$=$	$5x + (7 + 4x)$	associative property
2.		$=$	$5x + (4x + 7)$	commutative property
3.		$=$	$(5x + 4x) + 7$	associative property
4.		$=$	$(5 + 4)x + 7$	distributive property
5.		$=$	$9x + 7$	arithmetic $\square$

Technically speaking, step 5 is incorrect! The distributive property is really sort of a "left distributive property", telling us that we can factor out a common factor that is on the left of two other numbers. Let's fix that now with an exercise:

3. Prove the right distributive property: For all real numbers x, y and z, $xz + yz = (x + y)z$.

Now that we have taken care of this, we will take the distributive property to be either the original version stated (left) or this new version (right).

4. Prove that $7(3x) = 21x$. The proof is short, but still takes two steps!

5. Prove that $7x + 3(x + 5) = 10x + 5$

Defining and Using Subtraction

In the cases of addition and multiplication, we simply assumed that such operations existed. we will now create the binary operation of subtraction, denoted for any two real numbers x and y by $x - y$:

> **Definition of Subtraction**: For any two real numbers x and y we define $x - y = x + (-y)$.

Let's do a few exercise concerning subtraction, then we'll move on!

6. Prove that $x-4-x+10 = 6$, using only the properties of addition and multiplication, and the definition of subtraction. *Be sure to consider order of operations, and you will need to use properties 3 and 4!*

7. It can be proven that $0x = 0$, which you will need for the following. Refer to it as the "multiplication by zero property." Supply reasons for the steps in the following proof, where both a and x are real numbers:

	Statement	Reason
1.	$ax + (-a)x = [a + (-a)]x$	
2.	$= 0x$	
3.	$= 0$	

This shows that $(-a)x$ is the inverse of ax, or $-ax = (-a)x$. let's call this the "subtraction of a product property", and its use is now fair game.

8. Prove that $8x + 2 - 3x = 5x + 2$

9. Prove that the distributive property holds for subtraction as well. That is, prove that for any real numbers x, y and z, $x(y - z) = xy - xz$.

10. Prove that $2x + 3(x - 4) = 5x - 12)$.

11. Prove that $7 - 4(x - 2) = -4x + 15$.

4.2 Definitions

Performance Criteria:

4. (b) Demonstrate that a mathematical object meets the conditions of a definition.

In the rest of this chapter our goal is to validate mathematical truths, like

the product of two odd numbers is odd.

We take this statement to mean the product of *any* two odd numbers is odd. If we meant just two *particular* odd numbers, we could simply multiply them and see that the result is odd. Moreover, if the set of odd numbers was finite, we could simply multiply together all pairs of odd numbers and check to see if the result was odd in each case. *But the set of odd numbers is not finite, so we can't take this approach to show that the above statement is true.*

Our approach to verifying the truth of the statement will be to start with two numbers that we know are odd, but about which we know nothing else. This is what covers the words "*any* two odd numbers". We then multiply those numbers, and check to see if the result is odd. In order to do this we will need to have a very precise definition of an odd number. We all think we know what an odd number is, but what is it exactly that makes a number odd? In this section we will look at some definitions that we will be using. Good definitions meet three basic conditions:

- Every definition is a biconditional statement. For example, the definition of a cat tells us what characteristics something should have if we know it is a cat to begin with. Conversely, if we have something with the characteristics of a cat, it must be a cat.
- A definition should be precise, in the sense that it should contain previously defined terms or *commonly accepted* undefined terms that convey the idea of the definition clearly.
- A definition should be concise, using the minimum amount of words necessary to clearly convey the idea.

The concise nature of definitions sometimes leaves a person wondering if they really understand what is being defined. It is always helpful when encountering a new definition to gather some examples of the thing being defined, *along with some non-examples.*

For a mathematical example, the definition of a subset can be given as follows:

$$A \subseteq B \text{ if, and only if, } x \in A \text{ implies } x \in B$$

The fact that this is a biconditional means that

(a) if we know A is a subset of B, then every x that is in A must be in B as well, and

(b) if every x that is in A is in B also, then $A \subseteq B$.

Note that the definition appeals to two previously undefined ideas, set and element.

NOTE: Every definition is technically a biconditional statement, so it should include the phrase "if, and only if" when it is stated in words. It gets a bit cumbersome to write this every time we make a definition, so it has become standard practice to write definitions as conditional statements, with the understanding that each such statement should really be taken to be biconditional. From this point on these notes will follow this custom. Here are some definitions that we will use in this chapter:

- ⋄ **Even Number:** A number x is even if there exists an integer m such that $x = 2m$.
- ⋄ **Odd Number:** A number x is odd if there exists an integer m such that $x = 2m+1$.
- ⋄ **Rational Number** A number x is a rational number if there exist integers m and $n \neq 0$ such that $x = \frac{m}{n}$.
- ⋄ **Closed Set Under a Binary Operation:** A set A is closed under a binary operation $\star$ if the following holds: For all $a, b \in A$ for which $a \star b$ is defined, $a \star b \in A$.
- ⋄ An integer m is divisible by another integer $n \neq 0$ if there exists a third integer p such that $pn = m$. We also say that "n divides m".

You probably don't need examples and non-examples of even and odd numbers, but we should note one thing: Since the integers are closed under multiplication and addition, the definitions of even and odd numbers imply that both are themselves integers. You have already seen a number of examples and non-examples of sets that are closed or not closed under certain operations.

1. Show that each of the following is a rational number. That is, find two integers m and n such that the number can be written as $\frac{m}{n}$.

   (a) -7 (b) $3\frac{1}{4}$ (c) 12.83

2. It turns out that every repeating decimal is a rational number. This exercise will show how to find m and n. Consider the number $5.283838383...$

   (a) If we let $x = 5.2838383...$, what is $100x$?
   (b) Find $100x - x$.
   (c) Solve the equation $100x - x =$ your answer to (b) for x. That shows that $5.2838383...$ is a rational number!

3. Use a procedure similar to that just used to find the fraction form of $12.34123412341234...$

The above exercise shows that integers, "mixed numbers" and both terminating and repeating decimals are rational numbers. Real numbers that are not rational are called **irrational numbers**. Examples are $\sqrt{2}, \pi$ and e.

4. Determine whether or not each of the following is true. For any that are, explain why, *based on the definition*.

   (a) 17 is an odd number.

(b) 1.5 is a rational number.

(c) $\sqrt{2}$ is a rational number.

(d) 18 is divisible by 5.

(e) 45 is divisible by 15.

(f) $\frac{16}{3}$ is an even number.

(g) $5\frac{1}{3}$ is a rational number.

(h) 1000 is an even number.

4.3 Proofs From Definitions

Performance Criteria:

4. (c) Write a conditional statement in if-then form.
   (d) Prove a conditional statement involving definitions.

Up to this point we have made quite a few observations about sets and operations on them, and about numbers. Most of these observations can be summarized by statements, often conditional statements. The observations themselves are gathered by looking at a number of examples.

1. Determine whether the sum of an even number and an odd number is even or odd. Explain how you determined this.

In the above exercise you should have reached some sort of conclusion, most likely based on some examples. Such a conclusion can be stated in some general way, like

the product of two odd numbers is odd.

When trying to prove that such a statement is true, it is most convenient to have it in if-then form with specific letters for the two unknown odd numbers. For example, the above statement translates to

If x and y are odd, then xy is odd.

It is important that we are considering ANY two odd numbers, but we need to assign names to them (x and y) so that we can talk about them and work with them algebraically. Finally, note that the above statement is equivalent to saying

the odd numbers are closed under multiplication.

2. Translate each of the following into if-then statements.

   (a) The sum of an even number and an odd number is an odd number.
   (b) A multiple of three is a multiple of six.
   (c) The rational numbers are closed under subtraction.
   (d) A power of four is a power of two.

3. One of the above statements is actually false. Which is it? How did you find it?

We now look at how we would prove that

the sum of an even number and an odd numbers is an odd number.

The key to proving this is the same idea that you used in Exercise 1. You should have taken an even number and an odd number and added them. Then you checked to see if the result was odd. This reasoning is based on your understanding that a conditional statement "if P, then Q" is true if P being true causes Q to be true. So to prove that "if P, then Q" is true, *we assume P is true, and show that this causes Q to be true.* For our particular statement we must

1) consider *any* even number and *any* odd number,
2) add them,
3) see if the sum is odd.

Since we won't have a specific even number and odd number, we must rely on what we know about even and odd numbers in general when carrying out the above. *This requires appealing to the definitions of even and odd numbers.* This will actually be done twice, once when describing what our two given numbers look like and again when showing that the sum is odd. Here is the proof of the statement we have been considering:

	Statement	Reason
1.	Let x be even and y be odd	assumption
2.	$x = 2m$ and $y = 2n + 1$ for some $m, n \in \mathbb{Z}$	definitions of even and odd
3.	$x + y = 2m + (2n + 1) =$ $2m + 2n + 1 = 2(m + n) + 1$	axioms of real numbers
4.	$m + n \in \mathbb{Z}$	$\mathbb{Z}$ is closed under addition
5.	$x + y$ is odd	3, 4, definition of odd $\square$

NOTE: From this point on we will assume that $\mathbb{N}$, $\mathbb{W}$ and $\mathbb{Z}$ are closed under addition and multiplication, and $\mathbb{Z}$ is closed under subtraction. *Indicate when you are using any of these facts by saying so.* Also, we will not go through the specifics of how the axioms of the real numbers are used in computations like step 3.

4. Suppose that a person were to replace the second statement of the above proof with

$$x = 2m \text{ for some } m \in \mathbb{Z} \text{ and } y = 2m + 1 \text{ for some } m \in \mathbb{Z}$$

There is a problem with this; write the conditional statement that will actually be proved by this. Note that it is a less general statement than the original. That is, it applies only to more specific situations than the original.

5. In this exercise you will see how to prove that if x is odd, then $x + 1$ is even. As you work through this you should be modeling your proof on the one above.

   (a) Begin by assuming that x is odd - write on your paper "1. Let x be any odd number", giving *assumption* as the reason. Now, by definition, what does that mean? Write this as the second line of your proof.

(b) Based on what you wrote, what form must $x + 1$ have? Write this as the third line of your proof. $x + 1 =$ ______.

(c) You must now conclude that $x+1$ is even. This means that you need to make it look like two times something, where the something is an integer. Continue your algebra from part (b) to make $x + 1$ look like two times something.

(d) You *must* state that, and verify why, the "something" is an integer. Do this.

(e) You have now shown that $x + 1$ is even, but you need to state that and tell how you know.

6. Finish the statement "The product of an odd number and an even number is ______." Then translate it into "if-then" form and prove it.

7. Prove that if m is divisible by 6, then m is divisible by 3.

8. Translate "The rational numbers are closed under multiplication" into an if-then statement and prove it.

4.4 Subset Proofs From Definitions

Performance Criteria:

4. (e) Prove a subset relation.
   (f) Prove subset relation whose proof involves cases.

1. The definition of the union of two sets is

$$x \in A \cup B \text{ if, and only if, } x \in A \text{ or } x \in B$$

Note that $A \cup B$ is a set, and to define a set we must tell how to determine whether or not an object is in the set. The definition given does that. Use this as a model to write definitions of

(a) $A \cap B$, (b) $A - B$, (c) A'.

Now suppose that we wish to show that one set A is a subset of another set B. To do this we must show that A and B satisfy the definition, which means that we must show that

$$\text{if } x \in A, \text{ then } x \in B.$$

Thus proving one set is a subset of another boils down to proving a conditional statement. As in the previous section, we do such a proof by assuming the "if" part of the conditional statement and showing that leads to the "then" part being true.

2. We will prove that $A \cap B \subseteq A$.

   (a) Assume $x \in A \cap B$. As you have been doing, write this assumption down as the first step of the proof, giving *assumption* as the reason.
   (b) What follows immediately from this, and why? This is the next statement and reason from your proof.
   (c) We are essentially done, but let's state that $x \in A$. For a reason we will use *the meaning of and*.
   (d) The next statement should be that $A \cap B \subseteq A$. This should have a reason.

3. Prove that $A \subset A \cup B$. **The proof should have four steps.**

4. Provide reasons for the steps in the following proof that $(A - B)' \subseteq A' \cup B$.

   Statement

   1. Let $x \in (A - B)'$
   2. $x \notin (A - B)$
   3. Not $(x \in A$ and $x \notin B)$
   4. $x \notin A$ or $x \in B$
   5. $x \in A' \cup B$
   6. $(A - B)' \subseteq A' \cup B$

5. The first two steps of the proof that $(A \cap B) \cap C \subseteq A \cap (B \cap C)$ are given below. Supply the reasons and complete the proof. *Note that you must work with one intersection at a time, as shown in step 2.*

   1. Let $x \in (A \cap B) \cap C$
   2. $x \in (A \cap B)$ and $x \in C$

In some cases, in the course of doing a proof, one knows that one of two statements Q or R is true. At this point, the proof proceeds as two proofs, one for the case that Q is true, and the other for the case that R is true. The next exercise illustrates how this is done. The method I have used for the two cases is not necessarily standard.

6. Note how we prove that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$, as shown below. Then supply the reasons; you do not need to give reasons for steps 3 and 4, just 3(a),(b) and .

   Statement

   1. Let $x \in A \cup (B \cap C)$
   2. $x \in A$ or $x \in B \cap C$
   3. Suppose $x \in A$
      (a) $x \in A \cup B$ and $x \in A \cup C$
      (b) $x \in (A \cup B) \cap (A \cup C)$
   4. Suppose $x \in (B \cap C)$
      (a) $x \in B$ and $x \in C$
      (b) $x \in A \cup B$ and $x \in A \cup C$
      (c) $x \in (A \cup B) \cap (A \cup C)$
   5. $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

   Note that since both cases that arise in step 2 lead to $x \in (A \cup B) \cap (A \cup C)$, we can then conclude what we were trying to prove.

7. Use a procedure similar to the one above to prove that $A' \cup B' \subseteq (A \cap B)'$.

8. (a) Suppose that $x \in A \cup B$ and $x \notin A$. What can you then conclude, and why?

   (b) Prove that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. The proof of this involves cases as well, but is a bit tricky. Begin as shown below, but then use the two cases that either $x \in A$ or $x \notin A$.

9. Prove that $(A \cup B) \cup C \subseteq A \cup (B \cup C)$

4.5 Set Proofs From Set Properties

Performance Criteria:

4. (g) Prove set equalities using previously known equalities.

As we prove more and more things, we build up a collection of *theorems*, which are facts that can be proven to be true. Below is a list of such theorems for sets, some of which we have proved, some not. For this section we will take all of these to be true, whether we have proved them or not. We will use these to then prove other theorems about sets. To do this, we will simply begin with the left side of each inequality and create a sequence of expressions, each equal to the one before it because of one of the above theorems, until we arrive at the right hand side. Many of you have done this before when you verified trigonometric identities in a trigonometry class.

