Linear Algebra I

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0 Introduction

This book is an attempt to make the subject of linear algebra as understandable as possible, for a first time student of the subject. I developed the book from a set of notes used to supplement a standard textbook used when I taught the course in the past. At the end of the term I surveyed the students in the class, and the vast majority of them thought that the supplemental notes that I had provided would have been an adequate resource for them to learn the subject. Encouraged by this, I put further work in to correcting errors, adding examples and including material that I had left to the textbook previously. Here is the result, in its fourth edition.

You should look at the table of contents and page through the book a bit at first to see how it is organized. Note in particular that each section begins with statements of the **performance criteria** addressed in that section. Performance criteria are the specific things that you will be expected to do in order to demonstrate your skills and your understanding of concepts. Within each section you will find examples relating to those performance criteria, and at the end of each chapter are exercises for you to practice your skills and test your understanding.

It is my belief that you will likely forget many of the details of this course after you leave it, so it might seem that developing the ability to perform the tasks outlined in the performance criteria is in vain. That causes me no dismay, nor should it you! My first goal is for you to learn the material well enough that if you are required to recall or relearn it in other courses you can do so easily and quickly, and along the way I hope that you develop an appreciation for the subject of linear algebra. Most of all, my aim is for you to develop your skills beyond their current level in the areas of mathematical reasoning and communication. In support of that, you will find the first (well, "zeroth"!) **outcome** and its associated performance criteria below. Outcomes are statements of goals that are perhaps a bit nebulous, and difficult to measure directly. The performance criteria give specific, measurable tasks by which success in the overarching outcome can be determined.

Learning Outcome:

0. Use the subject of linear algebra to develop sophistication in understanding of mathematical concepts and connections, and in the communication of that understanding.

   Performance Criteria:

   (a) Apply skills and knowledge associated with specific performance criteria to problems related to, but not specifically addressed by, those performance criteria.

   (b) Communicate mathematical ideas clearly by correctly using written English and proper mathematical notation.

Enough talk - let's get to it!

Gregg Waterman
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1 Systems of Linear Equations

Learning Outcome:

1. Solve systems of linear equations using Gaussian elimination, use systems of linear equations to solve problems.

   Performance Criteria:

   (a) Determine whether an equation in n unknowns is linear.

   (b) Determine what the solution set to a linear equation represents geometrically.

   (c) Determine whether an n-tuple is a solution to a linear equation or a system of linear equations.

   (d) Solve a system of two linear equations by the addition method.

   (e) Find the solution to a system of two linear equations in two unknowns *graphically*.

   (f) Give the coefficient matrix and augmented matrix for a system of equations.

   (g) Determine whether a matrix is in row-echelon form. Perform, by hand, elementary row operations to reduce a matrix to row-echelon form.

   (h) Determine whether a matrix is in reduced row-echelon form. Use a calculator or software to reduce a matrix to reduced row-echelon form.

   (i) For a system of equations having a unique solution, determine the solution from either the row-echelon form or reduced row-echelon form of the augmented matrix for the system.

   (j) Use a calculator to solve a system of linear equations having a unique solution.

   (k) Use systems of equations to solve curve fitting and temperature equilibrium problems.

1.1 Linear Equations and Systems of Linear Equations

Performance Criteria:

1. (a) Determine whether an equation in n unknowns is linear.
   (b) Determine what the solution set to a linear equation represents geometrically.
   (c) Determine whether an n-tuple is a solution to a linear equation or a system of linear equations.

Linear Equations and Their Solutions

It is natural to begin our study of linear algebra with the process of solving systems of **linear equations**, and applications of such systems. Linear equations are ones of the form

$$2x-3y=7 \qquad 5.3x+7.2y+1.4z=15.9 \qquad a_{11}x_1+a_{12}x_2+\cdots+a_{1(n-1)}x_{n-1}+a_{1n}x_n=b_1$$

where x, y, z, x_1, ..., x_n are all unknown values. In the third example, a_{11}, a_{12}, ..., a_{1n}, b_1 are all known numbers, just like the values $2, -3, 5.3, 7.2, 1.4$ and 15.9. Although you may have used x and y, or x, y and z as the unknown quantities in the past, we will often use x_1, x_2, ..., x_n. One obvious advantage to this is that we don't have to fret about what letters to use, and there is no danger of running out of letters! You will eventually see that there is also a very good mathematical reason for using just x (or some other single letter), with subscripts denoting different values.

When we look at two or more linear equations together "as a package," the result is something called a **system of linear equations**. Here are a couple of examples:

$$\begin{aligned} x+3y-2z &= -4\\ 3x+7y+z &= 4\\ -2x+y+7z &= 7 \end{aligned}$$

$$\begin{aligned} a_{11}x_1+a_{12}x_2+\cdots+a_{1,n-1}x_{n-1}+a_{1n}x_n &= b_1\\ a_{21}x_1+a_{22}x_2+\cdots+a_{2,n-1}x_{n-1}+a_{2n}x_n &= b_2\\ &\vdots\\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{m,n-1}x_{n-1}+a_{mn}x_n &= b_m \end{aligned}$$

The system above and to the left is a system of three equations in three unknowns. We will spend a lot of time with such systems because they exhibit just about everything that we would like to see but are small enough to be manageable to work with. The second system above is a general system of m equations in n unknowns. m and n need not be the same, and either can be larger than the other.

Our objective when dealing with a system of linear equations is usually to solve the system, which means to *find a set of values for the unknowns for which all of the equations are true.* If we find such a set, it is called a **solution** to the system of equations. A solution is then a list of numbers, in order, which can be substituted for x_1, x_2, ... to make *EVERY* equation in the system true. In some cases there is more than one such set, so there can be many solutions, and sometimes a system of equations will have no solution. That is, there is no set of values for x_1, x_2, ... that makes every one of the equations in the system true.

1.2 The Addition Method

Performance Criteria:

1. (d) Solve a system of two linear equations by the addition method.
   (e) Find the solution to a system of two linear equations in two unknowns *graphically*.

Consider the system $\begin{array}{rcr} x-3y & = & 6 \\ -2x+5y & = & -5 \end{array}$ of linear equations. In this case a solution to the system is an **ordered pair** (x,y) that makes *both* equations true. A large part of linear algebra concerns itself with methods of solving such systems, and ways of interpreting solutions or lack of solutions.

In the past you should have learned two methods for solving such systems, the **addition method** and the **substitution method**. The method we want to focus on is the addition method. In this case we could multiply the first equation by two and add the resulting equation to the second. (The first equation itself is left alone.) The result is $\begin{array}{rcr} x-3y & = & 6 \\ -y & = & 7 \end{array}$; from this we can see that $y=-7$. This value is then substituted into the first equation to get $x=-15$.

Sometimes we have to do something a little more complicated:

◇ **Example 1.2(a):** Solve the system $\begin{array}{rcr} 2x-4y & = & 18 \\ 3x+5y & = & 5 \end{array}$ using the addition method.

Here we can eliminate x by multiplying the first equation by 3 and the second by -2, then adding:

$$\begin{array}{rcr} 2x-4y & = & 18 \\ 3x+5y & = & 5 \end{array} \quad \Longrightarrow \quad \begin{array}{rcr} 6x-12y & = & 54 \\ -6x-10y & = & -10 \\ \hline -22y & = & 44 \\ y & = & -2 \end{array}$$

Now we can substitute this value of y back into either equation to find x:

$$\begin{array}{rcr} 2x-4(-2) & = & 18 \\ 2x+8 & = & 18 \\ 2x & = & 10 \\ x & = & 5 \end{array}$$

The solution to the system is then $x=5$, $y=-2$, which we usually write as the ordered pair $(5,-2)$. It can be easily verified that this pair is a solution to both equations.

It is useful for the future to understand a way to multiply *only one* equation by a factor before adding. In the next example we see how that is done, using the same system of equations as in Example 1.2(a).

◇ **Example 1.2(b):** Solve the system $\begin{aligned} 2x - 4y &= 18 \\ 3x + 5y &= 5 \end{aligned}$ using the addition method.

This time we eliminate x by multiplying the first equation by $-\frac{3}{2}$ and then adding the result to the second equation:

$$\begin{aligned} 2x - 4y &= 18 \\ 3x + 5y &= 5 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} -3x + 6y &= -27 \\ 3x + 5y &= 5 \\ \hline y &= -2 \end{aligned}$$

As before, we substitute this value of y back into either equation to find x:

$$\begin{aligned} 2x - 4(-2) &= 18 \\ 2x + 8 &= 18 \\ 2x &= 10 \\ x &= 5 \end{aligned}$$

The solution to the system is $x = 5$, $y = -2$.

The previous two examples were two linear equations with two unknowns. Now we consider the following system of *three linear equations in three unknowns.*

$$\begin{aligned} x + 3y - 2z &= -4 \\ 3x + 7y + z &= 4 \\ -2x + y + 7z &= 7 \end{aligned} \tag{1}$$

We can use the addition method here as well; first we multiply the first equation by negative three and add it to the second. We then multiply the first equation by two and add it to the third. This eliminates the unknown x from the second and third equations, giving the second system of equations shown below. We can then add $\frac{7}{2}$ times the second equation to the third to obtain a new third equation in which the unknown y has been eliminated. This "final" system of equations is shown to the right below.

$$\begin{aligned} x + 3y - 2z &= -4 \\ 3x + 7y + z &= 4 \\ -2x + y + 7z &= 7 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x + 3y - 2z &= -4 \\ -2y + 7z &= 16 \\ 7y + 3z &= -1 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x + 3y - 2z &= -4 \\ -2y + 7z &= 16 \\ \tfrac{55}{2}z &= 55 \end{aligned} \tag{2}$$

We now solve the last equation to obtain $z = 2$. That result is then substituted into the second equation in the last system to get $y = -1$. Finally, we substitute the values of y and z into the first equation to get $x = 3$. The solution to the system is then the **ordered triple** $(3, -1, 2)$. The process of finding the last unknown first, substituting it to find the next to last, and so on, is called **back substitution**. The word "back" here means that we find the last unknown (in the order they appear in the equations) first, then the next to last, and so on.

You might note that we could eliminate any of the three unknowns from any two equations, then use the addition method with those two to eliminate another variable. However, we will always follow a process that first uses the first equation to eliminate the first unknown from all equations but the first one itself. After that we use the second equation to eliminate the second unknown from all equations from the third on, and so on. One reason for this is that if we were to create a computer algorithm to solve systems, it would need a consistent method to proceed, and what we have done is as good as any.

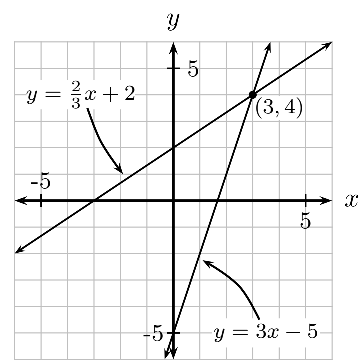
Geometric Interpretation

At the start of this section we saw that the system $\begin{aligned} x-3y &= 6 \\ -2x+5y &= -5 \end{aligned}$ has the solution $(-15,-7)$. You should be aware that if we graph the equation $x-3y=6$ we get a line. Technically speaking, what we have graphed is the **solution set**, the set of all pairs (x,y) that make the equation true. *Any pair (x,y) of numbers that makes the equation true is on the line, and the (x,y) representing any point on the line will make the equation true.* If we plot the solution sets of both equations in the system $\begin{aligned} x-3y &= 6 \\ -2x+5y &= -5 \end{aligned}$ together in the coordinate plane we will get two lines. Since $(-15,-7)$ is a solution to both equations, *the two lines cross at the point with those coordinates!* We could use this idea to (somewhat inefficiently and possibly inaccurately) solve a system of two equations in two unknowns:

◇ **Example 1.2(c):** Solve the system $\begin{aligned} 2x-3y &= -6 \\ 3x-y &= 5 \end{aligned}$ graphically.

We begin by solving each of the equations for y; this will give us the equations in $y=mx+b$ form, for easy graphing. The results are

$$y=\tfrac{2}{3}x+2 \qquad \text{and} \qquad y=3x-5$$



If we graph these two equations on the same graph, we get the picture to the right. Note that the two lines cross at the point $(3,4)$, so the solution to the system of equations is $(3,4)$, or $x=3$, $y=4$.

Now consider the system

$$\begin{aligned} x+3y-2z &= -4 \\ 3x+7y+z &= 4 \\ -2x+y+7z &= 7 \end{aligned}$$

having solution $(3,-1,2)$. What is the geometric interpretation of this? Since there are three unknowns, the appropriate geometric setting is three-dimensional space. *The solution set to any equation $ax+by+cz=d$ is a plane*, as long as not all of a, b and c are zero. Therefore, a solution to the system is a point that lies on each of the planes representing the solution sets of the three equations. For our example, then, the planes representing the three equations intersect at the point $(3,-1,2)$.

It is possible that two lines in the standard two-dimensional plane might be parallel; in that case a system consisting of the two equations representing those lines will have no solution. It is also possible that two equations might actually represent the same line, in which case the system consisting of those two equations will have infinitely many solutions. Investigation of those two cases will lead us to more complex considerations that we will avoid for now.

In the study of linear algebra we will be defining new concepts and developing corresponding notation. We begin the development of notation with the following. The set of all real numbers is denoted by $\mathbb{R}$, and the set of all ordered pairs of real numbers is $\mathbb{R}^2$, spoken as "R-two." Geometrically, $\mathbb{R}^2$ is the familiar Cartesian coordinate plane. Similarly, the set of all ordered triples of real numbers is the three-dimensional space referred to as $\mathbb{R}^3$, "R-three."

All of the algebra that we will be doing using equations with two or three unknowns can easily be done with more unknowns. In general, when we are working with n unknowns, we will get solutions that are **n-tuples** of numbers. Any such n-tuple represents a location in n-dimensional space, denoted $\mathbb{R}^n$. Note that a linear equation in two unknowns represents a line in $\mathbb{R}^2$, in the sense that the set of solutions to the equation forms a line. We consider a line to be a one-dimensional object, so the linear equation represents a one-dimensional object in two-dimensional space. The solution set to a linear equation in three unknowns is a plane in three-dimensional space. The plane itself is two-dimensional, so we have a two-dimensional "flat" object in three dimensional space.

Similarly, when we consider the solution set of a linear equation in n unknowns, its solution set represents an $n-1$-dimensional "flat" object in n-dimensional space. When such an object has more than two dimensions, we usually call it a **hyperplane**. Although such objects can't be visualized, they certainly exist in a mathematical sense.

Section 1.2 Exercises

1. Solve each of the following systems by the addition method.

(a) $\begin{aligned} 2x - 3y &= -7 \\ -2x + 5y &= 9 \end{aligned}$ (b) $\begin{aligned} 2x - 3y &= -6 \\ 3x - y &= 5 \end{aligned}$ (c) $\begin{aligned} 4x + y &= 14 \\ 2x + 3y &= 12 \end{aligned}$

(d) $\begin{aligned} 7x - 6y &= 13 \\ 6x - 5y &= 11 \end{aligned}$ (e) $\begin{aligned} 5x + 3y &= 7 \\ 3x - 5y &= -23 \end{aligned}$ (f) $\begin{aligned} 5x - 3y &= -11 \\ 7x + 6y &= -12 \end{aligned}$

2. Solve each of the following systems by graphing, as done in Example 1.2(b).

(a) $\begin{aligned} 3x - 4y &= 8 \\ x + 2y &= 6 \end{aligned}$ (b) $\begin{aligned} 4x - 3y &= 9 \\ x + 2y &= -6 \end{aligned}$ (c) $\begin{aligned} 5x + y &= 12 \\ 7x - 2y &= 10 \end{aligned}$

1.3 Solving With Matrices

Performance Criteria:

1. (f) Give the coefficient matrix and augmented matrix for a system of equations.

   (g) Determine whether a matrix is in row-echelon form. Perform, by hand, elementary row operations to reduce a matrix to row-echelon form.

   (h) Determine whether a matrix is in reduced row-echelon form. Use a calculator or software to reduce a matrix to reduced row-echelon form.

   (f) For a system of equations having a unique solution, determine the solution from either the row-echelon form or reduced row-echelon form of the augmented matrix for the system.

   (g) Use a calculator to solve a system of linear equations having a unique solution.

Note that when using the addition method for solving the system of three equations in three unknowns in the previous section, the symbols x, y and z and the equal signs are simply "placeholders" that are "along for the ride." To make the process cleaner we can simply arrange the constants a, b, c and d for each equation $ax+by+cz=d$ in an array form called a **matrix**, which is simply a table of values like

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & 7 & 1 & 4 \\ -2 & 1 & 7 & 7 \end{bmatrix}.$$

Each number in a matrix is called an **entry** of the matrix. Each horizontal line of numbers in a matrix is a **row** of the matrix, and each vertical line of numbers is a **column**. The **size** or **dimensions** of a matrix is (are) given by giving first the number of rows, then the number of columns, with the $\times$ symbol between them. The size of the above matrix is 3×4, which we say as "three by four."

Suppose that the above matrix came from the system of equations

$$\begin{aligned} x+3y-2z &= -4 \\ 3x+7y+z &= 4 \\ -2x+y+7z &= 7 \end{aligned}$$

When a matrix represents a system of equations, as this one does, it is called the **augmented matrix** of the system. The matrix consisting of just the coefficients of x, y and z from each equation is called the **coefficient matrix**:

$$\begin{bmatrix} 1 & 3 & -2 \\ 3 & 7 & 1 \\ -2 & 1 & 7 \end{bmatrix}$$

We are not interested in the coefficient matrix at this time, but we will be later. The reason for the name "augmented matrix" will also be seen later.

Once we have the augmented matrix, we can perform a process called **row-reduction**, which is essentially what we did in the previous section, but we work with just the matrix rather than the system of equations. The following example shows how this is done for the above matrix.

◇ **Example 1.3(a):** Solve the system (1) from the previous section by row-reduction.

We begin by adding negative three times the first row to the second, and put the result in the second row. Then we add two times the first row to the third, and place the result in the third. Using the notation R_n (not to be confused with $\mathbb{R}^n$!) to represent the nth row of the matrix, we can symbolize these two operations as shown in the middle below. The matrix to the right below is the result of those operations.

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & 7 & 1 & 4 \\ -2 & 1 & 7 & 7 \end{bmatrix} \quad \begin{array}{c} -3R_1 + R_2 \to R_2 \\ \Longrightarrow \\ 2R_1 + R_3 \to R_3 \end{array} \quad \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 7 & 3 & -1 \end{bmatrix}$$

Next we finish with the following:

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 7 & 3 & -1 \end{bmatrix} \quad \begin{array}{c} \frac{7}{2}R_2 + 2R_3 \to R_3 \\ \Longrightarrow \end{array} \quad \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 0 & \frac{55}{2} & 55 \end{bmatrix}$$

The process just outlined is called **row reduction**. At this point we return to the equation form

$$\begin{aligned} x + 3y - \ 2z &= -4 \\ 0x - 2y + \ 7z &= 16 \\ 0x + 0y + 55z &= 110 \end{aligned}$$

and perform back-substitution as before (see the top of page 6) to obtain $z = 2$, $y = -1$ and $x = 3$.

The final form of the matrix before we went back to equation form is something called **row-echelon** form. (The word "echelon" is pronounced "esh-el-on.") The first non-zero entry in each row is called a **leading entry**; in this case the leading entries are the numbers 1, -2 and $\frac{55}{2}$. To be in row-echelon form means that

- any rows containing all zeros are at the bottom of the matrix and
- the leading entry in any row is to the right of any leading entries above it.

◇ **Example 1.3(b):** Which of the matrices below are in row-echelon form?

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & 0 & 3 & -5 \\ 0 & 7 & -10 & -1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 6 & -1 & 9 & 5 \\ 0 & 0 & -8 & 1 & -3 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 7 & -12 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 1 & 8 \end{bmatrix}$$

The leading entries of the rows of the first matrix are 1, 3 and 7. Because the leading entry of the third row (7) is not to the right of the leading entry of the second row (3), the first matrix *is not* in row-echelon form. In the third matrix, there is a row of zeros that is not at the bottom of the matrix, so it *is not* in row-reduced form. The second matrix *is* in row-reduced form.

It is possible to continue with the matrix operations to obtain something called **reduced row-echelon form**, from which it is easier to find the values of the unknowns. The requirements for being in reduced row-echelon form are the same as for row-echelon form, with the addition of the following:

- All leading entries are ones.
- The entries above any leading entry are all zero *except perhaps in the last column*.

Obtaining reduced row-echelon form requires more matrix manipulations, and nothing is really gained by obtaining that form if you are doing this by hand. However, when using software or a calculator it is most convenient to obtain reduced row-echelon form. Here are two examples of matrices in reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 6 & 0 & 9 & 5 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For the first matrix above, one can easily see that if it came from the augmented matrix of a system of three equations in three unknowns, then $(3, -7, 4)$ would be the solution to the system. We will have to wait a bit before we are ready to interpret what the second matrix would be telling us if it came from a system of equations.

Occasionally we need to exchange two rows when performing row-reduction. The following example shows a situation in which this applies.

◇ **Example 1.3(c):** Row-reduce the matrix $\begin{bmatrix} 1 & 3 & -2 & -4 \\ 2 & 6 & -1 & -13 \\ -1 & 4 & -8 & 3 \end{bmatrix}$.

We begin by adding negative two times the first row to the second, and put the result in the second row. Then we add two times the first row to the third, and place the result in the third. Using the notation R_n (not to be confused with $\mathbb{R}^n$!) to represent the nth row of the matrix, we can symbolize these two operations as shown in the middle below. The matrix to the right below is the result of those operations.

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 2 & 6 & -1 & -13 \\ -1 & 4 & -8 & 3 \end{bmatrix} \quad \begin{matrix} -2R_1 + R_2 \to R_2 \\ \Longrightarrow \\ R_1 + R_3 \to R_3 \end{matrix} \quad \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & 0 & 3 & -5 \\ 0 & 7 & -10 & -1 \end{bmatrix}$$

We can see that the matrix would be in row-echelon form is we simply switched the second and third rows (which is equivalent to simply rearranging the order of our original equations), so that's what we do:

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & 0 & 3 & -5 \\ 0 & 7 & -10 & -1 \end{bmatrix} \quad \begin{matrix} R_2 \longleftrightarrow R_3 \\ \Longrightarrow \end{matrix} \quad \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & 7 & -10 & -1 \\ 0 & 0 & 3 & -5 \end{bmatrix}$$

The act of rearranging rows in a matrix is called **permuting** them. In general, a **permutation** of a set of objects is simply a rearrangement of them.

Row Reduction Using Technology

The two technologies that we will use in this course are your graphing calculator and the computer software called MATLAB. The row reduction process can be done on a TI-83 calculator as follows; if you have a different calculator you will need to refer to your user's manual to find out how to do this. Practice using the matrix from the Example 1.3(a).