Here are the theorems that you will use:

Universal set properties: (i) $A \cup U = U$, (ii) $A \cap U = A$

Empty set properties: (i) $A \cup \emptyset = A$, (ii) $A \cap \emptyset = \emptyset$

Identity Properties: (i) $A \cup A = A$, (ii) $A \cap A = A$ **Complement properties:** (i) $A \cup A' = U$, (ii) $A \cap A' = \emptyset$, (iii) $(A')' = A$

Commutative properties: (i) $A \cup B = B \cup A$, (ii) $A \cap B = B \cap A$

Associativity properties: (i) $(A \cup B) \cup C = A \cup (B \cup C)$, (ii) $(A \cap B) \cap C = A \cap (B \cap C)$

Distributive properties: (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

DeMorgan's Laws: (i) $(A \cup B)' = A' \cap B'$, (ii) $(A \cap B)' = A' \cup B'$

Relative complement: $A - B = A \cap B'$

Absorption laws: (i) $A \cap (A \cup B) = A$, (ii) $A \cup (A \cap B) = A$

1. Below are the statements for the proof that $A - (B - A) = A$. Give the reason for each step, choosing from the above list.

	Statement		
1.	$A - (B - A)$	$=$	$A - (B \cap A')$
2.		$=$	$A \cap (B \cap A')'$
3.		$=$	$A \cap (B' \cup A'')$
4.		$=$	$A \cap (B' \cup A)$
5.		$=$	$A \cap (A \cup B')$
6.		$=$	A

Note that steps 2 and 3 have the same reason, so we could just skip step 2. I have shown it to be very clear about what has happened. We will sometimes skip steps where the commutative properties are used, like step 4.

2. Use the same method to prove each of the following:

(a) $A \cup (B - A) = A \cup B$

(b) $(A \cap B) \cup (A \cap B') = A$

(c) $A - (A - B) = A \cap B$

(d) $(A \cup B') \cap (A \cup B) = A$

(e) $A - (B \cap C) = (A - B) \cup (A - C)$

(f) $[A \cap (B \cup A')] \cup B = B$

4.6 More Set Proofs

Performance Criteria:

4. (h) Prove set equalities and conditional statements involving sets.

Consider the following definitions.

- ⋄ **Equal Sets:** Two sets A and B are equal if A is a subset of B and B is a subset of A.
- ⋄ **Disjoint Sets:** Two sets A and B are disjoint if $A \cap B = \emptyset$.

1. In Exercise 7 of Section 4.4 you proved that $A' \cup B' \subseteq (A \cap B)'$
   (a) What else needs to be done to prove that $(A \cap B)' \subseteq A' \cup B'$?
   (b) Do it!
2. Prove that $(A \cup B)' = A' \cap B'$.

This isn't that interesting, so let's move on! We now look at subset proofs, but where we are given that another subset relation holds. Suppose we wanted to verify that

$$\text{If } A \subset B, \text{ then } A \subset A \cap B.$$

We want to show that $A \subset A \cap B$. This doesn't hold in general, but here we are also told that $A \subset B$. The following shows how this proof is done.

3. Fill in the missing reasons for the following proof.If any statement follows from more steps than just the previous one, list all steps it follows from, along with a reason.

Statement	Reason
1. $A \subseteq B$	given
2. Let $x \in A$	assumption
3. $x \in B$	
4. $x \in A \cap B$	
5. $A \subseteq A \cap B$	definition of subset □

Previously we proved that $A \cap B \subseteq A$ *for all sets* A *and* B. Thus it clearly holds in the case that $A \subseteq B$, so from this and the previous exercise we now have

$$A \cap B \subseteq A \qquad \text{and} \qquad A \subseteq A \cap B.$$

Thus, by definition, $A = B$. We could then modify the above proof by adding the lines

6. $A \cap B \subseteq A$	previously proven
7. $A = B$	definition of equal sets

to prove that

$$\text{If } \; A \subseteq B, \; \text{ then } \; A \cap B = A.$$

4. Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

5. Prove that if $A \subseteq B$, then $A \cup B \subseteq B$.

4.7 Function Proofs

Performance Criteria:

4. (i) Prove that a function is one-to-one or onto.
   (j) Prove statements about the one-to-one and onto properties of f, g and $f \circ g$.
   (k) Prove statements about the images of two sets under a function f.

In order to prove things about functions, we need to have some definitions. Suppose that $f : A \to B$ is a function.

- ◇ **Domain:** A is the domain of f.
- ◇ **Range:** y is in the range of f if, and only if, there exists an $x \in A$ such that $f(x) = y$.
- ◇ **Image of a Set:** If $S \subset A$, the image $f(S)$ of S under f is the set $\{f(a) \mid a \in S\}$.
- ◇ **Preimage of a Set:** If $T \subseteq B$, the preimage $f^{-1}(T)$ of T under f is the set $\{a \in A \mid f(a) \in T\}$.
- ◇ **Onto:** f is onto if, and only if, for every $y \in B$ there exists an $x \in A$ such that $f(x) = y$.
- ◇ **One-to-one:** f is one-to-one if, and only if, $f(x) = f(y)$ implies $x = y$.
- ◇ **Inverse:** Suppose that f is a bijection. Then $g : B \to A$ is the inverse of f if, and only if, $(g \circ f)(x) = x$ for all $x \in A$ and $(f \circ g)(y) = y$ for all $y \in B$.

You know how to get an idea of whether a function is one-to-one or onto by examining how it works for several values. This does not constitute a proof, however, since a proof must hold for *all* values in the domain and/or range of the function. In this section we see how *prove* that functions are one-to-one or onto. We begin with one-to-one, using the definition given above.

a function f is one-to-one if $f(x) = f(y)$ implies $x = y$.

1. We will prove that $h(x) = 7x + 1$ is one-to-one.
   (a) We must show that $h(x) = h(y)$ implies $x = y$. We begin by assuming what?
   (b) Substitute into your answer to (a) what each of the function values are.
   (c) Do some algebra, *one step at a time*, until you arrive at $x = y$.
   (d) Say something to the effect of "We have shown that _______ implies _______, so h is one-to-one." You are then done.

2. Prove that each of the following functions is one-to-one.

(a) $f(x) = \dfrac{1}{x+3}$ (b) $g(x) = \sqrt{5-x}$

3. A student provides the following proof that $h(x) = x^2$ is one-to-one. What is wrong?

$$\begin{aligned} h(x) &= h(y) \\ x^2 &= y^2 \\ \sqrt{x^2} &= \sqrt{y^2} \\ x &= y \end{aligned}$$

Suppose that we wish to show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is onto. We must show that for any $y \in \mathbb{R}$ there is an $x \in \mathbb{R}$ such that$f(x) = y$. To do this we will work backward from y to see what x must be, then show that $f(x) = y$. The next paragraph is not the proof, but how we find the right x.

We want x such that $f(x) = y$. That is, we must have $x^3 = y$. So what must x be? Solving the last equation for x gives us $x = \sqrt[3]{y}$. so this is the value of x that we will use in our proof. The next paragraph is the proof itself.

Suppose that y is any real number, and let $x = \sqrt[3]{y}$. Clearly $x \in \mathbb{R}$ and $f(x) = (\sqrt[3]{y})^3 = y$. Therefore f is onto.

4. Prove that $g(x) = 7x + 1$ is onto $\mathbb{R}$.

5. Prove that $h(x) = \sqrt{5-x}$ is onto $[0, \infty)$. Where in your proof did you need the fact that $y \geq 0$?

6. Suppose that $f : A \to B$ and $g : B \to C$ are functions.

   (a) Prove that if f and g are both one-to-one, then $g \circ f$ is one-to-one.

   (b) Prove that if f and g are both onto, then $g \circ f$ is onto.

7. Suppose again that $f : A \to B$ and $g : B \to C$ are functions, and consider the statement from part (a) of the previous exercise.

   (a) Write the converse of that statement.

   (b) Show that the converse is not necessarily true.

   (c) It *IS* true that one of f or g must be one-to-one if $f \circ g$ is. Finish the statement "If $f \circ g$ is one-to-one, then ... .

   (d) Prove your statement from part (c).

8. Repeat the previous exercise for part (b) of Exercise 6.

9. Suppose that $f : A \to B$ is a function and S and T are subsets of A.

   (a) Prove that $f(S \cup T) = f(S) \cup f(T)$.

   (b) Prove that $f(S \cap T) \subseteq f(S) \cap f(T)$.

   (c) Give an example showing that we cannot replace $\subseteq$ in part (b) with $=$. That is, show that $f(S \cap T) \neq f(S) \cap f(T)$.

(d) Prove that $f(S \cap T) = f(S) \cap f(T)$ if, and only if, f is one-to-one.

10. Suppose that we have a function $f : A \to B$.

    (a) Prove that for all $S \subseteq A$, $S \subseteq f^{-1}[f(S)]$.
    (b) Prove that for all $T \subseteq B$, $f[f^{-1}(T)] \subseteq T$.
    (c) Show that $\subseteq$ in part (a) cannot be replaced by $=$.
    (d) Show that $\subseteq$ in part (b) cannot be replaced by $=$.

4.8 Chapter 4 Exercises

Performance Criteria:

4. (l) Prove statements about numbers, sets and functions.

1. Use the closure properties of the integers to prove that the rational numbers are closed under addition.
2. The rational numbers are closed under division.
3. Prove or disprove that the sum of two odd numbers is even. If you disprove it, formulate an alternative statement that is true, and prove it.
4. Prove or disprove that the product of two odd numbers is even. If you disprove it, formulate an alternative statement that is true, and prove it.
5. What can you say about the difference of two even numbers? Prove your statement.
6. Why does it make no sense to try to prove anything like the previous for quotients of odd and even numbers?
7. Prove that any power of four is a power of two.
8. Prove the first absorption law given in Section 4.4.
9. Prove that if $A \cap B = \emptyset$, then $A - B = A$.
10. $A \triangle B = A \cup B$ if, and only if, $A \cap B = \emptyset$.
11. This exercise illustrates a method of proof called **proof by contradiction**. This method consists of assuming the negation of what you want to prove is true, and showing that the assumption leads to a contradiction. (That is, you arrive at something we know is false.) This means that our assumption must have been bad, so what we wanted to prove is actually true. In this exercise we will prove that the even numbers and odd numbers are disjoint sets.
    (a) Begin by assuming that the even numbers and odd numbers are *not* disjoint. What must then be the case?
    (b) Write what it means that your number from (a) is even. Write what it means that it is odd.
    (c) Your two statements from (b) can be combined, since they are about the same number. The result is an equation.
    (d) Manipulate your equation until you get a contradiction. It will occur when one side has the form of an even number and the other side is clearly odd, or vice-versa.
    (e) State that since the assumption that the two sets are not disjoint leads to a contradiction, it must be the case that they *are* disjoint. You are done.

12. (a) Prove that for all natural numbers, $2^n \leq (n+1)!$. Since $0!$ is defined to be 1, this actually holds for all whole numbers.

    (b) A stronger statement than what you just proved would be $2^n < n!$. is this statement true for any natural numbers? In Chapter 7 you will prove what you just discovered.

13. (a) Prove that the product of any two consecutive natural numbers is divisible by 2.

    (b) Prove that the product of three consecutive natural numbers is divisible by 3.

14. (a) Compute $n^3 - n$ for $n = 0, 1, 2, 3, 4, 5, 6$. Make a conjecture concerning divisibility of all of the results. Test your conjecture with the case $n = 7$.

    (b) Prove your conjecture.

5 Recursion and Induction

5.1 Sequences

Performance Criteria:

5. (a) Given the first five or six terms of a sequence, write an explicit definition of the sequence.
   (b) Given an explicit definition of a sequence, write out the first five or six terms of the sequence.
   (c) Given the first five or six terms of a sequence, write the recursive definition of the sequence.
   (d) Write out the first five or six terms of a sequence for which a recursive definition is given.

A **sequence** is an infinite list of numbers, which we will call **terms** of the sequence, *in a particular order*. Some examples of sequences are

$2, -4, 6, -8, 10, ...$ $\qquad$ $1, 2, 4, 8, 16, ...$

$1, 0, 1, 0, 1, 0, ...$ $\qquad$ $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, ...$

$1, 1, 2, 3, 5, 8, 13, ...$ $\qquad$ $\pi, \sqrt{\pi}, \sqrt[3]{\pi}, \sqrt[4]{\pi}, ...$

$7, -53, 2, 17, 421, 0, -5, 0, -5, 28, ...$

In this course we will be primarily concerned with sequences of integers, but we may occasionally see sequences of rational or real numbers. Our sequences will generally have some sort of governing pattern, but the last example above illustrates that the terms of a sequence need not exhibit an obvious pattern or any pattern at all. Finally, make note of the fact that *sequences are not sets*. In a set the order of the elements is of no importance, but the terms of a sequence have a definite order. Also, terms in a sequence may be repeated, whereas elements in a set may not.

Consider the first sequence above. It has a first term of 2, a second term of -4, a third term of 6, and so on. When we talk about a "first", "second", "third", etc., we are essentially assigning to each each natural number 1, 2, 3, ... the corresponding term of the sequence. Thus every sequence can be thought of as a function f defined on the natural numbers. (The target set may vary, but if we consider only sequences with integer values the target set is $\mathbb{Z}$.)

1. Thinking of the first sequence above as a function $f : \mathbb{N} \to \mathbb{Z}$, our function f must make the following assignments:

$$f(1) = 2\ , \qquad f(2) = -4\ , \qquad f(3) = 6\ , ...$$

   Give the formula for the function, using n as the dummy variable, since it represents a natural number.

When working with sequences, we often abandon the function notation and replace $f(1)$ with a_1, $f(2)$ with a_2, and so on. Thus a general sequence is denoted by $a_1, a_2, a_3, a_4, ...$ The subscripts here are called indices (the plural of **index**) and we say that the first sequence is indexed by the natural numbers if we describe it by $f(n) = (-1)^{n+1}2n$ or $a_n = (-1)^{n+1}2n$. We could index the same sequence by the whole numbers instead if we defined $a_n = (-1)^n 2(n+1)$. (Try letting $n = 0, 1, 2, ...$ in this formula to convince yourself that it does indeed generate the sequence $2, -4, 6, -8, ...$) Should you use the natural numbers, or the whole numbers, to index a sequence? You can use whichever you wish; try to pick the one that makes the formula for the general term a_n the simplest. Either way, the set chosen for the task of indexing the sequence is called the **index set** for the sequence. In certain circumstances the integers are used as an index set, but we will stick with using the integers or the natural numbers.

One way of defining or describing a sequence is to give a formula for the general term a_n of the sequence, followed by the index set. For example, the sequence $2, -4, 6, -8, ...$ just discussed could be described by

$$a_n = (-1)^{n+1}2n, \quad n = 1, 2, 3, ...$$

or

$$a_n = (-1)^n 2(n+1), \quad n = 0, 1, 2, ...$$

We call these **explicit** representations of the sequence. *It is always assumed that the counter increases by one unit for each new term of a sequence.*

2. Give an explicit representation of the each of the sequences.

   (a) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, ...$ (b) $5, 8, 11, 14, ...$

3. Write out the first five terms of each sequence.

   (a) $-3(2)^n, \quad n = 0, 1, 2, ...$ (b) $5n + 1, \quad n = 1, 2, 3, ...$

When we represent a sequence explicitly we are telling how to find a term of a sequence based on the index of the term. There is another way to represent a sequence, which we call a **recursive** representation of the sequence. A recursive representation consists of giving the first term (or first several terms) of the sequence and then telling how to obtain each term from the one term (or several terms) preceding it. For example, we can define the sequence from Exercise 2(b) above by

$$a_1 = 5 \ , \qquad a_n = a_{n-1} + 3 \ .$$

You will see that some sequences are more easily represented explicitly, and others are easier to represent recursively.