- Find and press the *MATRIX* (or maybe *MATRX*) key.
- Select *EDIT*. At that point you will see something like 3×3 somewhere. This is the number of rows and columns your matrix is going to have. We want 3×4.
- After you have told the calculator that you want a 3×4 matrix, it will begin prompting you for the entries in the matrix, starting in the upper left corner. Here you will begin entering the values from the augmented matrix, row by row. You should see the entries appear in a matrix as you enter them.
- After you enter the matrix, you need to get to *MATH* under the *MATRIX* menu. Select *rref* (for reduced row-echelon form) and you should see *rref* (on your calculator screen.
- Select *NAMES* under the *MATRIX* menu. Highlight A and hit enter, then enter again.
- Pick off the values of x, y and z, or x_1, x_2 and x_3, depending on notation. *Use the same letters as are given in the exercise you are doing.*

We will put off learning how to do this with MATLAB until we have some more things we can do with it as well, but the command is the same. Only the method for entering the matrix differs.

Section 1.3 Exercises

1. Give the coefficient matrix and augmented matrix for the system of equations

$$\begin{aligned} x + y - 3z &= 1 \\ -3x + 2y - z &= 7 \\ 2x + y - 4z &= 0 \end{aligned} .$$

2. Determine which of the following matrices are in row-echelon form.

$$A = \begin{bmatrix} 3 & -7 & 5 & 0 & 2 & -4 \\ 0 & 0 & 0 & -2 & 5 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 3 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 4 & 4 \\ 0 & 1 & -3 & 2 \\ 6 & 1 & 3 & 5 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 3 & -5 & 10 & -7 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3. Determine which of the matrices in Exercise 2 are in reduced row-echelon form.

4. Perform the first two row operations for the augmented matrix from Exercise 1, to get zeros in the bottom two entries of the first column.

5. Fill in the blanks in the second matrix with the appropriate values after the first step of row-reduction. Fill in the long blanks with the row operations used.

(a)

$$\begin{bmatrix} 1 & 5 & -7 & 3 \\ -5 & 3 & -1 & 0 \\ 4 & 0 & 8 & -1 \end{bmatrix} \quad \begin{array}{c} \underline{\hspace{4cm}} \\ \Longrightarrow \\ \underline{\hspace{4cm}} \end{array} \quad \begin{bmatrix} \_\_ & \_\_ & \_\_ & \_\_ \\ 0 & \_\_ & \_\_ & \_\_ \\ 0 & \_\_ & \_\_ & \_\_ \end{bmatrix}$$

(b)

$$\begin{bmatrix} 2 & -8 & -1 & 5 \\ 0 & -2 & 0 & 0 \\ 0 & 6 & -5 & 2 \end{bmatrix} \quad \begin{array}{c} \Longrightarrow \\ \\ \underline{\hspace{4cm}} \end{array} \quad \begin{bmatrix} \_\_ & \_\_ & \_\_ & \_\_ \\ 0 & \_\_ & \_\_ & \_\_ \\ 0 & 0 & \_\_ & \_\_ \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 3 & 5 & -2 \\ 0 & 2 & -8 & 1 \end{bmatrix} \quad \begin{array}{c} \Longrightarrow \\ \\ \underline{\hspace{4cm}} \end{array} \quad \begin{bmatrix} \_\_ & \_\_ & \_\_ & \_\_ \\ 0 & \_\_ & \_\_ & \_\_ \\ 0 & 0 & \_\_ & \_\_ \end{bmatrix}$$

6. Find x, y and z for the system of equations that reduces to each of the matrices shown.

(a) $\begin{bmatrix} 1 & 6 & -2 & 7 \\ 0 & 8 & 1 & 0 \\ 0 & 0 & -2 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 6 & -2 & 7 \\ 0 & 2 & -5 & -13 \\ 0 & 0 & 3 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -4 & 8 \end{bmatrix}$

7. Use row operations (by hand) on an augmented matrix to solve each system of equations.

(a) $\begin{aligned} x - 2y - 3z &= -1 \\ 2x + y + z &= 6 \\ x + 3y - 2z &= 13 \end{aligned}$ (b) $\begin{aligned} -x - y + 2z &= 5 \\ 2x + 3y - z &= -3 \\ 5x - 2y + z &= -10 \end{aligned}$ (c) $\begin{aligned} x + 2y + 4z &= 7 \\ -x + y + 2z &= 5 \\ 2x + 3y + 3z &= 7 \end{aligned}$

8. Use the *rref* capability of your calculator to solve each of the systems from the previous exercise.

1.4 Applications: Curve Fitting and Temperature Equilibrium

Performance Criterion:

1. (k) Use systems of equations to solve curve fitting and temperature equilibrium problems.

Curve Fitting

Curve fitting refers to the process of finding a polynomial function of "minimal degree" whose graph contains some given points. We all know that any two distinct points (that is, points that are not the same) in $\mathbb{R}^2$ have exactly one line through them. In a previous course you should have learned how to find the equation of that line in the following manner. Suppose that we wish to find the equation of the line through the points $(2,3)$ and $(6,1)$. We know that the equation of a line looks like $y = mx + b$, where m and b are to be determined. m is the slope, which can be found by $m = \frac{3-1}{2-6} = \frac{2}{-4} = -\frac{1}{2}$. Therefore the equation of our line looks like $y = -\frac{1}{2}x + b$. To find b we simply substitute either of the given ordered pairs into our equation (the fact that both pairs lie on the line means that either pair is a solution to the equation) and solve for b: $3 = -\frac{1}{2}(2) + b \implies b = 4$. The equation of the line through $(2,3)$ and $(6,1)$ is then $y = -\frac{1}{2}x + 4$.

We will now solve the same problem in a different way. A student should understand that whenever a new approach to a familiar exercise is taken, there is something to be gained by it. Usually the new method is in some way more powerful, and allows the solving of additional problems. This will be the case with the following example.

◇ **Example 1.4(a):** Find the equation of the line containing the points $(2,3)$ and $(6,1)$.

We are again trying to find the two constants m and b of the equation $y = mx + b$. Here we substitute the values of x and y from each of the two points into the equation $y = mx + b$ (separately, of course!) to get two equations in the two unknowns m and b. The resulting system is then solved for m, then b.

$$\begin{array}{rcl} 3 &=& 2m+b \\ 1 &=& 6m+b \end{array} \implies \begin{array}{rcr} 2m+b &=& 3 \\ -6m-b &=& -1 \\ \hline -4m &=& 2 \\ m &=& -\frac{1}{2} \end{array} \implies \begin{array}{rcl} 2(-\frac{1}{2})+b &=& 3 \\ -1+b &=& 3 \\ b &=& 4 \end{array}$$

The equation of a line is considered to be a first-degree polynomial, since the power of x in $y = mx + b$ is one. Note that when we have two points in the xy-plane we can find a first-degree polynomial whose graph contains the point. Similarly, when given three points we can find a second-degree polynomial (quadratic polynomial) whose graph contains the three points. In general,

<u>Theorem 1.4.1</u>: Given n points in the plane such that (a) no two of them have the same x-coordinate and (b) they are not collinear, we can find a *unique* polynomial function of degree $n-1$ whose graph contains the n points.

Often in mathematics we are looking for some object (solution) and we wish to be certain that such an object exists. In addition, it is generally preferable that *only one* such object exists. We refer to the first desire as "existence," and the second is "uniqueness." If we have, for example, four points meeting the two conditions of the above theorem, there would be infinitely many fourth degree polynomials whose graphs would contain them, and the same would be true for fifth degree, sixth degree, and so on. But the theorem guarantees us that there is only one third degree polynomial whose graph contains the four points.

Now let's see how we find such a polynomial for degrees higher than two.

◇ **Example 1.4(b):** Find the equation of the third degree polynomial containing the points $(-1,7)$, $(0,-1)$, $(1,-5)$ and $(2,11)$.

A general third degree polynomial has an equation of the form $y = ax^3+bx^2+cx+d$; our goal is to find values of a, b, c and d so that the given points all satisfy the equation. Since the values $x=-1$, $y=7$ must make the general equation true, we have $7 = a(-1)^3+b(-1)^2+c(-1)+d = -a+b-c+d$. Doing this with all four given ordered pairs and "flipping" each equation gives us the system

$$\begin{aligned} -a+b-c+d &= 7 \\ d &= -1 \\ a+b+c+d &= -5 \\ 8a+4b+2c+d &= 11 \end{aligned}$$

If we enter the augmented matrix for this system in our calculators and *rref* we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

So $a=-1$, $b=2$, $c=-5$, $d=-1$, and the desired polynomial equation is $y = -x^3+2x^2-5x-1$.

Temperature Equilibrium

Consider the following hypothetical situation: We have a plate of metal that is perfectly insulated on both of its faces so that no heat can get in or out of the faces. Each point on the edge, however, is held at a constant temperature (constant at that point, but possibly differing from point to point). The temperatures on the edges affect the interior temperatures. If the plate is left alone for a long time ("infinitely long"), the temperature at each point on the interior of the plate will reach a constant temperature, called the "equilibrium temperature." This equilibrium temperature at any given point is a **weighted average** of the temperatures at all the boundary points, with temperatures at closer boundary points being weighted more heavily in the average than points that are farther away.

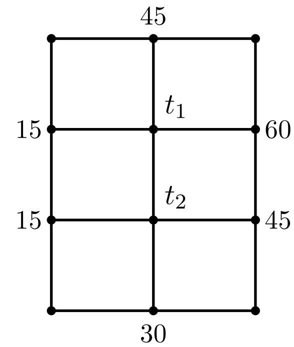
The task of trying to determine those interior temperatures based on the edge temperatures is one of the most famous problems of mathematics, called the **Dirichlet problem** (pronounced "dir-i-shlay"). Finding the exact solution involves methods beyond the scope of this course, but we will use systems of equations to solve the problem "numerically," which means to approximate the exact solution, usually by some non-calculus method. The key to solving the Dirichlet problem is the following:

THEOREM 1.4.2: **Mean Value Property**

The equilibrium temperature at any interior point P is the average of the temperatures of all interior points on any circle centered at P.

We will solve what are called **discrete** versions of the Dirichlet problem, which means that we only know the temperatures at a finite number of points on the boundary of our metal plate, and we will only find the equilibrium temperatures at a finite number of the interior points.

◇ **Example 1.4(c):** The temperatures (in degrees Fahrenheit) at six points on the edge of a rectangular plate are shown to the right. Assuming that the plate is insulated as described above and that temperatures in the plate have reached equilibrium, find the interior temperatures t_1 and t_2 at their indicated "mesh points."



The discrete version of the mean value property tells us that the equilibrium temperature at any interior point of the mesh is the average of the four adjacent points. This gives us the two equations

$$t_1 = \frac{15+45+60+t_2}{4} \qquad \text{and} \qquad t_2 = \frac{15+t_1+45+30}{4}$$

If we multiply both sides of each equation by four, combine the constants and get the t_1 and t_2 terms on the left side we get the system of equations $\begin{aligned} 4t_1 - \ t_2 &= 120 \\ -t_1 + 4t_2 &= \ \ 90 \end{aligned}$, which gives us $t_1 = 38$ and $t_2 = 32$. These can easily be shown to verify our discrete mean value property:

$$\frac{15+45+60+t_2}{4} = \frac{15+45+60+32}{4} = 38 = t_1\,,$$

$$\frac{15+t_1+45+30}{4} = \frac{15+38+45+30}{4} = 32 = t_2$$

Section 1.4 Exercises

1. Consider the four points $(-1,3)$, $(1,5)$, $(2,4)$ and $(4,-1)$. It turns out that there is a unique third degree polynomial of the form

$$y = a + bx + cx^2 + dx^3 \qquad (1)$$

whose graph contains those four points. The objective of this exercise is to find the values of the coefficients a, b, c and d.

   (a) Substitute the x and y values from the first ordered pair into (1) and rearrange the resulting equation so that it has all of the unknowns on the left and a number on the right, like all of the linear equations we have worked with so far.

   (b) Repeat (a) for the other 3 ordered pairs, and give the system of equations to be solved.

   (c) Give the augmented matrix for the system of equations.

(d) Use your calculator or an online tool to *rref* the augmented matrix. Give the values of the unknowns, each rounded to the nearest hundredth, based on the reduced matrix.

(e) Give the equation of the polynomial. Graph it using some technology, and make sure that it appears to go through the points that it is supposed to.

2. (a) Find the equation of the quadratic polynomial $y = ax^2 + bx + c$ whose graph is the parabola passing through the points $(-1, -4)$, $(1, 1)$ and $(3, 0)$.

   (b) Graph your answer on your calculator and use the trace function to see if the graph in fact goes through the three given points.

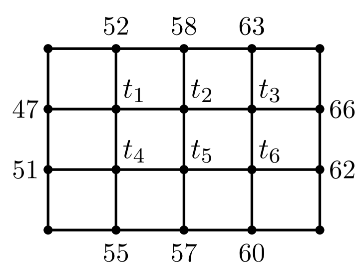
3. (a) Plot the points $(-4, 0)$, $(-2, 2)$, $(0, 0)$, $(2, 2)$ and $(3, 0)$ neatly on an xy grid. Sketch the graph of a polynomial function with the fewest number of turning points ("humps") possible that goes through all the points. What is the degree of the polynomial function?

   (b) Find a fourth degree polynomial $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ that goes through the given points.

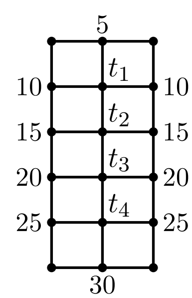
   (c) Graph your function from (b) on your calculator and sketch it, using a dashed line, on your graph from (a). Is the graph what you expected?

4. The equation of a plane in $\mathbb{R}^3$ can always be written in the form $z = a + bx + cy$, where a, b and c are constants and (x, y, z) is any point on the plane. Use a method similar to the above method for finding the equation of a line to find the equation of the plane through the three points $P_1(-5, 0, 2)$, $P_2(4, 5, -1)$ and $P_3(2, 2, 2)$. Use your calculator's *rref* command to solve the system. Round a, b and c to the thousandth's place.

5. Temperatures at points along the edges of a rectangular plate are as shown below and to the left. Find the equilibrium temperature at each of the interior points, to the nearest tenth.



Exercise 5



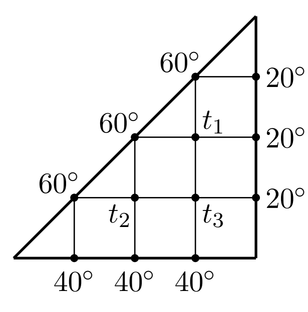
Exercise 6

6. Consider the rectangular plate with boundary temperatures shown above and to the right.

   (a) Intuitively, what do you think that the equilibrium temperatures t_1, t_2, t_3 and t_4 are?

   (b) Set up a system of equations and find the equilibrium temperatures. How was your intuition?

7. For the diagram to the right, the mean value property still holds, even though the plate in this case is triangular. Find the interior equilibrium temperatures, rounded to the nearest tenth.



1.5 Chapter 1 Exercises

1. Consider the system of equations below and to the right. Solve the system by Gaussian elimination (get in row-echelon form, then perform back substitution), **by hand** (no calculator). Show all steps, including what operation was performed for each step. **Hint:** You may find it useful to put the equations in a different order before forming the augmented matrix.

$$\begin{aligned} 5x - y + 2z &= 17 \\ x + 3y - z &= -4 \\ 2x + 4y - 3z &= -9 \end{aligned}$$

2. Find the equation of the parabola through the points $(0, 3)$, $(1, 4)$ and $(3, 18)$.

3. Consider the points $(1, 5)$, $(2, 2)$, $(4, 3)$ and $(5, 4)$.

   (a) What is the smallest degree polynomial whose graph will contain all of these points?

   (b) Find the polynomial whose graph contains all the points.

   (c) Check by graphing on your calculator.

4. Why would we not be able to find the equation of a line through $(0, 6)$, $(2, 3)$ and $(6, 1)$? We will see later what this means in terms of systems of equations, and we will resolve the problem in a reasonable way.

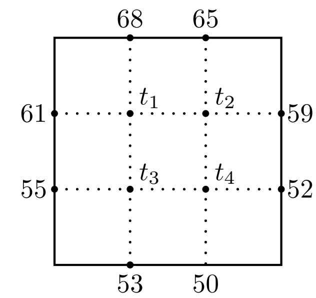
5. (Erdman) Consider the following two systems of equations.

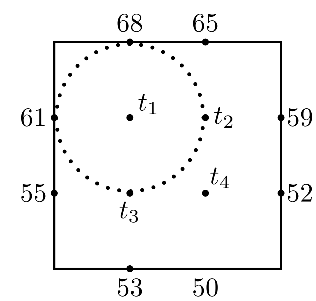
$$\begin{aligned} x + y + z &= 6 \\ x + 2y + 2z &= 11 \\ 2x + 3y - 4z &= 3 \end{aligned} \qquad\qquad \begin{aligned} x + y + z &= 7 \\ x + 2y + 2z &= 10 \\ 2x + 3y - 4z &= 3 \end{aligned}$$

   Note that both systems have the same left hand sides. It is often the case in applications that we wish to solve a number of systems, all having the same left hand sides, but with differing right hand sides. In practice this is usually done using something called an LU-factorization, but you will do something different here. Create an augmented matrix for the first system, then add the right side of the second system as a fifth column. Row reduce as usual, and you will get the solutions to both systems at the same time. What are they? (You should verify by substitution.)

6. There is exactly one plane through any set of three points in $\mathbb{R}^3$ as long as the points do not lie on the same line. The equation of a *non-vertical* plane in $\mathbb{R}^3$ can always be written in the form $z = a + bx + cy$, where a, b and c are constants and (x, y, z) is any point on the plane. This is a tiny bit different than the other form of the equation of a plane, but it is equivalent as long as the plane is not vertical. Find the equation of the plane through the three points $P_1(4, 1, -3)$, $P_2(0, -5, 1)$ and $P_3(3, 3, 2)$.

7. As described in the book, when a plate or solid object reaches a temperature equilibrium, the temperature at any interior point is the average of the temperatures at all points on any circle or sphere centered at that point and not extending outside the plate or object. Consider the plate of metal shown to the right, with boundary temperatures as indicated. The numbers t_1, t_2, t_3 and t_4 represent the equilibrium temperatures at the four interior points marked by dots. In the lower picture to the right I have drawn a circle centered at the point with temperature t_1.





(a) Write an equation that gives the temperature t_1 as the average of the four known or unknown temperatures on the circle. Multiply both sides by four to eliminate the fraction, and get all the unknowns on one side and all numbers on the other. You should end up with an equation of the form $at_1 + bt_2 + ct_3 + dt_4 = e$, where not all terms will be present on the left side.

(b) Repeat (a) for the other three interior points, averaging the temperatures on circles of the same size around each.

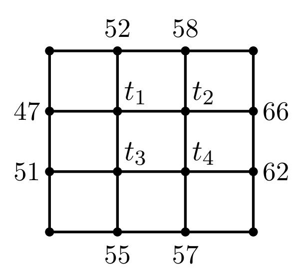
(c) Solve the resulting system of equations to determine each interior point, rounded to the nearest tenth.

8. Given a matrix A, we refer to the values in the matrix as entries, and they are each represented as a_{jk}, where i is the row of the entry, and j is the column of the entry. (The numbering for rows and columns begins with one for each, and at the upper left corner.)

(a) Set up systems of equations to solve Section 1.4 Exercises 5, 6, 7. Find the coefficient matrix in each case, and observe each carefully. You should see two or three things they all have in common. Use the notation just described for entries of a matrix to help describe what you see. **You should be able to summarize your observations in just a couple brief mathematical statements.**

(b) Solve each of the exercises listed in part (a). For each sheet, look at where the maximum and minimum temperatures occur.What can we say in general about the locations of the maximum and minimum temperatures? Can you see how this is implied by the Mean Value Property?

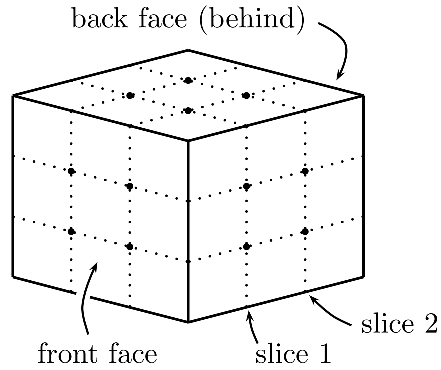
9. (a) A student is attempting to find the equilibrium temperatures at points t_1, t_2, t_3 and t_4 on a plate with a grid and boundary temperatures shown below and to the left. They get $t_1 = 50.3$, $t_2 = 67.4$, $t_3 = 53.6$, $t_4 = 60.5$. Explain in one *complete sentence* why their answer must be incorrect, *without finding the solution.*

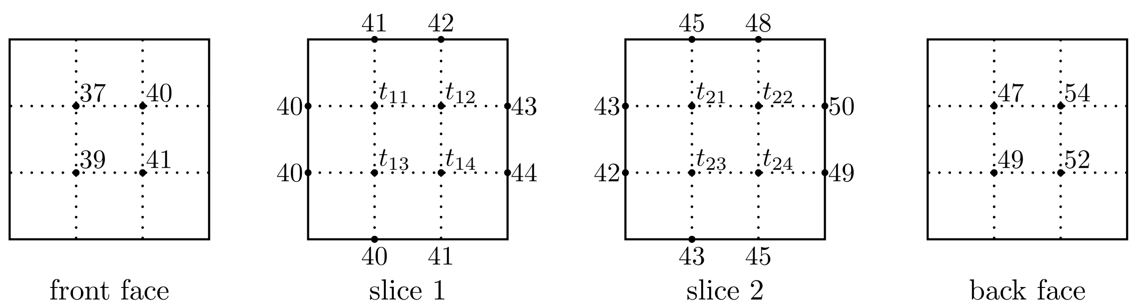


$$\begin{bmatrix} 4 & -1 & 0 & 0 & -1 & 0 & 103 \\ -1 & 4 & -1 & 0 & 0 & -1 & 92 \\ 0 & -1 & 4 & -1 & -1 & 0 & 110 \\ 0 & 0 & -1 & 4 & 0 & 0 & 98 \\ -1 & 0 & -1 & 0 & 4 & 0 & 105 \\ 0 & -1 & 0 & 0 & -1 & 4 & 107 \end{bmatrix}$$

(b) A different student is trying to solve another such problem, and their augmented matrix is shown above and to the right. How do we know that one of their equations is incorrect, *without setting up the equations ourselves?*

10. Given a cube of some solid material, it is possible to put a three-dimensional grid into the solid, in the same way that we put a two-dimensional grid on a rectangular plate. Given temperatures at all nodes on the exterior faces of the cube, we can find equilibrium temperatures at each interior node using a system of equations. Once again the key is the mean-value property. In this three dimensional case this property tells us that the equilibrium temperature at each interior node is equal to the average of all the temperatures at nodes of the grid that are immediately adjacent to the point in question. To the right I have shown a cube that has eight interior grid points. The word "slice" is used here to mean a cross section through the cube. The grids below show temperatures, known or unknown, at all nodes on the front face, each of the two slices, and the back face. Above and to the right I have "exploded" the cube to show the temperatures on the front and back faces, and the two slices. Of course each node on any slice is connected to the corresponding node on the adjacent slice or face.





(a) Using the Mean Value Property in three dimensions, the temperature at each interior point will *NOT* be the average of four temperatures, like it was on a plate. How many temperatures will be averaged in this case?

(b) Set up a system of equations to solve for the interior temperatures, and find each to the nearest tenth.