4. In each of the following, a sequence is defined either explicitly or recursively. Write out the first seven terms of the sequence.

   (a) $a_1 = 20 \ , \quad a_n = a_{n-1} - 3$ (b) $a_n = 3n + 2, \quad n = 0, 1, 2, ...$

   (c) $a_n = (-1)^{n+1}n^3, \quad n = 1, 2, 3, ...$ (d) $a_0 = 2 \ , \quad a_n = 3a_{n-1}$

   (e) $a_1 = 1, \ a_2 = 2, \ a_3 = 3 \ , \quad a_n = a_{n-1} - a_{n-2} + a_{n-3}$

5. Define each of the following sequences both explicitly and recursively, if possible. *Check each of your answers by writing out the first few terms of your definition for the sequence.*

   (a) $4, 7, 10, 13, ...$

   (b) $1, 2, 4, 8, 16, ...$

   (c) $1, 1, 2, 3, 5, 8, 13, 21, ...$

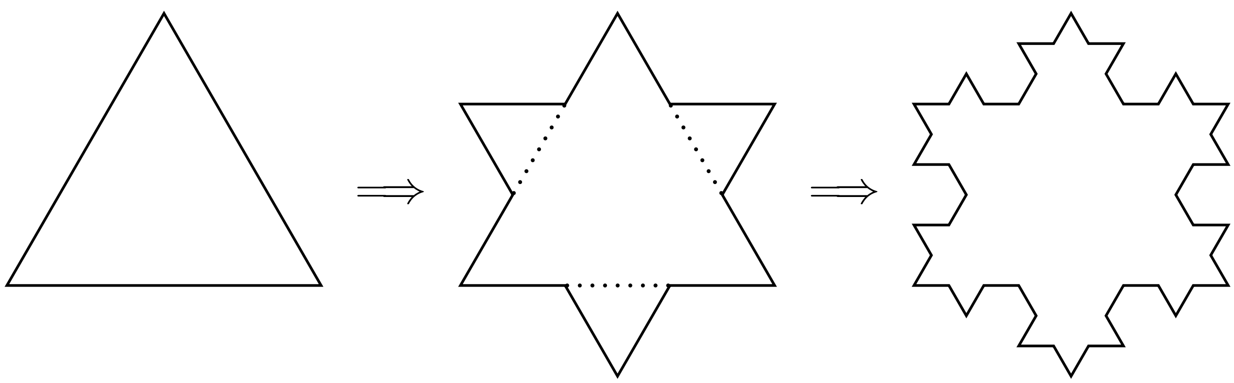
   (d) $5, -5, 5, -5, 5, ...$

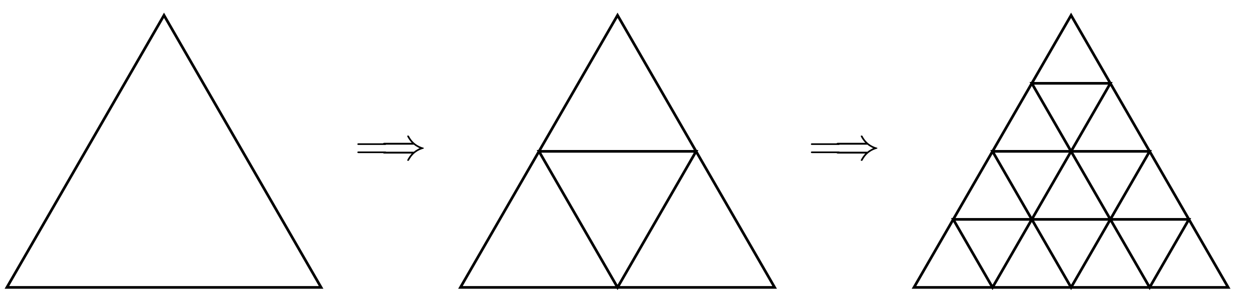
5.2 Recursively Defined Sequences, Iterative Processes

Performance Criteria:

5. (e) Write out terms of a recursively defined sequence.
   (f) Find an explicit representation of a recursively defined sequence.
   (g) Apply a given iterative process, and determine whether it converges.

A recursive or iterative process is one in which a sequence of objects is created by starting with one or several objects and obtaining all additional objects by applying the same process repeatedly to the last or last several objects. Here are a few interesting examples:





The first sequence above is interesting in that each of the given shapes is enclosed in a circle, so they all have finite area. However, the perimeters grow to be as large as one wants; even at the scale I have used, if you carry out the process long enough, the perimeter will grow to over a million miles, or a billion miles, or ...! The third sequence is constructed by taking the middle third out of the first segment, then taking the middle third out of each remaining segment, then the middle third out of each of the segments that then remain, and so on. There is a set of points that always remains, no matter how long you do this, and that set is called the *Cantor set*. It has many interesting mathematical properties.

In this chapter we will be dealing with sequences of numbers that are recursively generated, and we will look at iterations of functions. You saw the concept of recursively defined sequences in the last section.

1. Consider the sequence defined by the recursion formulas $a_0 = 3$ and $a_{n+1} = a_n + 5$.

   (a) Write out the first six terms of the sequence.

   (b) What do you think the term a_{42} would be? Give an explicit formula for a_n if you can.

You may or not have been successful at part (b) of this last exercise. Either way, there is a method for finding explicit formulas from recursive formulas that you should be familiar with. It basically amounts to good bookkeeping; each term of the sequence is written in a way that helps one to see a pattern. It begins with writing the terms of the sequence in the way that they are produced, *without carrying out any of the arithmetic operations that the recursive definition requires.* See the column on the left below.

$$\begin{aligned}
a_0 &= 3 \\
a_1 &= a_0 + 5 = 3 + 5 \\
a_2 &= a_1 + 5 = (3 + 5) + 5 \\
a_3 &= a_2 + 5 = [(3 + 5) + 5] + 5 \\
a_4 &= a_3 + 5 = \{[(3 + 5) + 5] + 5\} + 5
\end{aligned}$$

We now try to combine terms in a way that introduces the index of the term a_n into the arithmetic used to find a_n:

$$\begin{aligned}
a_0 &= 3 \\
a_1 &= 3 + 5 \\
a_2 &= (3 + 5) + 5 = 3 + 2(5) \\
a_3 &= a_2 + 5 = [3 + 2(5)] + 5 = 3 + 3(5) \\
a_4 &= a_3 + 5 = [3 + 3(5)] + 5 = 3 + 4(5)
\end{aligned}$$

From this we can see that $a_n = 3 + 5n$, and we have our explicit formula! Recall that we need also to state that this is for $n = 0, 1, 2, ...$

2. For each of the following recursively defined sequences,

   - write out the terms a_0 or a_1 through a_5,
   - find an explicit representation for the general term a_n,
   - test your explicit formula by finding a_5 and seeing if it agrees with what you got for it when you wrote out the terms.

   (a) $a_0 = 1$ and $a_{n+1} = 4a_n$

   (b) $a_0 = 3$ and $a_{n+1} = 2a_n$

   (c) $a_1 = 3$ and $a_{n+1} = 2a_n + 1$

   (d) $a_0 = 2$ and $a_{n+1} = a_n^2$

3. Write out the first six terms of the sequence defined by $a_0 = 1$, $a_1 = 2$ and $a_{n+1} = a_n + 2a_{n-1}$. Then find an explicit formula for a_n - you will probably not need the procedure outlined before the previous exercise.

4. (a) The sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, ...$ is a very famous sequence called the **Fibonacci sequence**. *Get familiar with this sequence - we will do more with it soon!* Write a recursive formula for this sequence.

   (b) Compute $\dfrac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$ for $n = 1, 2, 3$ without using your calculator. Try to compute the value for $n = 8$ with your calculator, using grouping symbols to avoid rounding at any step of the process. (If you are proficient with your calculator, you could enter the above equation with x in place of n and build a table of values.) I don't know about you, but I find the results of this to be fairly remarkable!

We now see how to create a sequence from a given function, using a process very similar to that which you have been doing. Take, for example, the function with domain $\{0, 1, 2, 3, ..., 20\}$ defined by

$$f(n) = \left\lfloor \frac{n(20-n)}{5} \right\rfloor .$$

We select a starting value for n, like $n = 4$. That is the first term of our sequence. The second term is obtained by taking $f(4) = 12$. Then, to get the third term we take $f(12) = f[f(4)]$, and so on. The first five terms of this sequence are $4, 12, 19, 3, 10, ...$ We will call this process **iterating** the function. The major difference between this sequence and the ones generated in Exercises 1 and 2 is that there is no apparent pattern to the terms of this sequence.

5. Using the same function, find the first five terms of the sequence whose first term is 6.

6. Consider the function $g : \{0, 1, 2, ..., 10\} \to \mathbb{N}$ defined by $g(n) = \left\lfloor \dfrac{2n(10-n)}{5} \right\rfloor$.

   (a) Create a sequence in the same manner that you did in Exercise 5, using the function g and starting with $n = 1$. Continue until it is apparent that you can stop (you will know when that is).

   (b) What happens if you start with $n = 5$? $n = 10$?

   The phenomenon you saw in part (a) is called a *cycle*; in part (b) the sequence heads for a *fixed point*, the number zero.

7. (a) Go back to your sequence for Exercise 5 and continue the sequence until it is apparent you should stop. Do you reach a fixed point, or a cycle?

   (b) Now create two sequences using the same function f, starting with $n = 2$ and $n = 3$. What happens?

   (c) The functions f and g are clearly quite similar. Based on what you have seen so far, if there is a fixed point we would expect it to have what value?

   (d) Using f again, create a sequence whose first term is $n = 5$.

8. Experiment with each of the following functions.

   (a) $h(n) = \left\lfloor \dfrac{n(8-n)}{2} \right\rfloor$ (b) $g(n) = \left\lfloor \dfrac{n(12-n)}{3} \right\rfloor$ (c) $h(n) = \left\lfloor \dfrac{4n(17-n)}{17} \right\rfloor$

5.3 Proof by Induction I

Performance Criteria:

5. (h) Give what is to be assumed and what is to be proved in the inductive step of a proof by induction.
   (i) Use induction to prove an explicit representation for a recursively defined sequence.
   (j) Use induction to prove other things.

Most of us have done the following: Stand a set of dominoes up on end in such a way that when we push the first one over, it in turn knocks over the second, which then knocks over the third, and so on. In this section we will see a method for proving facts about the whole or natural numbers that is analogous to that game. The process works as follows.

- Show that the fact is true for $n = 0$, or perhaps some other starting value.
- Show that if the fact is true for any whole or natural number k, then it holds for $k + 1$ as well.

The above process is sort of backward from the dominoes. The second item above is like setting up the dominoes in such a way that each will knock the next over if it goes over. It says that if the fact is true for 0, then it must be true for 1 as well. And if it is true for 1, then it is true for 2, and so on. Then we need only to start the process, which is the first item above. That is, we show the fact is true for $n = 0$ or some other starting value. The order that is shown above is that which is traditionally used in mathematics. This process is called **proof by induction**.

This is best shown by an example. Consider the sequence defined by

$$a_0 = 3\ , \qquad a_{n+1} = a_n + 5.$$

(This is Exercise 1(a) from the previous section.) We will prove that $a_n = 5n + 3$ for every whole number n, using induction. The two equations in the recursive definition of the sequence are taken as fact ("given").

Show that $a_n = 5n + 3$ when $n = 0$: $5(0) + 3 = 3 = a_0$, so $a_n = 5n + 3$ holds for $n = 0$.

Assume that $a_n = 5n + 3$ is true for $n = k$ and show that it must then be true for $n = k + 1$: Assume that $a_k = 5k + 3$. Using this and the recursive definition of the sequence, we have

$$a_{k+1} = a_k + 5 = (5k + 3) + 5 = (5k + 5) + 3 = 5(k + 1) + 3.$$

Thus $a_n = 5n + 3$ being true for $n = k$ leads to it being true for $n = k + 1$ as well. Since we have already shown that the formula holds for $n = 0$, it must now hold for every whole number n.

1. The second part of a proof by induction is called the *inductive step.* In the proof just done, it was where we assumed that $a_k = 5k+3$ and showed that $a_{k+1} = 5(k+1)+3$. In the next exercise you will prove each of the explicit representations given for the sequences in (a)-(d) below. For this exercise, tell what is assumed for the inductive step of each proof, and what needs to be shown.

   (a) Recursive (given): $a_0 = 1$ and $a_{n+1} = 4a_n$ Explicit (prove): $a_n = 4^n$

   (b) Recursive (given): $a_0 = 3$ and $a_{n+1} = 2a_n$ Explicit (prove): $a_n = 3(2)^n$

   (c) Recursive (given): $a_1 = 3$ and $a_{n+1} = 2a_n + 1$ Explicit (prove): $a_n = 2^{n+1} - 1$

   (d) Recursive (given): $a_0 = 2$ and $a_{n+1} = a_n^2$ Explicit (prove): $a_n = 2^{2^n}$

2. Prove each of the explicit representations of the recursively defined sequences from Exercise 1(a)-(d).

3. In Chapter 4 you (may have) showed that $2^n \le (n+1)!$ for all whole numbers. You then saw that $2^n < n!$ seems to hold for $n \ge 4$. Prove this, using induction.

5.4 Series

Performance Criteria:

5. (k) Find the first five or six partial sums of a series.
   (l) Write out the first five or six terms of a series that is given with summation notation.
   (m) Determine a formula for the nth partial sum of a series.

If we add all of the terms of a sequence together, we get something called a **series**. Here is an example of a series:

$$1 + 2 + 4 + 8 + 16 + \cdots$$

The individual numbers in a series are called the **terms** of the series. A series that keeps going indefinitely like this is called an infinite series; such series are very important in other branches of mathematics. Some of you may have worked with them in a calculus course. We will generally stop our series when the index reaches some arbitrary number n.

NOTE: From this point on we will be dealing at times with sequences, at other times with series. It is important that you distinguish between the two, using commas between the terms of a sequence, plus (and/or minus) signs between the terms of a series.

Associated with any series are values called **partial sums** of the series. The first partial sum is simply the first term of the series. The second partial sum is the sum of the first two terms, the third partial sum is the sum of the sum of the first three terms of the series, and so on. If our series is indexed starting with zero, as $a_0, a_1, a_2, ...,$ we will denote the partial sums by $s_0 = a_0,\ s_1 = a_0 + a_1,\ s_2 = a_0 + a_1 + a_2,$ and so on. If the first term of the series is $a_1,$ the first partial sum is then $s_1,$ followed by $s_2,\ s_3,$ etc.