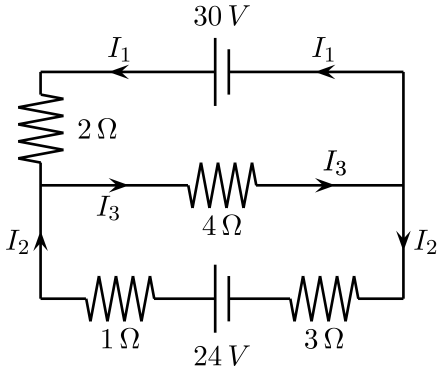
11. Do any of your observations from Exercise 8 change in the three dimensional case?

12. Suppose we are solving a system of three equations in the three unknowns x_1, x_2 and x_3, *with the unknowns showing up in the equations in that order.* It is possible to do row reduction in such a way as to obtain the matrix

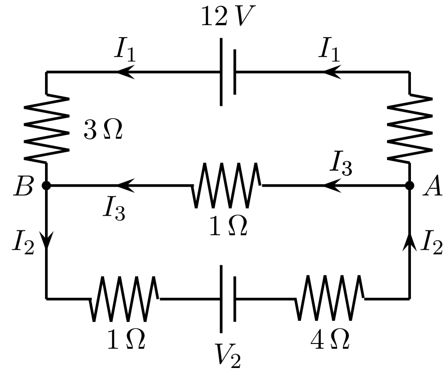
$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 3 & -2 & 0 & 7 \\ -1 & 5 & 2 & -3 \end{bmatrix}$$

Determine x_1, x_2 and x_3 *without row-reducing this matrix!* you should be able to simply set up equations and find values for the unknowns.

13. Find the currents I_1, I_2 and I_3 in the circuit with the diagram shown below and to the left.



Exercise 13



Exercise 14

14. Consider the circuit with the diagram shown above and to the right.

(a) Find the currents I_1, I_2 and I_3 when the voltage V_2 is 6 volts, rounded to the tenth's place.

(b) Does the current in the middle branch of the circuit flow from A to B, or from B to A?

(c) Find the currents I_1, I_2 and I_3 when the voltage V_2 is 24 volts, rounded to the tenth's place.

(d) Does the current in the middle branch of the circuit flow from A to B, or from B to A?

(e) Determine the voltage needed for V_2 in order that no current flows through the middle branch. (You might wish to row reduce by hand for this...)

2 More on Systems of Linear Equations

Learning Outcome:

2. Identify the nature of a solution, use systems of linear equations to solve problems, and approximate solutions iteratively.

   Performance Criteria:

   (a) Given the row-echelon or reduced row-echelon form of an augmented matrix for a system of equations, determine the rank of the coefficient matrix and the leading variables and free variables of the system.

   (b) Given the row-echelon or reduced row-echelon form for a system of equations:

   - Determine whether the system has a unique solution, and give the solution if it does.
   - If the system does not have a unique solution, determine whether it is inconsistent (no solution) or dependent (infinitely many solutions).
   - If the system is dependent, give the general form of a solution and give some particular solutions.

   (c) Use systems of equations to solve network analysis problems.

   (d) Approximate a solution to a system of equations using Jacobi's method or the Gauss-Seidel method.

In Section 1.1 there was a brief discussion about the geometric significance of a solution to a system of two linear equations in two unknowns. The graph of a single linear equation in two unknowns is a line, and the ordered pair solution to a system of two such equations represents the point where the two lines cross, *if in fact they do cross at one, and only one, point!* In the event that the the two lines are parallel, there will be no point on both lines, so the system will have no solution. If the two equations happen to describe the same line, there will be infinitely many solutions to the system of equations. (Those are the only three things that can happen - one solution, no solution, or infinitely many solutions.) The purpose of this chapter is to look at the cases of no solution or infinitely many solutions. We will seek to answer the following questions:

- How can we tell from the row-echelon or reduced row-echelon form of the augmented matrix whether the system has one solution, no solution or infinitely many solutions?
- In the event that there are infinitely many solutions, how do we describe all the solutions in general and give a few of them in particular?
- How does the possibility of infinitely many solutions appear in applied situations, and what does it mean?

After addressing these questions, we will see two iterative methods for approximating solutions to systems of linear equations.

2.1 "When Things Go Wrong"

Performance Criteria:

2. (a) Given the row-echelon or reduced row-echelon form of an augmented matrix for a system of equations, determine the rank of the coefficient matrix and the leading variables and free variables of the system.

   (b) Given the row-echelon or reduced row-echelon form for a system of equations:

   - Determine whether the system has a unique solution, and give the solution if it does.
   - If the system does not have a unique solution, determine whether it is inconsistent (no solution) or dependent (infinitely many solutions).
   - If the system is dependent, give the general form of a solution and give some particular solutions.

Consider the three systems of equations

$$\begin{aligned} x - 3y &= 6 \\ -2x + 5y &= -5 \end{aligned} \qquad \begin{aligned} 2x - 5y &= 3 \\ -4x + 10y &= 1 \end{aligned} \qquad \begin{aligned} 2x - 5y &= 4 \\ -4x + 10y &= -8 \end{aligned}$$

For the first system, if we multiply the first equation by 2 and add it to the second, we get $-y = 7$, so $y = -7$. This can be substituted into either equation to find x, and the system is solved!

When attempting to solve the second and third equations, things do not "work out" in the same way. In both cases we would likely attempt to eliminate x by multiplying the first equation by two and adding it to the second. For the second system this results in $0 = 7$ and for the third the result is $0 = 0$. So what is happening? Let's keep the unknown value y in both equations: $0y = 7$ and $0y = 0$. There is no value of y that can make $0y = 7$ true, so there is no solution to the second system of equations. We call a system of equations with no solution **inconsistent**.

The equation $0y = 0$ is true for *any* value of y, so y can be anything in the third system of equations. Thus we will call y a **free variable**, meaning it is free to have any value. *In this sort of situation we will assign another unknown, usually t, to represent the value of the free variable.* If there is another free variable we usually use s and t for the two free variables. Once we have assigned the value t to y, we can substitute it into the first equation and solve for x to get $x = \frac{5}{2}t + 2$.

What all this means is that any ordered pair of the form $(\frac{5}{2}t + 2, t)$ will be a solution to the third system of equations above. For example, when $t = 0$ we get the ordered pair $(2, 0)$, when $t = -6$ we get $(-13, -6)$. You can verify that both of these are solutions, as are infinitely many other pairs. At this point you might note that we could have made x the free variable, then solved for y in terms of whatever variable we assigned to x. *It is standard convention, however, to start assigning free variables from the last variable, and you will be expected to follow that convention in this class.* A system like this, with infinitely many solutions, is called a **dependent** system.

The fundamental fact that should always be kept in mind is this.

Solutions to a System of Equations

Every system of linear equations has either

- one unique solution
- no solution (the system is inconsistent)
- infinitely many solutions (the system is dependent)

In the context of both linear algebra and differential equations, mathematicians are always concerned with "existence and uniqueness." What this means is that when attempting to solve a system of equations or a differential equation, one cares about

1) whether at least one solution exists and

2) if there is at least one solution, is there exactly one; that is, is the solution unique?

We'll now see if we can learn to recognize which of the above three situations is the case, based on the row-echelon or reduced row-echelon form of the augmented matrix of a system. If the three systems we have been discussing are put into augmented matrix form and row reduced we get

$$\begin{bmatrix} 1 & 0 & -15 \\ 0 & 1 & -7 \end{bmatrix} \qquad \begin{bmatrix} 2 & -5 & 3 \\ 0 & 0 & 7 \end{bmatrix} \qquad \begin{bmatrix} 2 & -5 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

It should be clear that the first matrix gives us the unique solution to that system. The second line of the second matrix "translates" back to the equation $0x + 0y = 7$, which clearly cannot be true for any values of x or y. So that system has no solution.

If the row reduced augmented matrix for a system has any row with entries all zeros EXCEPT the last one, the system has no solution. The system is said to be **inconsistent**.

We now consider the third row reduced matrix. The last line of it "translates" to $0x + 0y = 0$, which is true for *any* values of x and y. That means we are free to choose the value of either one but, as discussed before, it is customary to let y be the free variable. So we let $y = t$ and substitute that into the equation $2x - 5y = 4$ represented by the first line of the reduced matrix. As before, that is solved for x to get $x = \frac{5}{2}t + 2$. The solutions to the system are then $x = \frac{5}{2}t + 2,\ y = t$ for all values of t.

We will now consider the system shown below and to the left; its augmented matrix reduces to the form shown below and to the right.

$$\begin{aligned} x_1 - x_2 + x_3 &= 3 \\ 2x_1 - x_2 + 4x_3 &= 7 \\ 3x_1 - 5x_2 - x_3 &= 7 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There is one observation we need to make, and we will develop some terminology that makes it easier to talk about what is going on. First we note that if we were to perform row-reduction on the coefficient matrix of the equation alone, the result would be the same as the row-reduced augmented matrix, but without the last column. We now make the following definitions:

- The **rank** of a matrix is the number of non-zero rows in its row-echelon or reduced row-echelon form.
- The **leading variables** are the variables corresponding to the columns of the reduced matrix containing the first non-zero entries (always ones for reduced row-echelon form) in each row. For the above system the leading variables are x_1 and x_2.
- Any variables that are not leading variables are **free variables**, so x_3 is the free variable in the above system. This means it is free to take any value.

You already know how to solve a system of equations with a single solution, from its reduced row-echelon matrix. If the last non-zero row of the reduced matrix is all zeros except its last entry, it corresponds to an equation with no solution, so the system has no solution. If neither of those is the case, then the system will have infinitely many solutions. It is a bit difficult to explain how to solve such a system, and it is probably best seen by some examples. However, let me try to describe it. Start with the last variable and solve for it if it is a leading variable. If it is not, assign it a parameter, like t. If the next to last variable is a leading variable solve for it, either as a number or in terms of the parameter assigned to the last variable. Continue in this manner until all variables have been determined as numbers or in terms of parameters.

◇ **Example 2.1(a):** Solve the system

$$\begin{aligned} x_1 - x_2 + x_3 &= 3 \\ 2x_1 - x_2 + 4x_3 &= 7 \\ 3x_1 - 5x_2 - x_3 &= 7 \end{aligned}$$

The row-reduced form of the augmented matrix for this system is $\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. In this case the leading variables are x_1 and x_2. Any variables that are not leading variables are free variables, so x_3 is the free variable in this case. If we let $x_3 = t$, the last non-zero row gives the equation $x_2 + 2t = 1$, so $x_2 = -2t + 1$. The first row gives the equation $x_1 + 3x_3 = 4$, so $x_1 = -3t + 4$ and the final solution to the system is

$$x_1 = -3t + 4, \qquad x_2 = -2t + 1, \qquad x_3 = t$$

We can also think of the solution as being any ordered triple of the form $(-3t + 4, -2t + 1, t)$.

◇ **Example 2.1(b):** A system of three equations in the four variables x_1, x_2, x_3 and x_4 gives the row-reduced matrix

$$\begin{bmatrix} 1 & 0 & 3 & 0 & -1 \\ 0 & 1 & -5 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Give the general solution to the system.

The leading variables are x_1, x_2 and x_4. Any variables that are not leading variables are the free variables, so x_3 is the free variable in this case. We can see that the last row gives us $x_4 = 4$. If we let $x_3 = t$, the second equation from the row-reduced matrix is $x_2 - 5t = 2$, so $x_2 = 5t + 2$. The first equation is $x_1 + 3t = -1$, giving $x_1 = -3t - 1$. The final solution to the system is then

$$x_1 = -3t - 1, \qquad x_2 = 5t + 2, \qquad x_3 = t, \qquad x_4 = 4,$$

or $(-3t - 1, 5t + 2, t, 4)$.

The solutions given in the previous two examples are called **general solutions**, because they tell us what any solution to the system looks like in the cases where there are infinitely many solutions. We can also produce some specific numbers that are solutions as well, which we will call **particular solutions**. These are obtained by simply letting any parameters take on whatever values we want.

◇ **Example 2.1(c):** Give three particular solutions to the system in Example 2.1(a).

If we take the easiest choice for t, zero, we get the particular solution $(4, 1, 0)$. Letting t equal negative one and one gives us the particular solutions $(7, 3, -1)$ and $(1, -1, 1)$.

The following examples show a situations in which there are two free variables, and one in which there is no solution.

◇ **Example 2.1(d):** A system of equations in the four variables x_1, x_2, x_3 and x_4 that has the row-reduced matrix

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Give the general solution and four particular solutions.

In this case, the rank of the matrix is two, the leading variables are x_1 and x_3, and the free variables are x_2 and x_4. We begin by letting $x_4 = t$; we have the equation $x_3 - 2t = 3$, giving us $x_3 = 2t + 3$. Since x_2 is a free variable, we call it something else. t has already been used, so let's say $x_2 = s$. The first equation indicated by the row-reduced matrix is then $x_1 + 2s - t = 2$, giving us $x_1 = -2s + t + 2$. The solution to the corresponding system is

$$x_1 = -2s + t + 2, \qquad x_2 = s, \qquad x_3 = 2t + 3, \qquad x_4 = t$$

If we let $s = 0$ and $t = 0$ we get the solution $(2, 0, 3, 0)$, and if we let $s = 2$ and $t = -1$ we get $(-3, 2, 1, -1)$. Letting $s = 0$ and $t = 1$ gives the particular solution $(3, 0, 5, 1)$ and letting $s = 1$ and $t = 0$ gives the particular solution $(0, 1, 3, 0)$.

The values used for the parameters in Examples 2.1(c) and (d) were chosen arbitrarily; any values can be used for t.

◇ **Example 2.1(e):** A system of equations in the four variables x_1, x_2, x_3 and x_4 has the row-reduced matrix

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Solve the system.

Since the last row is equivalent to the equation $0x_1 + 0x_2 + 0x_3 + 0x_4 = 5$, which has no solution, the system itself has no solution.

We conclude this section with a few examples concerning the idea of rank.

◇ **Example 2.1(f):** Give the ranks of the coefficient matrices from examples 2.1(a), (b), (d) and (e).

The row-reduced forms of the coefficient matrices for the systems are

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with the last being the row-reduced form of the coefficient matrices for both Example 2.1(d) and 2.1(e). The rank of the coefficient matrices for Examples 2.1(a), (e) and (f) are all two, and the rank of the coefficient matrix for Example 2.1(b) is three.

Section 2.1 Exercises

1. Consider the system of equations
$$\begin{aligned} 2x - 4y - z &= -4 \\ 4x - 8y - z &= -4 \\ -3x + 6y + z &= 4 \end{aligned}.$$

   (a) Determine which of the following ordered triples are solutions to the system of equations:

   $$(6,3,4) \qquad (3,-1,4) \qquad (0,0,4) \qquad (-2,-1,4) \qquad (5,2,0) \qquad (2,1,4)$$

   Look for a pattern in the ordered triples that *ARE* solutions. Try to guess another solution, and test your guess by checking it in all three equations. How did you do?

   (b) When you tried to solve the system using your calculator, you should have gotten the reduced echelon matrix as
   $$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
   Give the system of equations that this matrix represents. Which variable can you determine?

   (c) It is not possible to determine y, so we simply let it equal some arbitrary value, which we will call t. So at this point, $z = 4$ and $y = t$. Substitute these into the first equation and solve for x. Your answer will be in terms of t. Write the ordered triple solution to the system.

   NOTE: The system of equations you obtained in part (b) and solved in part (c) has infinitely many solutions, but we do know that every one of them has the form $(2t, t, 4)$. Note how this explains the results of part (b).

2. The reduced echelon form of the matrix for the system

$$\begin{aligned} 3x - 2y + z &= -7 \\ 2x + y - 4z &= 0 \\ x + y - 3z &= 1 \end{aligned} \qquad \text{is} \qquad \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) In this case, z cannot be determined, so we let $z = t$. Now solve for y, in terms of t. Then solve for x in terms of t.

(b) Pick a specific value for t and substitute it into your general form of a solution triple for the system. Check it by substituting it into all three equations in the original system.

(c) Repeat (b) for a different value of t.

3. The reduced echelon forms of some systems are given below. Find the solutions for any that have solutions. (Some may have single solutions, some may have infinitely many solutions, and some may not have solutions.)

(a) $\begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 2 & 0 & -5 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -3 & 0 & 1 & -4 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 & -2 & 1 & 6 \\ 0 & 1 & 3 & 5 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (h) $\begin{bmatrix} 1 & 4 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (i) $\begin{bmatrix} 1 & 5 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

4. Give four particular solutions from the general solution of Example 2.1(b).

5. Below is a system of equations and the reduced row-echelon form of the augmented matrix. Give the leading variables, free variables and the rank of the coefficient matrix.

$$\begin{aligned} x + y - 3z &= 1 \\ -3x + 2y - z &= 7 \\ 2x + y - 4z &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

6. For the system and reduced row-echelon matrix from the previous exercise, do one of the following:

- If the system has a unique solution, give it. If the system has no solution, say so.
- If the system has infinitely many solutions, give the general solution in terms of parameters s, t, etc., then give two particular solutions.

Then do the same for the systems whose augmented matrices row reduce to the forms shown below.

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2.2 Overdetermined and Underdetermined Systems

OK, let's talk about scientific and engineering reality for a bit. Systems of equations generally arise in situations where a number of variables are related in ways described by systems of linear equations. Those equations are generally based on measurements (data). In practice, it often is the case that there is a great deal of data, and we get more equations than unknowns. Usually when there are more equations than unknowns the system is inconsistent - there is no solution. This is no good! Systems like this are sometimes said to be **overdetermined**. This wording relates to the fact that in these situations there is too much information to determine a solution to the system.

You might think "Well, why not just use less data, so that the resulting system has a solution?" Well the additional data gives us some redundancy that can give us better results *if we know how to deal with it.* The way out of this problem is a method called **least-squares**, which you'll do later. It is a method for dealing with systems that don't have solutions. What it allows us to do is obtain values that are in some sense the "closest" values there are to an actual solution. Again, more on this later.

When there are fewer equations than unknowns there will be no hope of a unique solution - there will either be no solution or infinitely many solutions. Usually there will be infinitely many solutions and we call such a system **underdetermined**, meaning there is not enough information (data) to determine a unique solution.

In the next section we'll look at an application leading to an underdetermined system, and we'll "solve" such systems in the best sense possible, meaning our solutions will depend on some parameter that can take any value.

2.3 Application: Network Analysis

Performance Criterion:

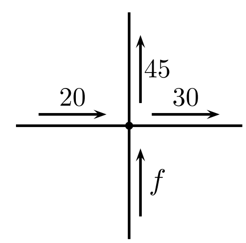
2. (c) Use systems of equations to solve network analysis problems.

A network is a set of junctions, which we'll call **nodes**, connected by what could be called pipes, or wires, but which we'll call **directed edges**. The word "directed" is used to mean that we'll assign a direction of flow to each edge. (In some cases we might then find the flow to be negative, meaning that it actually flows in the direction opposite from what we have designated as the direction of flow.) There will also be directed edges coming into or leaving the network. It is probably easiest to just think of a network of plumbing, with water coming in at perhaps several places, and leaving at several others. However, a network could also model goods moving between cities or countries, traffic flow in a city or on a highway system, and various other things.

Our study of networks will be based on one simple idea, known as **conservation of flow**:

At each node of a network, the flow into the node must equal the flow out.

◇ **Example 2.3(a):** A one-node network is shown to the right. Find the unknown flow f.

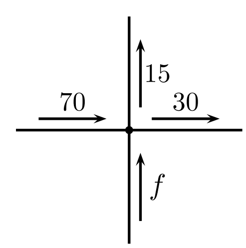


The flow in is $20 + f$ and the flow out is $45 + 30$, so we have

$$20 + f = 45 + 30.$$

Solving, we find that $f = 55$.

◇ **Example 2.3(b):** Another one-node network is shown to the right. Find the unknown flow f.



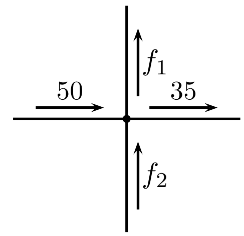
The flow in is $70 + f$ and the flow out is $15 + 30$, so we have

$$70 + f = 15 + 30.$$

Solving, we find that $f = -25$, so the flow at the arrow labeled f is actually in the direction opposite to the arrow.

There is nothing wrong with what we just saw in the last example. When setting up a network we must commit to a direction of flow for any edges in which the flow is unknown, but when solving the system we may find that the flow is in the opposite direction from the way the edge was directed initially. We may also have less information than we did in the previous two examples, as shown by the next example.

◇ **Example 2.3(c):** For the one-node network is shown to the right, find the unknown flow f_1 in terms of the flow f_2.



By conservation of flow,

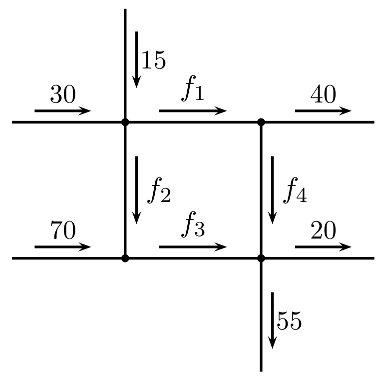
$$50 + f_2 = f_1 + 35.$$

Solving for f_1 gives us $f_1 = f_2 + 15$. Thus if f_2 was 10, f_1 would be 25 (look at the diagram and think about that), if f_2 was 45, f_1 would be 60, and so on.

The above example represents, in an applied setting, the idea of a free variable. In this example either variable can be taken as free, but if we know the value of one of them, we'll "automatically" know the value of the other. The way I worded the example, we were taking f_2 to be the free variable, with the value of f_1 then depending on the value of f_2.

The systems in these first three examples have been very simple; let's now look at a more complex system.

◇ **Example 2.3(d):** Determine the flows f_1, f_2, f_3 and f_4 in the network shown below and to the right.



Utilizing conservation of flow at each node, we get the equations

$$30 + 15 = f_1 + f_2, \qquad 70 + f_2 = f_3,$$

$$f_1 = 40 + f_4, \qquad f_3 + f_4 = 20 + 55$$

Rearranging these give us the system of equations shown below and to the left. The augmented matrix for this system reduces to the matrix shown below and to the right.

$$\begin{array}{rcl} f_1 + f_2 & = & 45 \\ f_2 - f_3 & = & -70 \\ f_1 - f_4 & = & 40 \\ f_3 + f_4 & = & 75 \end{array} \qquad \left[\begin{array}{rrrrr} 1 & 0 & 0 & -1 & 40 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 75 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

From this we can see that f_4 is a free variable, so lets say it has value t. The solution to the network is then

$$f_1 = 40 + t, \qquad f_2 = 5 - t, \qquad f_3 = 75 - t, \qquad f_4 = t,$$

where t is the flow f_4.

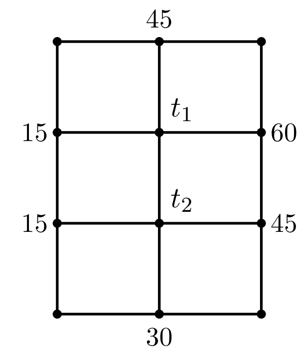
Let's think a bit more about this last example. Suppose that $f_4 = t = 0$. The equations given as the solution to the network then give us $f_1 = 40$, $f_2 = 5$, $f_3 = 75$. We can see this without even solving the system of equations. Looking at the node in the lower right, if $f_4 = 0$ one can easily see that f_3 must be 75 in order for the flow in to equal the flow out. Knowing f_3, we can go to the node in the lower left and see that $f_2 = 5$. Finally, $f_2 = 5$ gives is $f_1 = 40$, from the node in the upper left. This reasoning is essentially the process of back-substitution!