Note that if we then list the partial sums in order, we have a sequence. For example, the sequence of partial sums for the series $1 + 2 + 4 + 8 + \cdots$ is $1, 3, 7, 15, ...$

1. Find the first six terms of the sequence of partial sums of each of the following series. *Remember that the first partial sum is just the first term.*

   (a) $1 + 1 + 2 + 3 + 5 + 8 + 13 + \cdots$
   (b) $1 - 1 + 1 - 1 + 1 - 1 + \cdots$
   (c) $2 + 4 + 6 + 8 + 10 + \cdots$

There is a compact way to represent a series, called **summation notation**. An example is

$$1 + 2 + 4 + 8 + \cdots + 2^n = 2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^n = \sum_{j=0}^{n} 2^j\ .$$

The symbol $\sum$ is called sigma, and stands for "sum", or add. The letter j is called the **index** for the series. If we are given a series in summation form, the process of writing it out is called *expanding* the series. For example, the expansion of $\sum_{j=0}^{n}(3j-1)$ is

$$\sum_{j=0}^{n}(3j-1) = -1+2+5+8+\cdots+(3n-1).$$

2. Expand each of the following series, writing out the first five terms and the last term.

   (a) $\sum_{j=1}^{n}(2j-1)$ (b) $\sum_{j=1}^{n} j$ (c) $\sum_{j=1}^{n} j^2$ (d) $\sum_{j=1}^{n} j^3$

3. Write the series from Exercises 1(b),(c) using summation notation.

4. Find the first five partial sums of each of the series from Exercise 2.

Our main objective for this section and the next is to determine a formula for the nth partial sum of a series, and use induction to prove that our formula is correct. In many cases, determining the formula is more difficult than proving it holds.

5. Compute $\left[\frac{n(n+1)}{2}\right]^2$ for $n = 1, 2, 3, 4, 5$.

In the above exercise you should have obtained the values of the partial sums s_1, s_2, s_3, s_4 and s_5 from the series in Exercise 2(d). Thus

$$\sum_{j=1}^{n} j^3 = \left[\frac{n(n+1)}{2}\right]^2 \tag{1}$$

holds for at least $n = 1, 2, 3, 4, 5$. In the next section we will prove that it in fact holds for all values $n \geq 1$.

6. Compute $\frac{n(n+1)(2n+1)}{6}$ for $n = 1, 2, 3, 4, 5$. Then write something like (1) for one of the other series in Exercise 2.

7. You now have formulas for the nth partial sums of two of the series from Exercise 2. Find formulas for the other two. If you can't see the formula for the series in 2(b), use the following to help you:

   - Consider the sequence of partial sums for the series, as you wrote it out for Exercise 4. Write a new sequence for which each term is twice the corresponding term in the sequence of partial sums.
   - Find a formula for the terms in this sequence. (**Hint:** Think of each number as a product of two other numbers.)
   - What do you have to do to your formula from the modified sequence to get the desired formula? (**Hint:** You must undo what you did to get the new sequence form the original.) Do that.
   - Be sure to see if your final formula from "works" by testing it for several values of n.

8. Find a formula for the nth partial sum of $\sum_{j=1}^{n}\frac{1}{j(j+1)}$.

9. Recall the Fibonacci sequence, which is defined by $a_1 = 1,\;\; a_2 = 1,\;\; a_{n+1} = a_n + a_{n-1}$.

(a) Write out the first ten terms of the sequence.

(b) Find $a_1 + a_3$, $a_1 + a_3 + a_5$ and $a_1 + a_3 + a_5 + a_7$. Then make a conjecture about

$$a_1 + a_3 + a_5 + \cdots + a_{2n-1} = \sum_{j=1}^{n} a_{2j-1}\ .$$

(c) Repeat part (b) for the even terms of the Fibonacci sequence.

5.5 Proof by Induction II

Performance Criteria:

5. (n) Give the assumption and what needs to be shown for the inductive step of proving a formula for a sum of a series.
   (o) Prove a formula for a sum of a series, using induction.

In the previous section we saw that

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}\,, \qquad n = 1, 2, 3, ... \tag{1}$$

SEEMS to be true. It is our objective in this section to prove such statements are true, using induction. The process is exactly as before: we first need to show that (1) holds for $n = 1$, then show that if it holds for $n = k$ it in fact holds for $n = k + 1$ as well. Let's do the proof!

- Note that $\displaystyle\sum_{j=1}^{1} j^2 = 1^2 = 1$ and $\displaystyle\frac{(1)(1+1)(2(1)+1)}{6} = \frac{(1)(2)(3)}{6} = 1$. Thus (1) holds when $n = 1$.
- Next we assume that
$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$
and we show that this leads to
$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}\,.$$

When reading the following, look for where the assumption is used!

$$\begin{aligned}
\sum_{j=1}^{k+1} j^2 &= 1^2+2^2+3^2+\cdots+\cdots+k^2+(k+1)^2 \\
&= \sum_{j=1}^{k} j^2+(k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6}+\frac{6(k+1)^2}{6} \\
&= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} \\
&= \frac{(k+1)[k(2k+1)+6(k+1)]}{6} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}
\end{aligned}$$

Note two things:

(1) The inductive assumption is used to get from the second line to the third line.

(2) To get from the the fourth line to the sixth line, *I factored first, THEN expanded what was left.* If I would have expanded the quantity $k(k+1)(2k+1)+6(k+1)^2$ I would have gotten $2k^3+9k^2+13k+6$. I don't know about you, but I would not be able to see how to factor this!

1. (a) For Exercise 2(a) of the previous section you should have conjectured the formula $\sum_{j=1}^{n}(2j-1)=n^2$. Write the assumption and what is to be shown for the inductive step of proving this formula.

   (b) Repeat (a) for $\sum_{j=1}^{n} j=\frac{n(n+1)}{2}$.

   (c) Repeat (a) for $\sum_{j=1}^{n} j^3=\left[\frac{n(n+1)}{2}\right]^2$.

2. Use induction to prove each of the three formulas from the previous exercise.

3. Use induction to prove the formula that you found in Exercise 8 of the previous section.

4. Find and prove a formula for the sum $\sum_{j=0}^{n} 2^j$.

5. Use induction to prove your conjecture from Exercise 9(b) of the previous section. *You will need to use the recursion formula for the Fibonacci sequence for the proof*, along with the standard inductive step.

6. Use induction to prove your conjecture from Exercise 9(c) of the previous section.

5.6 Chapter 5 Exercises

1. Let $A = \mathbb{W} \times \mathbb{W} \times \mathbb{W}$. That is, A is the set of all ordered triples of whole numbers, like $(3, 19, 7)$ and $(189, 2, 54)$. We abbreviate this set by $\mathbb{W}^3$. Now define a function $f : A \to A$ by
$$f(n_1, n_2, n_3) = (|n_1 - n_2|, |n_2 - n_3|, |n_3 - n_1|).$$
For example,
$$f(7, 18, 3) = (|7 - 18|, |18 - 3|, |3 - 7|) = (11, 15, 4).$$
   (a) Keep iterating the function to get a sequence of ordered triples. Continue until it is clear you should stop.
   (b) Do the same thing again, but start with a different ordered triple. Do you get the same result?
   (c) Iterate f, starting with the ordered triple $(7, 7, 7)$. Then try starting with $(8, 18, 13)$.
   (d) Make a conjecture about what results are possible when iterating f, and about what characteristics the initial ordered triple must have in order to end up at each result.

2. Repeat the same thing, but for $f : \mathbb{W}^2 \to \mathbb{W}^2$ defined in a similar manner.

3. Repeat again, but for $f : \mathbb{W}^4 \to \mathbb{W}^4$. Do you have any sort of conjecture as to what would happen for functions like this defined on $\mathbb{W}^5$, $\mathbb{W}^6, ...$?

4. Now consider a function $f : \mathbb{N} \to \mathbb{N}$ defined as follows: For a natural number n, $f(n)$ is the sum of all the natural numbers *less than* n that divide n, including 1. For example, suppose $n = 6$. Then 1, 2 and 3 all divide 6, so $f(6) = 1 + 2 + 3 = 6$. $f(5) = 1$, since 1 is the only number less than 5 that divides 5.
   (a) Find $f(12)$, then $f(f(12))$, then $f(f(f(12)))$. Continue until it is logical to stop.
   (b) Repeat (a) for a few other numbers. Try starting with 28.

If you find this function interesting, you might try two things:

(1) Find out a bit about perfect numbers on the web. Note that 28 is a perfect number.

(2) Write a computer program that carries out the iteration of f beginning with all natural numbers up to 300 until either a fixed point or a cycle is obtained.

5. Prove that for all sets A, if $|A| = n$, then $|P(A)| = 2^n$.

6. Suppose we create a sequence by letting $a_1 = 1$ and obtaining additional terms by $a_{n+1} = a_n + (2n - 1)$. Let's call this a **recursive-explicit** definition of a sequence, since it uses both the previous term and its index to get the next term.
   (a) Write out the first five terms of this sequence and find a purely explicit representation of the same sequence.

(b) Prove the explicit representation from the recursive-explicit representation, using induction.

7. Repeat the previous exercise for the sequence defined by $a_1 = 1, \ a_{n+1} = a_n + n$. To find the explicit representation, use the following hints:

- Write out the first 5 terms of the sequence.
- Look at a new sequence $\{b_n\}_{n=1}^{\infty}$ obtained by doubling each term of the original sequence. If you can find an explicit representation for each b_n, then $a_n = \frac{1}{2}b_n$.

8. Do as the previous two exercises for the sequence defined by $a_1 = 1, \ a_{n+1} = a_n + 2n$. To find the explicit representation, look at the sequence $\{b_n\}_{n=1}^{\infty}$ obtained by subtracting one from each term of the original sequence.

9. Consider the statement $n^3 < 2^n$, and suppose that we wished to show that this was true, using induction.

(a) Is the above statement true for $n = 2$? Clearly, then, the statement may not be true for all values of n.

(b) Suppose that we assume that $n^3 < 2^n$ for some natural number n. We would then like to show that $(n+1)^3 < 2^{n+1}$. Expand ("multiply out") $(n+1)^3$, then factor out n^3. Using our assumption, we can now write

$$(n+1)^3 < \_\_\_\_\_\_ \cdot \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ .$$

(c) Since $2^{n+1} = 2 \cdot 2^n$, we will have $(n+1)^3 < 2^{n+1}$ if we can show that

$$\_\_\_\_\_\_\_\_\_\_\_\_ < \_\_\_\_\_\_ .$$

One way to do this is to show that each of ______, ______ and ______ are less than $\frac{1}{3}$. How large would n have to be for that to be the case?

(d) Is it true that $n^3 < 2^n$ for the value of n that you found in (c)? If so, then you have proved that $n^3 < 2^n$ for all values of n greater than or equal to that value.

(e) Show that $n^3 < 2^n$ actually holds for values of n less than what you found in (c).

10. Someday there will be an exercise here about something called **strong induction**. For the time being, you don't have to worry about it!

11. In Chapter 4 you (may have) showed that $n^3 - n$ is divisible by 6 for all natural numbers n. Use proof by induction to prove this. Here are two hints:

- At some point you should add zero in the form $-k + k$.
- You will need the fact that one of two consecutive natural numbers must be divisible by two.

12. Find a formula for $\displaystyle\sum_{j=0}^{n}(-1)^j$, and prove that it holds for $n = 0, 1, 2, ...$

6 Relations and Groups

6.1 Introduction

Performance Criteria:

6. (a) Tell whether or not two things are related by a given relation.
   (b) Determine whether a relation is reflexive, symmetric and/or transitive.

Like functions, relations can be thought of in more than one way. We will begin with the most basic way of looking at relations. The relations we will consider here are what are called binary relations. Essentially a **binary relation** is a rule for telling whether an element of a given set is related to another element of the same set. For example, we might say that a real number x is related to a real number y if, and only if, $x < y$. So $5\frac{1}{2}$ is related to 8, 4.7 is not related to π. *Note that x and y are taken in that order - order is always taken into account when working with binary relations.* Thus 8 is *not* related to $5\frac{1}{2}$. (We will see that for some relations order really doesn't matter, but we must assume it does until we know otherwise.)

NOTE: Be careful not to confuse a binary relation with a binary operation, which we studied earlier. A binary relation says whether or not two things are related, a binary operation takes two things and creates one from them.

We generally name relations, with the capital letter R being the generic name for a relation, in the same way that f is the generic name for a function. Thus we would define the relation above by

$$x\,R\,y \text{ if, and only if, } x < y$$

and we write $5\frac{1}{2}\,R\,8$ to indicate that $5\frac{1}{2}$ and 8 are related. To again emphasize that order is critical, note that $8\,R\,5\frac{1}{2}$ is not true; that is, 8 and $5\frac{1}{2}$ are *NOT* related.

1. A number of relations are defined below. (Recall that when it is clear that a statement is a definition, the words "if, and only if" are replaced by "if" - we use that convention from this point on.) For each, give one pair that is related and one pair that is not.

   (a) A real number x relates to a real number y if $x \geq y$.
   (b) An integer m relates to an integer n if $m - n$ is a multiple of five.
   (c) A set A relates to a set B if $A \cap B \neq \emptyset$.
   (d) Two integers are related if they are either both even or both odd.
   (e) Two integers are related if they are relatively prime.
   (f) An integer m relates to an integer n if m divides n.
   (g) A set A relates to a set B if $A \subseteq B$.
   (h) A function $f : \mathbb{R} \to \mathbb{R}$ is related to a function $g : \mathbb{R} \to \mathbb{R}$ if $f(1) = g(1)$.

2. (a) Consider the relation from Exercise 1(a). Is it true that every real number relates to itself? That is, does $x\,R\,x$ hold for all real numbers x?

(b) Any relation for which every object is related to itself is said to be **reflexive**. Which of the relations from Exercise 1 are reflexive?

3. (a) Find a pair of sets A and B that satisfy the relation from Exercise 1(c). That is A is related to B. Is it also the case that B is related to A?

   (b) If any time that two objects are related in a given order, they are also related in the reverse order, the relation is said to be **symmetric**. Is the relation from Exercise 1(c) symmetric?

   (c) Which of the relations from Exercise 1 are symmetric? For those that are not, provide counterexamples.

4. (a) Consider again the relation from Exercise 1(c). Give two sets A and B such that $A R B$. If $A R C$, find new A, B, C such that A does *NOT* relate to C. If your original A did not relate to C, find new sets A, B, C such that it is true that $A R C$.

   (b) When $x R y$ and $y R z$ imply (always!) that $x R z$ for a relation R, it is said to be a **transitive** relation. Thus you just showed that the relation from Exercise 1(c) is *NOT* transitive. (Make sure you see why.) Which of the remaining relations from Exercise 1 are transitive? For those that are not, provide counterexamples.

6.2 Equivalence Relations and Equivalence Classes

Performance Criteria:

6. (c) Determine whether a relation is an equivalence relation.
   (d) Tell how many equivalence classes there are for a given equivalence relation.
   (e) Give some elements of an equivalence class containing a given element.

There are various important types of relations in mathematics. We will now look at a very important kind of relation called an equivalence relation. A relation is an **equivalence relation** if, and only if, it is reflexive, symmetric and transitive.

1. Which relations from Exercise 1 of the previous section are equivalence relations?

Suppose that R is an equivalence relation on a set A, and suppose that x is an element of A. The set of all elements of A that are related to x is a subset of A that we call an **equivalence class**.

Note: The concept of equivalence classes only makes sense for an *equivalence* relation.

2. For each relation from Exercise 1 of the previous section that is an equivalence relation,

   (i) give four elements that are in one equivalence class, and

   (ii) tell how many equivalence classes there are. (Your answer here will sometimes be "infinitely many".)

3. Is it possible that two different equivalence classes have an element in common? If so, give an example; if not, explain.