2.4 Approximating Solutions With Iterative Methods

Performance Criterion:

2. (d) Approximate a solution to a system of equations using Jacobi's method or the Gauss-Seidel method.

Consider again the rectangular metal plate of Example 1.3(b), as shown to the right, with boundary temperatures specified at a few points. Remember that our goal was to find the temperatures t_1 and t_2 at the two interior points. Applying the mean-value property we arrived at the equations



$$\begin{aligned} t_1 &= \frac{15+45+60+t_2}{4} = \frac{120+t_2}{4}\,, \\ t_2 &= \frac{15+t_1+45+30}{4} = \frac{90+t_1}{4} \end{aligned} \tag{1}$$

In Example 1.4(c) we rearranged these equations and solved the system of two equations in two unknowns by row-reduction. Now we will use what is called an **iterative method** to approximate the solution to this problem. An iterative method is one in which successive approximations to a solution are generated in sequence, with each approximation getting closer to the true solution.

The method we'll use first is called the **Jacobi method**. Although it is more laborious than simply solving the system by row reduction in this case, and does not even produce an exact solution, iterative methods like the Jacobi method are at times preferable to direct methods like row-reduction. For very large systems the Jacobi method can be less computationally costly than row reduction. This is especially true for matrices with lots of zeros, as many that arise in practice are.

The general idea behind the Jacobi method is to begin by guessing values for both t_1 and t_2. We'll denote these first two values by $t_1(0)$ and $t_2(0)$. The guessed value $t_2(0)$ is inserted into the first equation above for t_2, and the guessed value $t_1(0)$ is inserted into the second equation for t_1. This allows us to compute new values of t_1 and t_2, denoted by $t_1(1)$ and $t_2(1)$. These new values are then inserted into the equations (1) for t_1 and t_2, as was done with the first guesses. This process is repeated over and over, resulting in new values for t_1 and t_2 each time. These values will eventually approach the exact solution values of t_1 and t_2. Let's do it!

◇ **Example 2.4(a):** Use the Jacobi method to find the third approximations $t_1(3)$ and $t_2(3)$ to the above problem.

To begin we need guesses for $t_1(0)$ and $t_2(0)$. We can guess any values we want, and people sometimes use zero for initial guesses. A better guess would be something between the lowest and highest boundary temperatures, so lets use the average of those two, 37.5, for our guesses for both $t_1(0)$ and $t_2(0)$. Putting each of those into the equations (1) in order to find $t_1(1)$ and $t_2(1)$, we get

$$t_1(1) = \frac{120+t_2(0)}{4} = \frac{120+37.5}{4} = 39.375$$

and

$$t_2(1) = \frac{90 + t_1(0)}{4} = \frac{90 + 37.5}{4} = 31.875$$

We then take each of these new values for t_1 and t_2 and put them into equations (1) again:

$$t_1(2) = \frac{120 + t_2(1)}{4} = \frac{120 + 31.875}{4} = 37.96875$$

and

$$t_2(2) = \frac{90 + t_1(1)}{4} = \frac{90 + 39.375}{4} = 32.34375$$

And once more:

$$t_1(3) = \frac{120 + t_2(2)}{4} = \frac{120 + 32.34375}{4} = 38.0859375$$

and

$$t_2(3) = \frac{90 + t_1(2)}{4} = \frac{90 + 37.96875}{4} = 31.9921875$$

In Example 1.3(b) we found that the exact solution to this problem is $t_1 = 38$ and $t_2 = 32$, so after only three iterations the Jacobi method has given approximations for t_1 and t_2 that are very close to the exact solution.

Now we will illustrate another iterative method for approximating solutions, called the **Gauss-Seidel method**. Note that in the Jacobi method we guessed initial values for *both* t_1 and t_2, then used those to find the first approximations of both *at the same time*. In the Gauss-Seidel method we guess a value for *just one of* t_1 *or* t_2 and put it into one of the equations (1) to find the other. Say, for example, we guess initially that $t_2(0) = 37.5$. We put this into the first equation (1) to obtain a first value for t_1, which we'll denote by $t_1(1)$, the first approximation for t_1. (We could just as well have called this $t_1(0)$.) This value is then used in the second equation to obtain the first approximation $t_2(1)$ for t_2. That value is then put back into the first equation to obtain a second approximation for t_1, and so on.

◇ **Example 2.4(b):** Use the Gauss-Seidel method to find the third approximations $t_1(3)$ and $t_2(3)$ to the system of equations

$$t_1 = \frac{120 + t_2}{4}, \qquad t_2 = \frac{90 + t_1}{4} \tag{1}$$

We begin with the guess $t_2(0) = 37.5$, and put that into the first of the above equations to find $t_1(1)$:

$$t_1(1) = \frac{120 + t_2(0)}{4} = \frac{120 + 37.5}{4} = 39.375$$

This value is then used in the second equation above to find $t_2(1)$:

$$t_2(1) = \frac{90 + t_1(1)}{4} = \frac{90 + 39.375}{4} = 32.34375$$

We now find $t_1(2)$

$$t_1(2) = \frac{120 + t_2(1)}{4} = \frac{120 + 32.34375}{4} = 38.0859375$$

which, in turn, is used to find $t_2(2)$:

$$t_2(2) = \frac{90 + t_1(2)}{4} = \frac{90 + 38.0859375}{4} = 32.02148438$$

Continuing, we have

$$t_1(3) = \frac{120 + t_2(2)}{4} = \frac{120 + 32.02148438}{4} = 38.00537109$$

and

$$t_2(3) = \frac{90 + t_1(3)}{4} = \frac{90 + 38.00537109}{4} = 32.00134277$$

As with the Jacobi method, the Gauss-Seidel method has given us a solution that is very close to the exact solution.

Section 2.4 Exercises

1. Consider the system $\begin{aligned} 5x - 2y + \ z &= -10 \\ 2x + 3y - \ z &= -3 \\ -x - \ y + 2z &= 5 \end{aligned}$ of equations. This exercise will lead you through solving the system using the Jacobi method.

   (a) Solve the system using row-reduction on your calculator or *MATLAB*.

   (b) Solve the first equation for x, the second for y and the third for z.

   (c) Create a table like the one below, or just use the one provided here. Note that there is an **index** n that goes from zero to eight, and could go on farther if we wished. Our objective is to create a sequence $x(0), x(1), x(2), ...$ of x values, and similarly for y and z, such that the values **converge** to the solution to the system. Begin by letting $x(0) = y(0) = z(0) = 0$, filling in the table accordingly.

n:	0	1	2	3	4	5	6	7	8
$x(n)$:									
$y(n)$:									
$z(n)$:									

   (d) Find $x(1)$ by substituting the values of $y(0)$ and $z(0)$ into your equation from (b) that was solved for x. You should get $x(1) = -2$; put that value into the correct spot on the table. Then substitute $x(0)$ and $z(0)$ into your second equation from (b), the one where you solved for y. This will give you $y(1)$. Then do the same to get $z(1)$. You now have the **first iteration** values for x , y and z.

   (e) Use your values of x, y and z after the first iteration to find the second iteration values, *rounded to two places past the decimal*, and place them in the appropriate column.

   (f) Repeat this process, using the first iteration values to find the second iteration values

   (g) Use *Excel* to create and finish the table. By the time you reach the eight iteration, your values should be very close to the actual solution. You have just used the **Jacobi method** to approximate a solution to the system of equations.

2. For this exercise you will again work with the system from the previous exercise, but this time you will the Gauss-Seidel method to approximate the solution to the system.

    (a) On the same *Excel* worksheet, but farther down, make another table like the one above. Leave $x(0)$ blank and let $y(0) = z(0) = 0$. Find $x(1)$ in the same way you did before. To find $y(1)$ we want to use the "most recent" values of x and z, so use $x(1)$ and $z(0)$. Then, to find $z(1)$ you will use the most recent values of x and y, which are $x(1)$ and $y(1)$.

    (b) You can now copy the column with the $n = 1$ formulas across the table. Compare the results with the table for the Jacobi method.

3. For this exercise you'll be working with the system
$$\begin{aligned} x + 2y + 4z &= 7 \\ -x + y + 2z &= 5 \\ 2x + 3y + 3z &= 7 \end{aligned}.$$

    (a) Use row-reduction to solve the system.

    (b) Using the Jacobi method with $x(0) = y(0) = z(0) = 0$ to find $x(1)$, $y(1)$, $z(1)$ and $x(2)$, $y(2)$, $z(2)$. Do the values obtained appear to be approaching the solution?

2.5 Chapter 2 Exercises

1. Do one of the following for each of the systems whose augmented matrices row reduce to the forms shown below. **Assume that the unknowns are** x_1**,** x_2**, ...**

   - If the system has a unique solution, give it. If the system has no solution, say so.
   - If the system has infinitely many solutions, give the general solution in terms of parameters s, t, etc., then give two particular solutions.

(a) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & -1 & 0 & -4 \\ 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

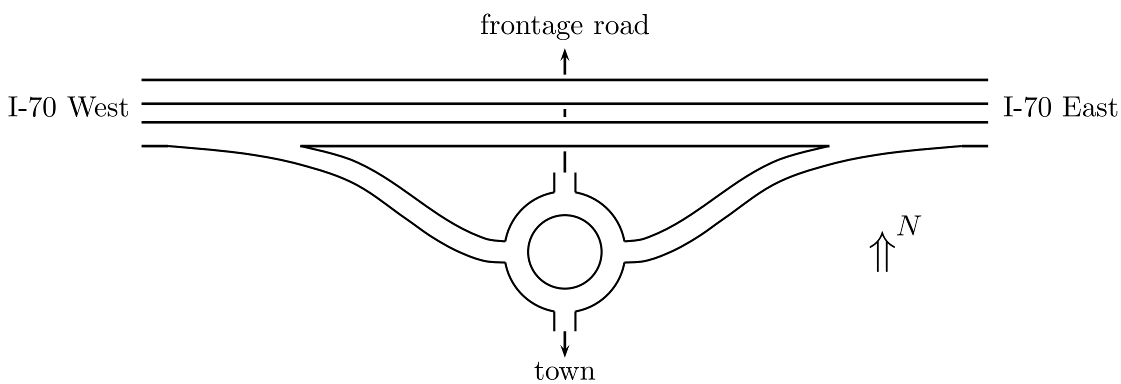
2. Consider the row-echelon augmented matrix $\begin{bmatrix} 1 & -1 & 3 & -2 & 4 \\ 0 & 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 2 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   (a) Give the general solution to the system of equations that this arose from.

   (b) Give three specific solutions to the system.

   (c) Change one entry in the matrix (cross it out on this sheet and write its replacement nearby) so that the system of equations would have no solution.

3. Vail, Colorado recently put in traffic "round-a-bouts" at all of its exits off Interstate 70. Each of these consists of a circle in which traffic is only allowed to flow counter-clockwise (do that all turns are right turns), and four points at which the circle can be entered or exited. See the diagram below.



It is known that at 7:30 AM the following is occurring:

- 22 vehicles per minute are entering the roundabout from the west. (These are the workers who cannot afford to live in Vail, and commute on I-70 from towns 30 and 40 miles west.)
- 4 vehicles per minute are exiting the roundabout to go east on I-70. (These are the tourists headed to the airport in Denver.)
- 7 vehicles per minute are exiting the roundabout toward town and 11 per minute are exiting toward the frontage road.

Solve the system and answer the following:

(a) What is the minimum number of cars per minute passing through the southeast quarter of the roundabout?