4. Let $A = \{1, 2, 3, 4\}$ and define a relation on $P(A)$ (remember, this is the power set of A) by $A_1, A_2 \in P(A)$ are related if $|A_1| = |A_2|$. It should be clear that this is an equivalence relation. Give all the equivalence classes by listing all the elements of each.

Suppose we have an equivalence relation R on a set A. Every element of A is in some equivalence class, containing at least itself. Moreover, your answer to Exercise 3 indicates that the equivalence classes are disjoint. Both of these things should be evident in your answer to Exercise 4. This implies the most important fact about an equivalence relation on a set A: *the equivalence classes form a partition of* A. In the next section you will see how we can use a binary operation on the set A to create a new binary operation that acts on the equivalence classes themselves.

6.3 Modular Arithmetic

Performance Criteria: 6. (f) For a given n, determine whether two integers are equal modulo n. (g) Perform modular arithmetic.

Consider the relation defined on $\mathbb{Z}$ by

$$xRy \text{ if, and only if, } x - y \text{ is a multiple of 3.}$$

We can easily prove that this is an equivalence relation:

- Let x be any integer; does xRx? Since $x - x = 0$ and zero is divisible by three, R is reflexive.
- Suppose that xRy; that is, there exists an integer m such that $x - y = 3m$. Note that $-m$ is an integer as well and

  $$y - x = -(x - y) = -3m = 3(-m)\ ,$$

  so yRx. Thus R is symmetric.
- Suppose that x, y, z are such that xRy and yRz. Then there exist integers m and n such that

  $$x - y = 3m \qquad \text{and} \qquad y - z = 3n\ .$$

  We then see that

  $$x - z = (x - y) + (y - z) = 3m + 3n = 3(m + n)\ .$$

  But the integers are closed under addition so $m + n \in \mathbb{Z}$ and xRz. Therefore R is transitive.

1. (a) Let x be any integer. We will denote by $[x]$ the equivalence class that contains x. For the relation just described, give $[6]$ using set builder notation. Your answer should look like $[6] =$ __________.
   (b) Give $[5]$ using set builder notation.
   (c) Find an integer $x \neq 1$ such that $[x] = [1]$.
   (d) How many equivalence classes does this relation have?

You should have found that the relation has only three equivalence classes, which can be denoted by $[0]$, $[1]$ and $[2]$. The formal definition of the equivalence class of some element $x \in A$ is

$$[x] = \{y \in A \mid yRx\}.$$

One might think the use of [ ] for equivalence classes might cause some confusion with interval notation. There are two reasons that this should not be a problem: 1) It should be clear from the context when we are dealing with an equivalence class. 2) The notation [7] has no meaning as interval notation. If we want the set of real numbers consisting of just the the number 7, we write $\{7\}$.

We now define the operations of addition and multiplication on the equivalence classes of our example by

$$[x]+[y]=[x+y] \qquad \text{and} \qquad [x]\cdot[y]=[xy]\ . \tag{1}$$

Thus, for example,

$$[1]+[2]=[1+2]=[3]=[0] \qquad \text{and} \qquad [1]\cdot[2]=[1\cdot 2]=[2]\ .$$

We can illustrate the result of using addition and multiplication on all of the equivalence classes by writing out addition and multiplication tables. (Do you remember "learning your times table"?) Here they are:

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

·	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

Note that we could have included the classes $[3],[4],[5],...$ in the tables above. We would see that $[1]+[4]=[2]$ and so on. This is redundant, however, because $[3]=[0],[4]=[1],....$ The class of any integer is equal to one of the classes $[0],[1],[2]$.

NOTE: It gets rather tiring to write the brackets around numbers indicating that we are dealing with equivalence classes, so we will dispense of them from now on, it with the understanding that we are not dealing with numbers, but with equivalence classes of those numbers.

2. Consider the relation defined on the integers by

$$xRy \text{ if, and only if, } x-y \text{ is a multiple of 5.}$$

   (a) Prove that this relation is an equivalence relation. Try not to look at the previous such proof, but use it to check yourself.
   (b) Give all of the equivalence classes for this relation *by listing* (this will make the next thing you have to do easier), beginning with [0], then [1], etc. *DO use the bracket notation here.*
   (c) Define addition and multiplication of the equivalence classes by (1) and write the the addition and multiplication tables for the equivalence classes. *Go ahead and dispense with the bracket notation now.*

There is another way to look at all that we have been doing; it is called **modular arithmetic**. To repeat the example from the last exercise using the modular arithmetic approach, we begin with the set $\{0,1,2,3,4\}$, and we define addition as follows. To add two numbers, we add them in the usual sense, *but our answer is then the remainder after dividing the result of the addition by five.* For example, $2+4=1$ because we ordinarily have $2+4=6$ and the remainder when six is divided by five is one. Multiplication is defined in the same way. Here we are doing what we call arithmetic modulo five, often shortened to "mod five".

3. Suppose that we are doing arithmetic with the set $\{0, 1, 2, ..., 8\}$, modulo 9. Use the method just described to find each of the following:

   (a) $7 + 4$ (b) $3 \cdot 5$ (c) $4 + 5$ (d) $6 \cdot 0$

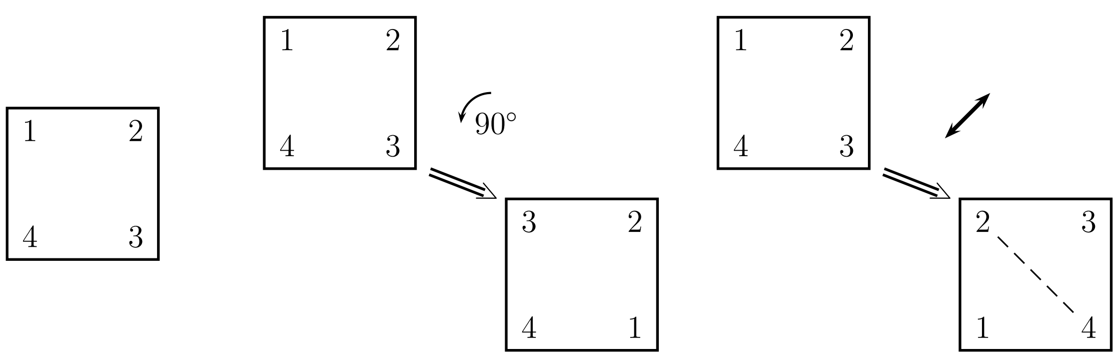
4. Give the addition modulo six and multiplication modulo six tables for the set $\{0, 1, 2, 3, 4, 5\}$.

6.4 Symmetries of a Square, Groups

Performance Criteria:

6. (h) Find the result of the $\star$ operation on two symmetries of a square.
   (i) Find the inverse of any symmetry of a square.
   (j) Determine whether a set, together with a given binary operation, forms a group.

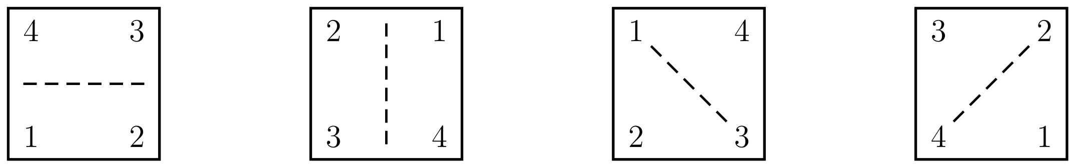
Our goal here is to take a set of "objects", define a binary operation on them, and study the properties of the set and objects taken together. We will begin with a square, which *is not one of our "objects".* The square is labeled at each corner, to identify one corner from another. (Note that each corner is labeled with the same number on both sides of the square.) Beginning with the square oriented as shown below and to the left (with the number one in the upper left corner), our "objects" will be all single movements of the square that land it in the same orientation as it began in, but with the corners in different positions than they were originally in. For example, we could rotate the square counter-clockwise by 90 degrees to obtain the orientation in the center below. (It is standard practice to rotate counter-clockwise, but I really don't know why - everything would work just fine if we were to rotate clockwise instead. The only reason I can see for rotating counter-clockwise is that it is the direction we go around the unit circle in trigonometry.) If we were to flip the square over the diagonal running from the upper left to lower right we would obtain the orientation shown below and to the right.



1. How many ways is it possible to rotate or flip the square so that it "lands" in the same orientation as it began in (ignoring the positions of the numbers)?

You probably came up with seven movements of the square, but there are in fact eight. You probably had three rotations of the square, by 90, 180 and 270 degrees. Note, however, that we could also rotate the square by zero degrees. We will consider this to be a movement, even though it doesn't seem like one at all. We will use the symbols R_0, R_{90}, R_{180} and R_{270} for the counter-clockwise rotations of 0, 90, 180 and 270 degrees, respectively.

There were also four "flips", or reflections, of the square. We will denote to the top-to-bottom reflection by V, for "vertical", and the left-to-right reflection by H, for "horizontal". The flip over the diagonal that runs from upper left to lower right we will refer to as D_1, and the flip over the diagonal that runs from the lower left to upper right will be denoted by D_2. See the diagrams below for the four reflections.



V H D_1 D_2

These eight movements will be our set of objects with which we will be dealing - they are traditionally called **symmetries of the square**. We now define a binary operation on these movements. Given two symmetries a and b, we define $a \star b$ to be the single movement that gives the same result as doing a followed by b. For example, one can see that $D_1 \star R_{90} = V$.

2. Find each of the following.

   (a) $H \star R_{270}$ (b) $V \star H$ (c) $R_{180} \star R_{270}$ (d) $R_{90} \star D$

3. Make a table with eight rows and eight columns as follows: List the eight symmetries across the top in this order:

$$R_0 \qquad R_{90} \qquad R_{180} \qquad R_{270} \qquad H \qquad V \qquad D_1 \qquad D_2$$

   Then list the symmetries down the left side, in the same order. To fill in any space in the table, take the symmetry at the left side of that row of the table, $\star$ it with the symmetry at the top of that column, and put the resulting symmetry in the space. (You are just making a $\star$ table, in the same manner that you made $+$ and $\cdot$ tables in Section 6.3.) For example, your answer Exercise 2(a) will go in the fifth row, fourth column.

4. Use your table from Exercise 3 to answer each of the following.

   (a) Note that when we $\star$ any two of our eight symmetries, the result is still one of our eight symmetries. What does that tell us about the set of symmetries? (Think in terms of Chapter 2.)

   (b) There is a particular symmetry a such that $a \star b = b$ and $b \star a = b$ for *every* symmetry b. What is it? This symmetry is called the **identity element** for $\star$, or the $\star$-identity.

   (c) Consider the symmetry R_{90}. Is there a symmetry a such that both $a \star R_{90} = R_0$ and $R_{90} \star a = R_0$? This symmetry is called the **inverse element** of R_{90} under $\star$, or the $\star$-inverse of R_{90}.

   (d) Find the inverse element of R_{270} under $\star$.

   (e) Find the $\star$-inverse of D_1. Then make sure that you could find the $\star$-inverse of any symmetry.

(f) Find $(R_{180} \star V) \star D_2$ and $R_{180} \star (V \star D_2)$. What property does this indicate that $\star$ might have? Of course, to know this for certain we would have to check all pairs $(a \star b) \star c$ and $a \star (b \star c)$, and there are 512 such pairs! This property does in fact hold for all such pairs.

(g) You should be able to use your table to determine whether $\star$ is commutative. Is it?

Note that what we have here is a set and a binary operation such that

- the set is closed under the operation,
- there is an identity element for the operation,
- every element has an inverse under the operation,
- the operation is associative.

Any set/operation combination having these properties is called a **group**. *Note from our example that the operation need not be commutative, although it will be for some groups.* There are essentially infinitely many groups; the one above is just one example. We call this particular group D_4 You will see some other examples (as well as non-examples) soon. Groups in turn are just one type of **algebraic system** of importance in mathematics. Some others are things called rings, fields, vector spaces and Boolean algebras. Central to the study of algebraic systems are an understanding of binary operations and closure. There are also mathematical systems called topological systems. When algebraic systems have topological structure to them as well, the results are mathematical structures that are very useful in understanding and solving science and engineering problems.

5. In each of the following, a set and an operation is given. Decide whether the two together form a group in each case. If they don't, tell which of the items above do not hold. (Note that for each of the following, the set is always a set of real numbers and the operation is always addition or multiplication. Since we already know that addition and multiplication are associative for all real numbers, you do not need to check the fourth item above.)

   (a) The set is all odd numbers, the operation is addition.
   (b) The set is all even numbers, the operation is addition.
   (c) The set is all odd numbers, the operation is multiplication.
   (d) The set is all even numbers, the operation is multiplication.
   (e) The set is all *integer* powers of two, the operation is multiplication.

6. Repeat Exercise 5 for the following sets and operations. You will probably find it easiest to construct operation tables for the given set and operation.

   (a) The set is $\{0, 1, 2\}$ and the operation is addition mod three.
   (b) The set is $\{0, 1, 2, 3\}$ and the operation is addition mod four.
   (c) The set is $\{0, 1, 2\}$ and the operation is multiplication mod three.
   (d) The set is $\{0, 1, 2, 3\}$ and the operation is multiplication mod four.

Based on what you have just seen, you might guess that the set $\{0, 1, 2, ..., n-1\}$ with addition mod n is always a group, and the same set with multiplication mod n is not. That is in fact correct. We can always cut our set down (sometimes to a ridiculously small size) to get a group with multiplication mod n. The next exercise gives some indication of how this is done.

7. (a) Consider the set $\{0, 1, 2\}$ with multiplication mod three. Where does this combination first not meet the definition of a group? You can throw out an element of the set and fix the problem. Give the resulting operation table.

   (b) You can't do the same with $\{0, 1, 2, 3\}$ with multiplication mod four. What still goes wrong? Which other elements of the set need to be removed to solve the problem? Give the resulting operation table.

   (c) Make a conjecture about when the set $\{1, 2, ..., n-1\}$ with multiplication mod n is a group with multiplication mod n. Demonstrate with another operation table that has not been done so far (by you or in the notes).

   (d) In what cases do more values than just zero need to be removed, and which values need to be removed? Demonstrate as you did for (c).

6.5 More on Groups

Performance Criteria:

6. (k) Determine whether a subset of a group is a subgroup.
   (l) Find a proper subgroup of a given group.
   (m) Be able to work with the groups $SL(2,\mathbb{R})$, $\mathbb{Z}_n$ and $U(n)$.

1. In exercise 3(b) of the previous section you found that the set of even numbers with the operation of addition is a group. Determine which of the following subsets of the even numbers together with addition forms a group. For any that do not, tell why.

   (a) $\{0, 2, 4, 6, 8, ...\}$
   (b) $\{..., -12, -8, -4, 0, 4, 8, 12, ...\}$
   (c) $\{-4, -2, 0, 2, 4\}$
   (d) $\{..., -28, -18, -8, 0, 8, 18, 28, ...\}$

Whenever we have a group, any subset of the group elements that is itself a group under the given operation is called a **subgroup**. The only subgroup above is the subset in part (b). Can you see how to get a proper subgroup of the group we are considering? (Of course every group is a subgroup of itself. We will generally just be interested in proper subgroups.)

2. Can you find any proper subgroups of the group D_4 from the previous section? There are three of them, two of them being pretty small (meaning they have very few elements).

We now consider some other groups, as well as some non-groups.

3. Look at your tables for the operations of addition and multiplication modulo 5 in section 6.3.

   (a) The set $\{0, 1, 2, 3, 4\}$ together with addition modulo five is a group which we will call $\mathbb{Z}_5$. What is the identity? Give the inverse of each element.
   (b) The same set with multiplication modulo five is *NOT* a group. Why not?
   (c) If we consider only the set $\{1, 2, 3, 4\}$ with multiplication modulo 5 we have a group, which we will call $U(5)$. Find its identity, and the inverse for each element.

4. $\mathbb{Z}_6$ is the set $\{0, 1, 2, 3, 4, 5\}$ with the operation of addition modulo six. It is a group; give all of its proper subgroups.

5. (a) Tell why the set $\{1, 2, 3, 4, 5\}$ with multiplication modulo six is not a group.
   (b) What do you suppose it is about the set $\{1, 2, 3, 4\}$ with multiplication modulo five that makes it a group when $\{1, 2, 3, 4, 5\}$ with multiplication modulo six is not? Test your conjecture.

For our next example you will need to recall (or learn) a few basic facts about matrices. We will work only with two by two matrices, meaning two rows and two columns. Here is an example of how to multiply two such matrices together:

$$\begin{bmatrix} 1 & 5 \\ -4 & 7 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 1\cdot 2+5\cdot 3 & 1\cdot 0+5\cdot 6 \\ -4\cdot 2+7\cdot 3 & -4\cdot 0+7\cdot 6 \end{bmatrix} = \begin{bmatrix} 17 & 30 \\ 13 & 42 \end{bmatrix}$$

Note also that for any matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

This indicates that the matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix for two by two matrices with matrix multiplication. We will restrict our attention to those matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c and d are real numbers and $ad - bc = 1$. This of course includes the matrix I.

6. Give an example of such a matrix for which none of a, b, c or d is zero.

7. Prove that the set of all such matrices is closed under matrix multiplication. Recall that to do this you must begin with two unspecified matrices meeting this condition, multiply them, and show that the result is another matrix of the same form.

8. There is a general "recipe" for finding the multiplicative inverse of any two by two matrix. (Any matrix that has an inverse, anyway. Some do not have inverses.) Try to find it by determining values of a, b, c and d such that

$$\begin{bmatrix} 4 & -7 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

   When you think you have the recipe, test it on another matrix.

9. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is such that $ad - bc = 1$, show that its inverse meets this condition as well.

We now know that the set of matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c and d are real and $ad - bc = 1$

- is closed under multiplication,
- contains the identity matrix for two-by-two matrices,
- contains the inverse of any matrix that is already in the set.

Taking as fact that matrix multiplication of matrices is associative, the set of all these matrices is a group that we will denote by $SL(2, \mathbb{R})$

10. Could matrices of the form $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ be a subgroup of $SL(2, \mathbb{R})$? If not, why not? If so, what conditions would a and d have to meet?

11. Could matrices of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ be a subgroup of $SL(2, \mathbb{R})$? If not, why not? If so, what conditions would a and b have to meet?

12. Could matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ be a subgroup of $SL(2, \mathbb{R})$? If not, why not? If so, what conditions would a, b and d have to meet?

6.6 Chapter 6 Exercises

Use each of the relations below for Exercises 1-5.

(a) Define R on $\mathbb{R}$ by xRy if $\lfloor x \rfloor = \lfloor y \rfloor$.

(b) Define R on $\mathbb{R}$ by xRy if $\lceil x \rceil = \lfloor y \rfloor$.

(c) Define R on $\mathbb{R}$ by xRy if $|x| = |y|$.

(d) Define R on $\mathbb{Z}$ by xRy if $x + y$ is even.

(e) Define R on $\mathbb{Z}$ by xRy if $x - y$ is even.

(f) Define R on $\mathbb{Z}$ by xRy if $x + y$ is odd.

(g) Define R on $\mathbb{Z}$ by xRy if xy is even.

(h) Define R on $\mathbb{R} \times \mathbb{R}$ by $(u, v)R(x, y)$ if $v = y$.

(i) Define R on $\mathbb{R} \times \mathbb{R}$ by $(u, v)R(x, y)$ if $u = y$ and $v = x$.

(j) Define R on $\mathbb{Z} \times \mathbb{Z}$ by $(u, v)R(x, y)$ if $u + v = x + y$.

1. For each relation above, give two pairs of elements of the set that relate to each other and two pairs that don't. Give the pairs that relate by writing something like $5\,R\,7$, and indicate pairs that don't relate by writing $4 \not R\, 9$.
2. Determine whether each relation is reflexive.
3. Determine whether each relation is symmetric. For those that aren't, provide a counterexample by writing something of the form $3\,R\,7$ but $7 \not R\, 3$.
4. Determine whether each relation is transitive. Provide counterexamples for those that are not by writing $3\,R\,6$ and $6\,R\,12$, but $3 \not R\, 12$.
5. Determine which of the relations is an equivalence relation. For those that are, describe the equivalence classes in some way that makes it clear what every class is. For relations on Cartesian products of sets it might be easiest to show an equivalence class with a graph.
6. Define a relation R on $\mathbb{R} \times \mathbb{R}$ by $(u, v)R(x, y)$ if $u + v = x + y$. This is clearly an equivalence relation.

   (a) Choose any ordered pair and find five other ordered pairs that are related to it. Because R is an equivalence relation, all of these pairs are related to each other, and form an equivalence class. Plot all of those pairs on a coordinate grid, and speculate as to what the graph of the entire equivalence class looks like.

   (b) Describe the equivalence class from (a) using set builder notation.

   (c) Repeat (a) and (b) for a different equivalence class.

   (d) What does a general equivalence class look like?

   (e) Define a new relation R on $\mathbb{R} \times \mathbb{R}$ by $(u, v)R(x, y)$ if $\lfloor u + v \rfloor = \lfloor x + y \rfloor$. This is also an equivalence relation. Draw the graph of one equivalence class.

7. Define a relation R on $\mathbb{R} \times \mathbb{R}$ by $(u, v)R(x, y)$ if $\lfloor u \rfloor = \lfloor y \rfloor$ and $\lfloor v \rfloor = \lfloor x \rfloor$. Determine whether this is an equivalence relation; if it is, draw the graphs of two equivalence classes. If it is not, tell why and provide appropriate counterexamples.

8. Define a relation R on $\mathbb{R} \times \mathbb{R}$ by $(u, v)R(x, y)$ if $\lfloor u \rfloor = \lfloor x \rfloor$ and $\lfloor v \rfloor = \lfloor y \rfloor$. Determine whether this is an equivalence relation; if it is, draw the graphs of two equivalence classes. If it is not, tell why and provide appropriate counterexamples.

9. Define R on $\mathbb{Z}$ by xRy if x and y have the same sign or both are zero. Give equivalence classes, define $+$ and $\cdot$ on the classes. Is this isomorphic to $\mathbb{Z}$ mod 2?

Index of Symbols

Solutions to Exercises

Section 1.1

2. (a) yes (b) no

Section 1.2

1. (a) S (b) S (c) NS (d) S (e) NS (f) S

2. (b) and (f) are unconditionally true statements.

3. (a) true for $x = -1, -2$, false for all other values of x
   (d) true if $x = y = 2$ or $x = \frac{3}{2}$, $y = 3$, and a bunch of other xy-pairs (can you find some more?). False for $x = y = 1$.

5. If $x = -1$ or $x = -2$, then $x^2 + 3x + 2 = 0$.

8. (a) F, 15 is divisible by 3, but not by 6 (b) T (c) T
   (d) F, $5 < 6$ and $-3(5) > -3(6)$

Section 1.3

1. (a) $B = \{x \in \mathbb{R} \mid 3 \leq x < 7\}$ (b) $[3, 7)$

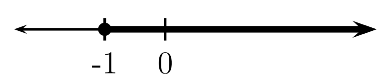
2. (a) $\{8, 9, 10, 11, ...\}$ (b) $\{x \in \mathbb{N} \mid x > 7\}$ or $\{x \in \mathbb{N} \mid x \geq 8\}$

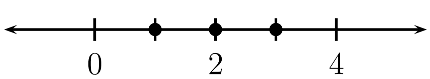
3. The set of all real numbers less than or equal to 2. $\{x \in \mathbb{R} \mid x \leq 2\}$

4. (a) $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ...\}$ (b) $\{..., -11, -7, -3, 1, 5, 9, ...\}$

5. (a) $\{n^2 \mid n \in \mathbb{N}\}$ (b) $\{7n \mid n \in \mathbb{Z}\}$ (c) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$
   (d) $\{3^n \mid n \in \mathbb{Z}\}$ (e) $\{2n + 1 \mid n \in \mathbb{Z}\}$ or $\{2n - 1 \mid n \in \mathbb{Z}\}$

6. (a) 

   (b) 

7. 

8. (a) C, I (b) D, I (c) C, I (d) D, F
   Note that the set in part (a) is an infinite set, but a finite interval. The set in (c) is an infinite set that is also an infinite interval.

Section 1.4

1. (a) $C \subseteq D$ (b) $C \nsubseteq D$ because $5 \in C$ and $5 \notin D$. (c) $C \subseteq D$

(d) $C \nsubseteq D$ because $1 \in C$ and $1 \notin D$. Note that it is sufficient to provide just *one* element in C that is not in D.

(e) $C \subseteq D$ (f) $C \nsubseteq D$ because $1.5 \in C$ and $1.5 \notin D$.

2. Every set is a subset of itself.
3. (a) and (c) are correct, (b) is not
4. The largest possible subset of A is A itself.
5. $\emptyset, \{a\}, \{b\}, \{a, b\}$
6. $|A| \leq |B|$

Section 1.5

1. (a) 3 (b) 18 (c) -9 (d) 9 (e) 1
   (f) does not exist (g) 4.1 (h) 9
2. Both are 5.
3. (a) Multiplication is commutative, subtraction and division are not.
   (b) All of gcd, lcm, min and max are commutative.
4. (a) Addition and multiplication are associative, subtraction and division are not.
   (b) All of gcd, lcm, min and max are associative.
5. (a) $5^2 = 25$, $2^5 = 32$, so $5^2 \neq 2^5$ (b) $4^2 = 16$, $2^4 = 16$, so $4^2 = 2^4$
   (c) $\star$ is defined for all natural numbers.
   (d) $\star$ is not defined for a^b with $a = 0$ and at the same time $b \leq 0$.
6. (a) True, both statements are true. (b) False, the second statement is false.
   (c) False, both statements are false.
7. P and Q is true only if both P and Q are true.
8. (a) True, both statements are true. (b) True, the first statement is true.
   (c) False, both statements are false.
9. P or Q is true if either P or Q is true.

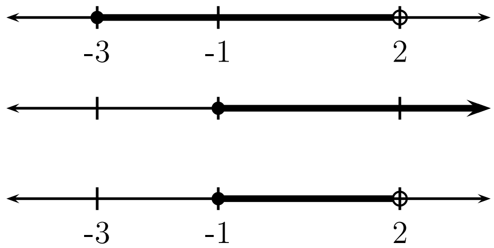
Section 1.6

1. $A \cap B = \{4, 6\}$, $A \cup B = \{2, 3, 4, 5, 6, 8, 10\}$
2. (a) $A' = \{1, 3, 5, 7, 9\}$, $B' = \{1, 2, 7, 8, 9, 10\}$, $A - B = \{2, 8, 10\}$
   (b) $A' \cap B' = \{1, 7, 9\}$, $(A \cap B)' = \{1, 2, 3, 5, 7, 8, 9, 10\}$. The two sets are not the same.

(c) C can be any set of elements of U other than 2, 4, 6, 8, 10.

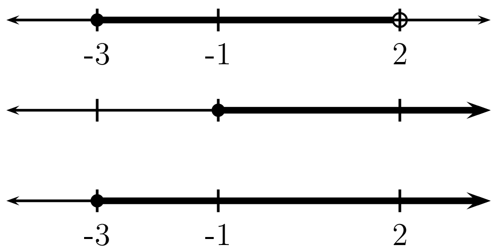
3. (a) If $A \subseteq B$, then $A \cap B = A$. (b) If $B \subseteq A$, then $A \cup B = A$.
   (c) If $A = B$, then $A \cap B = A \cup B$.

4. (a)



$[-1, 2)$ - This is obtained by shading only the part of the number line that is shaded on the graphical representations of *BOTH* of the two original sets.

(b)



$[-3, \infty)$ - This is obtained by shading the part of the number line that is shaded on the graphical representations of *EITHER* of the two original sets.

5. $A \cap B = [1, 4.1]$, $A \cup B = (-2, 8]$, $A' = (-\infty, -2] \cup (4.1, \infty)$, $B' = (-\infty, 1) \cup (8, \infty)$,
   $A' \cap B' = (-\infty, -2] \cup (8, \infty)$, $(A \cap B)' = (-\infty, 1) \cup (4.1, \infty)$, $A - B = (-2, 1)$.

6. (a) (b) $[-2, 3) \cup (3, \infty)$

7. $A \cup B = [3, 5] \cup \{6, 7\}$, $A \cap B = \{4, 5\}$, $A - B = (3, 4) \cup (4, 5)$, $B - A = \{3, 6, 7, \}$

8. (a) $(-\infty, 2) \cup (2, \infty)$ (b) $\{x \in \mathbb{R} \mid x \neq 2\}$ (c) All real numbers except 2.

9. (a) Union and intersection are commutative, difference is not.
   (b) Union and intersection are associative, difference is not.

10. (a) $\leq$, $+$ (b) If A and B are disjoint, then $|A \cup B| = |A| + |B|$
    (c) $|A \cup B| = |A| + |B| - |A \cap B|$

Section 2.1

2. (a) T (b) F (c) F (d) T (e) T (f) T (g) T

3. (b) Let $x = 3$ and $y = 2$. Then $x - y = 1$ and $x + y = 5$ and $x - y$ is not greater than $x + y$.

(c) Let $U = \{1, 2, 3, 4\}$, $A = \{1\}$ and $B = \{2\}$. Then $A' = \{2, 3, 4\}$ which is NOT a subset of B.

4. (a) Let $x = 3$ and $y = -2$. Then $x - y = 5$ and $x + y = 1$, so $x - y > x + y$ for those values.
   (d) This is actually true for all $m, n \in \mathbb{N}$, so it doesn't matter which two you choose!
   (f) The statement is true whenever $A = B$. Let $A = \{2\}$ and $B = \{2\}$; then $A \cup B = \{2\}$ and $A \cap B = \{2\}$, so $A \cup B = A \cap B$.

5. The statement is true only when $x = 0$.

6. (a) $A \not\subseteq C$ (b) x is not a real number (c) x is not divisible by four

7. $x \leq 7$

8. (a) $2 \in \mathbb{R}$ and $2 + 3 \neq 7$ (b) No, it is false because $4 \in \mathbb{R}$ and $4 + 3 = 7$.
   (c) The statement in (b) is not the negation of the statement in (a) because they are both false.

9. See answer to 8(a).

10. (a) There exist sets A and B such that $A \cap B \not\subseteq A$.
    (b) For all sets A and B, $A \cap B \neq A$.
    (c) For all $x, y \in \mathbb{R}$, $x - y \leq x + y$.
    (d) There exist sets A and B such that $A' \not\subseteq B$.
    (e) There exist $m, n \in \mathbb{N}$ such that $m + n \notin \mathbb{N}$.
    (f) For all sets A and B, $A \cup B \not\subseteq A \cap B$.

11. (a) $x = 10$, $x = -3$ (b) $x = 6$, $x = -5$
    (c) For the statement to be false we must insure that $x \leq 7$ and $x \neq -3$.

12. (a) "Not P or not Q", because both P and Q must be true to make "P and Q" true, so if either P or Q is false the whole statement "P and Q" will be false.
    (b) $n = 14$ (c) N is odd or $n \leq 10$ (d) It makes it false.
    (e) $n = 13$ makes the statement in (b) false, and it makes the answer from (c) true.

13. (a) $x \notin A$ and $x \notin B$ (b) $x \notin A$ or $x \in B$
    (c) n is not divisible by two or n is divisible by three.

14. (a) $x \notin A$ and $x \notin B$ (b) $x \notin A$ or $x \in B$

15. The statement is false because 6 is divisible by 2 and 6 is not divisible by 4.

16. (a) $A \subseteq B$ and $A \cap B \neq B$.
    (b) $x < y$ and $x + 3 \geq y + 3$.
    (c) $A \subseteq B$ and $B' \not\subseteq A'$.
    (d) m is an integer and m is not even and m is not odd.
    (e) $x + y = 7$ and either $x \neq 4$ or $x \neq 3$.

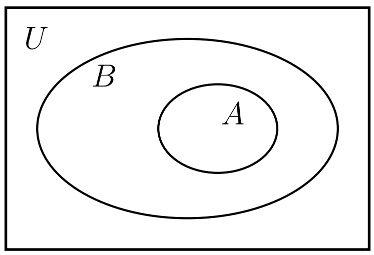
17. For all $x, y \in \mathbb{R}$, if $x < y$ then $x + 3 < y + 3$. There exist $x, y \in \mathbb{R}$ such that $x < y$ and $x + 3 \geq y + 3$.

18. (c) is true, (e) is false because choosing $x = 2$ and $y = 5$ gives $x + y = 7$ and either $x \neq 4$ or $y \neq 3$. (Both, actually, but either one would be enough for the original statement to be false.)

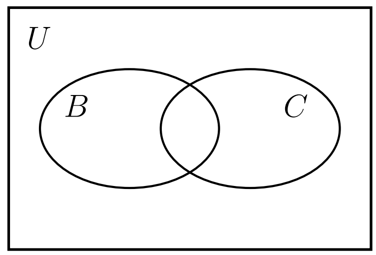
Section 2.2

1. (a) $A \cap B$ (b) $A - B$ (c) $(A \cup B)'$

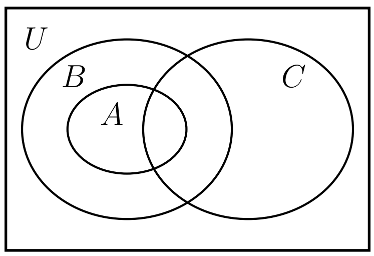
2. (a)



(b)

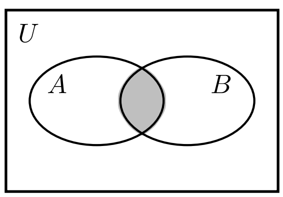


(c)

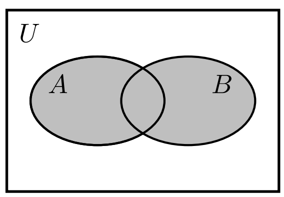


4. There is one in which $A \subseteq B$, one in which $B \subseteq A$, one in which $A = B$ and one in which A and B are disjoint.

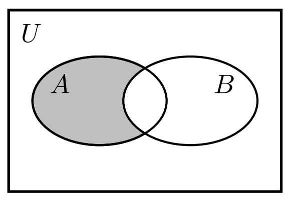
6. (a)



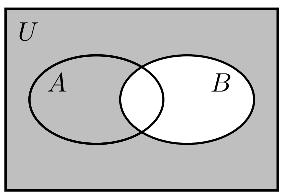
(b)



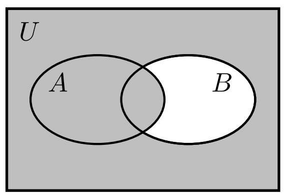
(c)



(d)



(e)



7. (a) $A \cap B \subseteq A$ (b) $A \cap B = A$ (c) Yes.

8. $A \subseteq A \cup B$

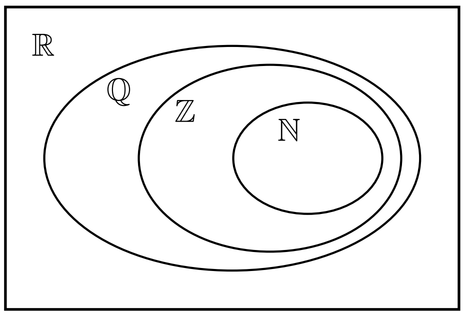
9. (a) $A \cup B = B$ if $A \subseteq B$. (b) $A - B = \emptyset$ if $A \subseteq B$.

10. $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$

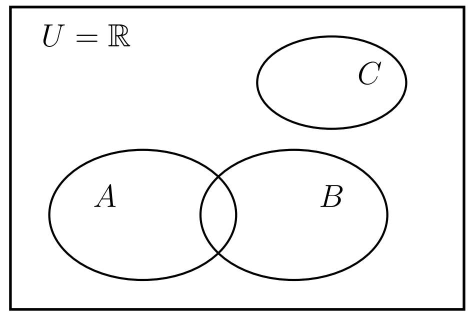
11. If $A \subseteq B$, then $B' \subseteq A'$.

12. (a) $A - B = A \cap B'$ $(A - B)' = B \cup A'$

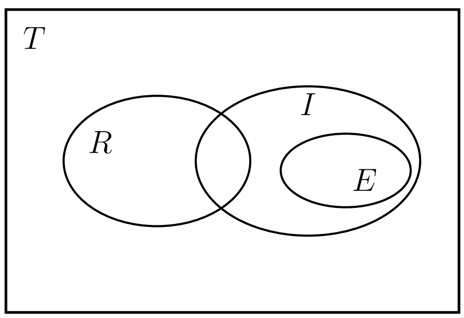
13.



14.



15. (a)



(b) $E \subseteq I$

(c) The intersection of the right triangles and the equilateral triangles is the empty set.

(d) The intersection of the right triangles and the isosceles triangles are the 45-45-90 triangles.

Section 2.3

1. (a) $-1,\ 0,\ 1$ (b) $-2,\ -1,\ 0,\ 1,\ 2$

2. (a) C (closed) (b) NC (not closed), $-1, 1 \in A$, $-1 - 1 = -2 \notin A$.
   (c) NC (d) NC (e) C (f) NC (g) C
   (g) C (h) C (i) NC (j) C (k) NC

3. (a) C, C (b) NC, C

4. (a) C, C (b) NC, NC (c) $\{1, 5, 7\}$ and $\{4, 6, 24\}$, for example.

5. *Any* set of real numbers is closed under min and max.

Section 2.4

1. (a) and (b) The whole numbers are closed under addition and multiplication.
   (c) The whole numbers are not closed under subtraction; $3 - 7 = -4 \notin \mathbb{W}$. The smallest set containing the whole numbers that IS closed under subtraction is $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$

2. The integers are closed under addition and multiplication, but not division: $5 \div 2 = \frac{5}{2} \notin \mathbb{Z}$.

3. square root, cube root, absolute value, ...

Section 2.5

1. $|A| = 5$

2. (a) iii (b) i (c) ii (d) iv (e) i (f) ii

3. (a) A is not closed under union; $\{1\} \cup \{2\} = \{1, 2\} \notin A$
   (b) A is closed under intersection.
   (c) A is not closed under difference; $\{2, 3\} - \{2\} = \{3\} \notin A$

4. (a) 4 (b) B is closed under union, intersection and difference.
   (c) If the empty set is removed from B, it is no longer closed under intersection ($[0, 1) \cap [1, 2) = \emptyset$) or difference ($[0, 1) - [0, 1) = \emptyset$).

5. (a) $P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$
   (b) $P(B) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$, $|P(B)| = 16$
   (c) $|P(C)| = 2$ (d) $|P(D)| = 4$ (e) $|P(A)| = 2^{|A|}$ (f) $|\emptyset| = 1 = 2^0$

6. (a) Not a partition because $A_1 \cap A_2 = \{1\} \neq \emptyset$.
   (b) Partition.
   (c) Not a partition because intersections of some of the sets are non-empty AND because one of the sets is the empty set.
   (d) Not a partition because the union of all the sets is not the entire set A.

7. Include the sets $\{3\}$ and $[4, 6]$, for one example.

8. $A \times B = \{(a, 1), (a, 2), (a, 3), (a, 4), (b, 1), (b, 2), (b, 3), (b, 4), (c, 1), (c, 2), (c, 3), (c, 4)\}$

9. In general, $A \times B \neq B \times A$. It IS true in the case that $A = B$.

10. $|A \times B| = mn$

12. (a) $\{(x, y) \in \mathbb{Z} \times \mathbb{Z}\}$ or $\{(x, y) \mid x, y \in \mathbb{Z}\}$ (c) $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = 3\}$
    (e) $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid -1 < x \leq 3 \text{ and } 2 \leq y < 4\}$ (f) $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \geq 2\}$

Section 2.6

1. (a) If m is divisible by 6, then m is divisible by 2 and by 3.
   (b) If $m+n$ is even, then m and n are odd.
   (c) If $A \cap B = A$, then $A \subseteq B$.

2. (a) The statements from 1(a) and (c) are true, and their converses are as well. The statement from 1(b) is true, but its converse is not.
   (b) No, as evidenced by the statement from 1(b).

3. $A = B$ if, and only if, $A \cap B = A \cup B$

4. (a) If $A \subseteq B$, then $A \cap B = B$; If $A \cap B = B$, then $A \subseteq B$. The biconditional statement is true.
   (b) If $A \subseteq B$, then $B' \subseteq A'$; If $B' \subseteq A'$, then $A \subseteq B$. The biconditional statement is true.
   (c) If $x + y = 7$, then $x = 4$ and $y = 3$; If $x = 4$ and $y = 3$, then $x + y = 7$. The biconditional statement is false because the first of these two statements is false.

5. (a) If $m + 1$ is not odd, then m is not even.
   (b) If $A \cap B \neq B$, then $A \not\subseteq B$.
   (c) If $x + 3 \geq y + 3$, then $x \geq y$.
   (d) If $A \cap B \neq A$, then $A \not\subseteq B$.

6. (a) The statements from Exercises 5(a),(c) and (d) are all true, and so are all of their contrapositives.
   (b) It seems that if a statement is true, its contrapositive is as well.

7. (a) If $m + n$ is not even, then m is not odd or n is not odd.
   (b) If m is not even and m is not odd, then m is not an integer.
   (c) If $x \neq 4$ or $y \neq 3$, then $x + y \neq 7$.

Section 3.1

1. (a) The rule is a function.
   (b) The rule is not a function because the element $c \in A$ is not assigned an element in B and because d is assigned two values in B. (Of course either one of these alone is enough to show that the rule is not a function.)
   (c) The rule is a function.
   (d) The rule is not a function because the element $b \in A$ is assigned more than one element in B.
   (e) The rule is not a function because the element $b \in A$ is not assigned an element in B.
   (f) The rule is a function.

2. $\text{Ran}(f) = \{n \in \mathbb{N} \mid 1 \leq n \leq 26\}$ and $\text{Ran}(g) = \{0, 1\}$.

3. (a) $\text{Dom}(f) = (-\infty, \infty)$ and $\text{Ran}(f) = (-\infty, 0]$.
   (b) $\text{Dom}(g) = (-\infty, \infty)$ and $\text{Ran}(g) = (-\infty, \infty)$.
   (c) $\text{Dom}(h) = (-\infty, 5]$ and $\text{Ran}(h) = [0, \infty)$.
   (d) $\text{Dom}(f) = (-\infty, -3) \cup (-3, \infty)$ and $\text{Ran}(f) = (-\infty, 0) \cup (0, \infty)$.

4. (a) $\text{Dom}(f) = \mathbb{R}$ and $\text{Ran}(f) = \{x \in \mathbb{R} \mid x \leq 0\}$.
   (b) $\text{Dom}(g) = \mathbb{R}$ and $\text{Ran}(g) = \mathbb{R}$.
   (c) $\text{Dom}(h) = \{x \in \mathbb{R} \mid x \leq 5\}$ and $\text{Ran}(h) = \{x \in \mathbb{R} \mid x \geq 0\}$.
   (d) $\text{Dom}(f) = \{x \in \mathbb{R} \mid x \neq -3\}$ and $\text{Ran}(f) = \{x \in \mathbb{R} \mid x \neq 0\}$.

5. (a) The range of f is $[0, \infty)$ and the range of g is $\{1, 4, 9, 16, 25, ...\}$, which can also be written as $\{n^2 \mid n \in \mathbb{N}\}$.
   (b) If we restrict the domain of f to $[0, \infty)$ the range remains unchanged. There are many other ways to restrict the domain without restricting the range, the most notable of which is to $(-\infty, 0]$.

6. (a) $\text{Dom}(f) = \mathbb{Z}$ and $\text{Ran}(f) = \{..., -9, -4, -1, 0\}$.
   (b) $\text{Dom}(g) = \mathbb{Z}$ and $\text{Ran}(g) = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.
   (c) $\text{Dom}(h) = \{..., 2, 3, 4, 5\}$ and $\text{Ran}(h) = \{0, 1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, ...\}$.
   (d) $\text{Dom}(f) = \{n \in \mathbb{Z} \mid n \neq -3\}$ and $\text{Ran}(f) = \{1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, ...\}$.

Section 3.2

1. (a) $\{..., -3, -2, -1, 0, 1, 2\}$, the largest element is 2
   (b) $\{..., -6, -5, -4, -3\}$, the largest element is -3
   (c) $\{..., 2, 3, 4, 5, 6, 7\}$, the largest element is 7
   (d) $\{..., -2, -1, 0, 1, 2, 3\}$, the largest element is 3

2. (a) 3 (b) -7 (c) -11 (d) 1

3. $\lfloor x \rfloor = x$ when x is an integer.

4. (a) The range of the floor function is the integers.
   (b) $\text{Ran}(f) = \mathbb{Z}$ and $\text{Ran}(g) = \{..., -9, -6, -3, 0, 3, 6, 9, ...\}$

5. (a) $\text{Ran}(f) = \{0, 1, 2, 3, 4\}$ (b) Yes, $\text{Ran}(f) = \{0, 1, 2, 3, 4, 5\}$ (c) No
   (d) Yes, $\text{Ran}(f) = \{0, 1, 2, 3, 4, 5\}$

6. (a) 9 (b) -5 (c) 7 (d) 4

7. $\lfloor x + 1 \rfloor = \lceil x \rceil$ if $x \in \mathbb{R} - \mathbb{Z}$. If $x \in Z$, $\lfloor x + 1 \rfloor = \lceil x \rceil + 1$.

8. (a) $\lfloor -x \rfloor = -\lceil x \rceil$ (b) $\lceil -x \rceil = -\lfloor x \rfloor$

9. $\lceil x \rceil = -\lfloor -x \rfloor$

Section 3.3

1. (a) $f([3,7]) = [1,9]$ and $f(\{1,2,3\}) = \{-3,-1,1\}$
   (b) $g([3,7]) = [9,49]$ and $g([-1,5]) = [0,25]$
   (c) $h([3,7]) = \{3,4,5,6,7\}$

4. (a) $f^{-1}(\{1,2,3,4\}) = \{3, 3.5, 4, 4.5\}$ and $f^{-1}([4,7]) = [4.5, 6]$
   (b) $g^{-1}(\{7\}) = \{-\sqrt{7}, \sqrt{7}\}$ and $g^{-1}([4,49]) = [-7,-2] \cup [2,7]$

5. $g^{-1}(\{3\}) = [3,4)$ and $g^{-1}(\{-5,9,10\}) = [-5,-4) \cup [9,11)$

Section 3.4

1. $A \cup B, A \cap B,\ A - B,\ B - A,\ A \triangle B$ (symmetric difference)

2. Subtraction and division are *not* commutative, so $f - g \neq g - f$ and $\frac{f}{g} \neq \frac{g}{f}$.

3. (a) $(f+g)(j) = f(j) + g(j) = 10 + 1 = 11$ (b) $(fg)(e) = f(e)g(e) = (5)(0) = 0$
   (c) $\left(\dfrac{g}{f}\right)(c) = \dfrac{g(c)}{f(c)} = \dfrac{1}{3}$ (d) $\left(\dfrac{g}{f}\right)(i) = \dfrac{g(i)}{f(i)} = \dfrac{0}{9} = 0$
   (e) $\left(\dfrac{f}{g}\right)(i)$ is undefined, since $g(i) = 0$

4. (a) $(f+g)(x) = \dfrac{x^3 - 2x^2 + x + 3}{x - 2}$ (b) $(f-g)(x) = \dfrac{x^3 - 2x^2 - x - 3}{x - 2}$
   (c) $(fg)(x) = \dfrac{x^3 + 3x^2}{x - 2}$ (d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{x^3 - 2x^2}{x + 3}$

5. (a) All but $\dfrac{f}{g}$ can be evaluated for all values in A.
   (b) $\dfrac{f}{g}$ cannot be evaluated for all values in A. The domain of $\dfrac{f}{g}$ is all the consonants.

6. (a) $\text{Dom}(f) = \{x \in \mathbb{R} \mid x \neq 3\}$ and $\text{Dom}(g) = [-2, \infty)$.
   (b) The domains are all $[-2,3) \cup (3,\infty)$,.
   (c) $\text{Dom}\left(\dfrac{f}{g}\right) = (-2,3) \cup (3,\infty)$.

7. The domains of $f+g$, $f-g$ and fg are all $\text{Dom}(f) \cap \text{Dom}(g)$. The domain of $\dfrac{f}{g}$ is $\text{Dom}(f) \cap \text{Dom}(g) - g^{-1}(\{0\})$.

8. $f[g(7)] = f[5] = 15$

9. (a) $(g \circ f)(-1) = g[f(-1)] = g[-3] = -5$
   (b) $(g \circ f)(7) = g[f(7)] = g[21] = 19$, $f \circ g$ and $g \circ f$ are not the same function.
   (c) $(g \circ f)(x) = g[f(x)] = g[3x] = 3x - 2$

10. (a) $(g \circ f)(9) = 4$, $(f \circ g)(3) = 2$, $(f \circ g)(-5)$ does not exist because we can't find $f(-4)$.
   (b) The domain of f is $[0, \infty)$ and its range is the same. Both the domain and range of g are all real numbers.
   (c) The range of g must be a subset of the domain of f.
   (d) If the domain of g is restricted to $[-1, \infty)$, then $f \circ g$ can be found for every element in that set.
   (e) Because the range of f is contained in the domain of g.
11. The domain of $f \circ g$ is $\{x \in \mathbb{R} \mid x \neq 0\}$ and the domain of $g \circ f$ is $\{x \in \mathbb{R} \mid x \neq 7\}$.
12. $\text{Dom}(f \circ g) = g^{-1}(\text{Dom}(f))$.

Section 3.5

1. (a) $\text{Ran}(f) = [0, \infty)$ and $\text{Ran}(g) = \mathbb{R}$.
   (b) For every y in the range of f *except zero*, there are two values of x $(\pm\sqrt{y})$ such that $f(x) = y$. $f(x) = 0$ only when $x = 0$.
   (c) For every y in the range of g there is exactly one value of x such that $g(x) = y$. For example, to obtain $y = 173$, x must be 85.
2. (a) one-to-one but not onto (b) not one-to-one, but onto
   (c) one-to-one and onto
3. (a) neither one-to-one nor onto (b) not one-to-one, but onto
   (c) neither one-to-one nor onto
4. (a) not one-to-one or onto (c) one-to-one and onto
   (f) one-to-one, not onto
5. A function is one-to-one if and only if the preimage of any singleton subset (set containing only one element) of the range is also a singleton set.
6. $[0, \infty)$ also
7. (a) $\{\text{e}, \text{g}, \text{i}\}$ (c) $\{\text{e}, \text{f}, \text{g}, \text{h}\}$ (f) $\{\text{f}, \text{g}, \text{h}, \text{i}\}$
8. $[0, \infty)$ or $(-\infty, 0]$. There are others, like $[0, 5) \cup (-\infty, -5]$.
9. (a) The range of f is $(-\infty, 0]$. f is not one-to-one, but it is if the domain is restricted to either $[0, \infty)$ or $(-\infty, 0]$.
   (b) f is one-to-one and onto, with domain and range $\mathbb{R}$.
   (c) h is one-to-one and onto, with domain and range $\mathbb{R}$.
   (d) f has range $[0, \infty)$. It is one-to-one.
   (e) g has range $\mathbb{R} - \{0\}$, and it is one-to-one.
   (f) h has range $[0, 1)$. It is not one-to-one, but it is if the domain is restricted to either $[0, \infty)$ or $(-\infty, 0]$.

10. $h(x) = \dfrac{x-3}{2}$.

11. $(g \circ h)(x) = g[h(x)] = g\left[\dfrac{x-3}{2}\right] = 2\left(\dfrac{x-3}{2}\right) + 3 = x$ $\quad (h \circ g)(x) = h[g(x)] = h(2x-3) = \dfrac{(2x-3)+3}{2} = x$

12. (a) $f^{-1}(x) = \sqrt{x}$ (b) $f^{-1}(x) = -\sqrt{x}$

13. (a) $f : [0, \infty) \to (-\infty, 0]$, $f^{-1}(x) = \sqrt{-x}$ or
$f : (-\infty, 0] \to (-\infty, 0]$, $f^{-1}(x) = -\sqrt{-x}$
(b) $g : \mathbb{R} \to \mathbb{R}$, $g^{-1}(x) = \frac{1}{7}x$
(c) $h : \mathbb{R} \to \mathbb{R}$, $h^{-1}(x) = x^3$
(d) $f : (-\infty, 5] \to [0, \infty)$, $f^{-1}(x) = 5 - x^2$
(e) $g : \mathbb{R} \to \mathbb{R} - \{3\}$, $g^{-1}(x) = \frac{1}{x} - 3$
(f) $h : [0, \infty) \to [0, 1)$, $h^{-1}(x) = \sqrt{\dfrac{x}{1-x}}$ or
$h : (-\infty, 0] \to [0, 1)$, $h^{-1}(x) = -\sqrt{\dfrac{x}{1-x}}$

Section 3.6

3. (a) $g(n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{1}{n} & \text{if } n \text{ is even.} \end{cases}$
(b) $h(x) = |x|$ (c) $f(n) = (-1)^n$, $g(n) = (-1)^{n+1}$

4. $f(n) = \begin{cases} (-3)(\frac{n-1}{2}) & \text{if } n \text{ is odd,} \\ 3(\frac{n}{2}) & \text{if } n \text{ is even.} \end{cases}$

7. (a) $\text{Dom}(g) = [0, 5) \cup (5, \infty)$, $\text{Ran}(g) = \mathbb{R}$
(b) $\text{Dom}(h) = \mathbb{R}$, $\text{Ran}(h) = [0, 1)$

8. (a) $\chi_{[2,5]}(7) = 0$, $\chi_{[2,5]}(-4.5) = 0$, $\chi_{[2,5]}(\pi) = 1$
(b) $g(x) = \frac{1}{x}\chi_A(x)$, where A is the set of even numbers, or the set of even natural numbers (either is OK).

Section 4.1

1. (a) $x \in A \cap B$ if, and only if, $x \in A$ and $x \in B$
(b) $x \in A - B$ if, and only if, $x \in A$ and $x \notin B$
(c) $x \in A'$ if, and only if, $x \notin A$

2. $A = B$ if $A \subseteq B$ and $B \subseteq A$

3. (a) 17 is odd because $17 = 2(8) + 1$ and $8 \in \mathbb{Z}$

(b) 1.5 is a rational number because $1.5 = \frac{3}{2}$ and $3, 2 \in \mathbb{Z}$

(e) 45 is divisible by 15 because $45 = 3(15)$ and $3 \in \mathbb{Z}$

Section 4.2

1. The sum of an even number and an odd number is an odd number. I determined this by considering $4 + 7 = 11$.

2. (a) If x is even and y is odd, then $x + y$ is odd.

   (b) If x is a multiple of 3, then x is a multiple of 6.

   (c) If x and y are both rational, then $x - y$ is rational.

   (d) If $x = 4^m$, then $x = 2^n$ for some n.

Section 5.1

1. $f(n) = (-1)^{n+1}2n$

2. (a) $\frac{1}{3^n}$, $n = 0, 1, 2, 3, ...$ (b) $3n + 2$, $n = 1, 2, 3, ...$

3. (a) $-3, -6, -12, -24, -48, ...$ (b) $6, 11, 16, 21, 26, ...$

4. (a) $20, 17, 14, 11, 8, 5, 2, ...$ (b) $2, 5, 8, 11, 14, 17, 20, ...$

   (c) $1, -8, 27, -64, 125, -216, 343, ...$ (d) $2, 6, 18, 54, 162, 486, 1458, ...$

   (e) $1, 2, 3, 2, 1, 2, 3, ...$

5. (a) $1 + 3n$, $n = 1, 2, 3, ...$ $a_1 = 4$, $a_{n+1} = a_n + 3$

   (b) 2^n, $n = 0, 1, 2, ...$ $a_1 = 1$, $a_{n+1} = 2a_n$

   (c) Hmmmmm.... the amazing thing is that this sequence DOES have an explicit representation, but it is pretty complicated (and involves $\sqrt{5}$)! Its recursive definition is $a_1 = 1$, $a_2 = 1$, $a_{n+1} = a_n + a_{n-1}$

   (d) $5(-1)^n$, $n = 0, 1, 2, ...$ $a_1 = 5$, $a_{n+1} = -a_n$

Section 5.2

1. (a) 3, 8, 13, 18, 23, 28, ...

   (b) The a_{42} term would be $3+42(5) = 3+210 = 213$. $a_n = 3+5n$, $n = 0, 1, 2, 3, ...$

2. (a) $1, 4, 16, 64, 256, 1024, ...$, $a_n = 4^n$, $n = 0, 1, 2, 3, ...$

   (b) $3, 6, 12, 24, 48, 96, ...$, which is the same as $3(1), 3(2), 3(4), 3(8), 3(16), ...$, so $a_n = 3(2^n)$, $n = 0, 1, 2, 3, 4, ...$

   (c) $3, 7, 15, 31, 63, 127, ...$, which is the same as $4 - 1, 8 - 1, 16 - 1, 32 - 1, 64 - 1, ...$, so $a_n = 2^{n+1} - 1$, $n = 1, 2, 3, 4, ...$

   (d) $2, 4, 16, 256, 65536, ...$, which is the same as $2^1, 2^2, 2^4, 2^8, 2^{16}, ...$, so $a_n = 2^{2^n}$, $n = 0, 1, 2, 3, ...$

3. $1, 2, 4, 8, 16, 32, ...,$ $\quad a_n = 2^n,\ n = 0, 1, 2, 3, 4, ...$

4. (a) $a_1 = 1,\ a_2 = 1,\ a_n = a_{n-1} + a_{n-2}$

5. $6, 16, 12, 19, 4, ...$

6. (a) $1, 3, 8, 6, 9, 3, 8, 6, 9, ...$ (b) $5, 10, 0, 0, 0, ...,$ $\quad 0, 0, 0, 0, ...$

7. (a) $6, 16, 12, 19, 3, 10, 20, 0, 0, 0, ...$
   (b) $2, 7, 19, 3, 10, 20, ...$ This has a fixed point of zero.
   $3, 10, 20, 0, 0, 0, ...$ This also has a fixed point of zero.
   (c) If there is a fixed point, it seems to be zero.
   (d) $5, 15, 15, 15, 15, ...$ Hmmmm.... I guess there can be other fixed points besides zero!

Section 5.3

1. (a) Assume $a_k = 4^k$, show $a_{k+1} = 4^{k+1}$.
   (b) Assume $a_k = 3(2)^k$, show $a_{k+1} = 3(2)^{k+1}$.
   (c) Assume $a_k = 2^{k+1} - 1$, show $a_{k+1} = 2^{(k+1)+1} - 1$.
   (d) Assume $a_k = 2^{2^k}$, show $a_{k+1} = 2^{2^{k+1}}$.

Section 5.4

1. (a) $1, 2, 4, 7, 12, 20, ...$ (b) $1, 0, 1, 0, 1, 0, ...$ (c) $2, 6, 12, 20, 30, 42, ...$

2. (a) $1 + 3 + 5 + 7 + 9 + \cdots + (2n - 1)$ (b) $1 + 2 + 3 + 4 + 5 + \cdots + n$
   (c) $1 + 4 + 9 + 16 + 25 + \cdots + n^2$ (d) $1 + 8 + 27 + 64 + 125 + \cdots + n^3$

3. (b) $\sum_{j=0}^{n} (-1)^j$ OR $\sum_{j=1}^{n} (-1)^{j+1}$ (c) $\sum_{j=1}^{n} 2j$

Section 5.5

1. (a) Assume $\sum_{j=1}^{k} (2j - 1) = k^2$, show that $\sum_{j=1}^{k+1} (2j - 1) = (k + 1)^2$.
   (b) Assume $\sum_{j=1}^{k} j = \frac{k(k + 1)}{2}$, show that $\sum_{j=1}^{k+1} j = \frac{(k + 1)[(k + 1) + 1]}{2}$
   (c) Assume $\sum_{j=1}^{k} j^3 = \left[\frac{k(k + 1)}{2}\right]^2$, show that $\sum_{j=1}^{k+1} j^3 = \left[\frac{(k + 1)[(k + 1) + 1]}{2}\right]^2$

Section 6.1

2. (a) Yes, every real number relates to itself.
   (b) The relations (a), (b), (c), (d), (f), (g) and (h) are all reflexive.
3. (a) If A is related to B, then B is related to A as well.
   (b) The relations (b), (c), (d), (e) and (h) are all symmetric.
4. (b) The relations (a), (b), (d), (f), (g) and (h) are all transitive.

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