(b) If 18 vehicles per minute are passing through the southeast (SE) quarter of the roundabout per minute, how many are passing through each of the other quarters (NW, NE, SW)?

4. Consider the system $\begin{aligned} x_1 - x_2 - 4x_3 &= 6 \\ 5x_1 + x_2 - 2x_3 &= 18 \\ 2x_1 + 4x_2 + 10x_3 &= 0 \end{aligned}$.

   (a) Use your calculator or an online tool to reduce the matrix to reduced row echelon form. Write the system of two equations represented by the first two rows of the reduced matrix. (The last equation is of no use, so don't bother writing it.)

   (b) The second equation contains x_2 and x_3. Suppose that $x_3 = 1$ and compute x_2 using that equation. Then use the values you have for x_2 and x_3 in the first equation to find x_1.

   (c) Verify that the values you obtained in (b) are in fact a solution to the original system given.

   (d) Now let $x_3 = 0$ and repeat the process from (b) to obtain another solution. Verify that solution as well.

   (e) Let $x_3 = 2$ to find yet another solution.

   (f) Because there is no equation allowing us to determine x_3, we say that it is a **free variable**. What we will usually do in situations like this is let x_3 equal some **parameter** (number) that we will denote by t. That is, we set $x_3 = t$, which is really just renaming it. Substitute t into the second equation from (a) and solve for x_2 in terms of t. Then substitute that result into the first equation for x_2, along with t for x_3, and solve for x_1 in terms of t. Summarize by giving each of x_1, x_2 and x_3 in terms of t, all in one place.

   (g) Substitute the number one for t into your answer to (f) and check to see that it matches what you got for (b). If doesn't, you've gone wrong somewhere - find the error and fix it.

5. Solve each of the following systems of equations that have solutions. Do/show the following things:

   - Enter the augmented matrix for the system in your calculator.
   - Get the row-reduced echelon form of the matrix using the *rref* command. **Write down the resulting matrix**.
   - Write the system of equations that is equivalent to the row-reduced echelon matrix.
   - Find the solutions, if there are any. *Use the letters that were used in the original system for the unknowns!* For those with multiple solutions, give them in terms of a parameter t or, when necessary, two parameters s and t.

(a) $\begin{aligned} x_1 - x_2 + 3x_3 &= -4 \\ -2x_1 + 3x_2 - 8x_3 &= 13 \\ 5x_1 - 3x_2 + 11x_3 &= -10 \end{aligned}$ (b) $\begin{aligned} c_1 + 3c_2 + 5c_3 &= 3 \\ 2c_1 + 7c_2 + 9c_3 &= 5 \\ 2c_1 + 6c_2 + 11c_3 &= 7 \end{aligned}$

(c) $\begin{aligned} x_1 + 3x_2 - 2x_3 &= -1 \\ -7x_1 - 21x_2 + 14x_3 &= 7 \\ 2x_1 + 6x_2 - 4x_3 &= -2 \end{aligned}$ (d) $\begin{aligned} c_1 - c_2 + 3c_3 &= -4 \\ -2c_1 + 3c_2 - 8c_3 &= 13 \\ 5c_1 - 3c_2 + 11c_3 &= 4 \end{aligned}$

(e) $\begin{aligned} x - 3y + 7z &= 4 \\ 5x - 14y + 42z &= 29 \\ -2x + 5y - 20z &= -16 \end{aligned}$ (f) $\begin{aligned} x + 3y &= 2 \\ 4x + 12y + z &= 1 \\ -x - 3y - 2z &= 12 \end{aligned}$

- Give three *specific* solutions to the system from part (a) above.
- Give three *specific* solutions to the system from part (c) above.
- Solve the system from part (b) above by hand, showing all steps of the row reduction and indicating what you did at each step.

6. Give the reduced row echelon form of an augmented matrix for a system of four equations in four unknowns x_1, x_2, x_3 and x_4 for which

   - $x_4 = 7$
   - x_2 and only x_2, is a free variable

7. (Erdman) Consider the system $\begin{aligned} x + ky &= 1 \\ kx + y &= 1 \end{aligned}$, where k is some constant.

   (a) Set up the augmented matrix and use a row operation to get a zero in the lower left corner.

   (b) For what value or values of k would the system have infinitely many solutions? What is the form of the general solution?

   (c) For what value or values of k would the system have no solution?

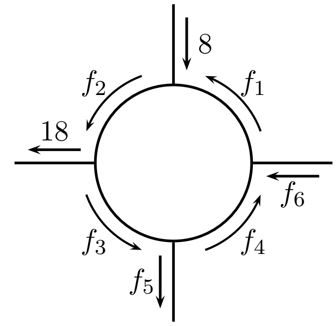
   (d) For all remaining values of k the system has a unique solution (that depends on the choice of k). What is the solution in that case? Your answer will contain the parameter k.

8. (Erdman) Consider the system $\begin{aligned} x - y - 3z &= 3 \\ 2x + z &= 0 \\ 2y + 7z &= c \end{aligned}$, where c is some constant.

   (a) Set up the augmented matrix and use a row operation to get a zero in the first entry of the second row.

   (b) Look at the second and third rows. For what value or values of c can the system be solved? Give the solution if there is a unique solution. Give the general solution if there are infinitely many solutions.

9. The network to the right represents a traffic circle. The numbers next to each of the paths leading into or out of the circle are the net flows in the directions of the arrows, in vehicles per minute, during the first part of lunch hour.



   (a) Suppose that $f_3 = 7$ and $f_5 = 4$. You should be able to work your way around the circle, eventually figuring out what each flow is. Do this.

   (b) Still assuming that $f_3 = 7$ and $f_5 = 4$, set up an equation at each junction of the circle, to get four equations in four unknowns. Solve the system. What do you notice about your answers?

   (c) Now assume that the only flows you know are the ones shown in the diagram. When you set up a system of equations, based on the flows in and out of each junction, how many equations will you have? How many unknowns? How many free variables do you expect?

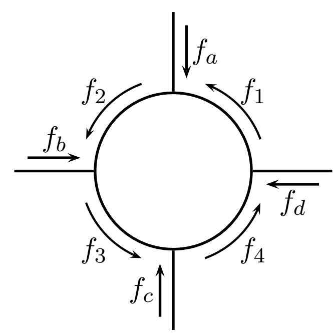
   (d) Go ahead and set up the system of equations. Give the augmented matrix and the reduced matrix (obtained with your calculator), and then give the general solution to the system. Were you correct in how many free variables there are?

   (e) Choose the value(s) of the parameter(s) that make $f_3 = 7$ and $f_5 = 4$, then find the resulting particular solution. What do you notice?

   (f) What restriction(s) is(are) there on the parameter(s), in order that all flows go in the directions indicated. (Allow a flow of zero for each flow as well.)

10. For another traffic circle, a student uses the diagram shown below and to the right and obtains the flows given below and to the left, in vehicles per minute.

$$f_1 = t - 8, \qquad f_2 = t + 3, \qquad f_3 = t - 5, \qquad f_4 = t$$



   (a) Determine the minimum value of t that makes each of f_1 through f_4 zero or greater. Give the minimum allowable values for each flow, in the form $f_i \geq a$, assuming that no vehicles ever go the wrong way around a portion of the circle. Remember that setting a value for any flow determines all the other flows. You may neglect units.

   (b) Give each of the flows f_1 through f_4 when the flow in the northeast quarter (f_1) is 12 vehicles per minute. You may neglect units.

   (c) Determine each of the flows f_a through f_d, still for $f_1 = 12$. You should be able to do this based only on the four equations given. At least one of them will be negative, indicating that the corresponding arrow(s) should be reversed.

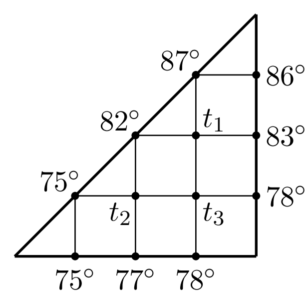
11. In a previous exercise, you may have attempted to find the equation of a parabola through the three points $(-1,-6)$, $(0,-4)$ and $(1,-1)$. You set up a system to find values of a, b and c in the parabola equation $y = ax^2 + bx + c$. There was a unique solution, meaning that there is only one parabola containing those three points. Expect the following to not work out as "neatly."

   (a) Use a system of equations to find the equation of a parabola that goes through just the two points $(-1,-6)$ and $(1,-1)$. Explain your results.

   (b) Use a system of equations to find the equation $y = mx + b$ of a line through the four points $(1.3, 1.5)$, $(0.8, 0.4)$, $(2.6, 3.0)$ and $(2.0, 2.0)$.

   (c) Plot the four points from (b) on a neat and accurate graph, and use what you see to explain your answer to (b). **You should be able to give your explanation in one or two complete sentences.**

12. The diagram to the right shows boundary temperatures for a triangular piece of metal, and a grid connecting some interior points with the boundary. In this exercise you will use the Jacobi Method to approximate the interior temperatures t_1, t_2 and t_3. You will see that an iterative method like the Jacobi method or Gauss-Seidel method work well for this sort of problem.



   (a) Write an equation for each of t_1, t_2 and t_3 as an average of the four surrounding temperatures. Combine any numbers that are added, and notice that you now have three equations in the correct form for an iterative method.

   (b) You will now use the Jacobi method to approximate values for t_1, t_2 and t_3, but you need starting values $t_1(0)$, $t_2(0)$ and $t_3(0)$. You could use just about anything, but the method will converge more quickly if a good choice of initial values is made. If you look at the boundary values, you can see that the average of all of them should be around $80°$. (We could find the exact average, but it wouldn't result in any quicker convergence.) So use $t_1(0) = t_2(0) = t_3(0) = 80$ and approximate the values of $t_1(1)$, $t_2(1)$, $t_3(1)$ and $t_1(2)$, $t_2(2)$, $t_3(2)$, **rounding to four places past the decimal when rounding is necessary**.

   (c) Repeat part (b), but using the Gauss-Seidel method.

3 Euclidean Space and Vectors

Outcome:

3. Understand vectors and their algebra and geometry in $\mathbb{R}^n$.

   Performance Criteria:

   (a) Recognize the equation of a plane in $\mathbb{R}^3$ and determine where the plane intersects each of the three axes. Give the equation of any one of the three coordinate planes or any plane parallel to one of the coordinate planes.

   (b) Find the distance between two points in $\mathbb{R}^n$.

   (c) Find the vector from one point to another in $\mathbb{R}^n$. Find the length of a vector in $\mathbb{R}^n$.

   (d) Multiply vectors by scalars and add vectors, algebraically. Find linear combinations of vectors algebraically.

   (e) Illustrate the parallelogram method and tip-to-tail method for finding a linear combination of two vectors.

   (f) Find a linear combination of vectors equalling a given vector.

   (g) Find the dot product of two vectors, determine the length of a single vector.

   (h) Determine whether two vectors are orthogonal (perpendicular).

   (i) Find the projection of one vector onto another, graphically or algebraically.

In the study of linear algebra we will be defining new concepts and developing corresponding notation. *The purpose of doing so is to develop more powerful machinery for investigating the concepts of interest.* We begin the development of notation with the following. The set of all real numbers is denoted by $\mathbb{R}$, and the set of all ordered pairs of real numbers is $\mathbb{R}^2$, spoken as "R-two." Geometrically, $\mathbb{R}^2$ is the familiar Cartesian coordinate plane. Similarly, the set of all ordered triples of real numbers is the three-dimensional space referred to as $\mathbb{R}^3$, "R-three." The set of all ordered ***n*-tuples** (lists of n real numbers in a particular order) is denoted by $\mathbb{R}^n$. Although difficult or impossible to visualize physically, $\mathbb{R}^n$ can be thought of as n-dimensional space. All of the $\mathbb{R}^n$s are what are called **Euclidean space**.

3.1 Euclidean Space

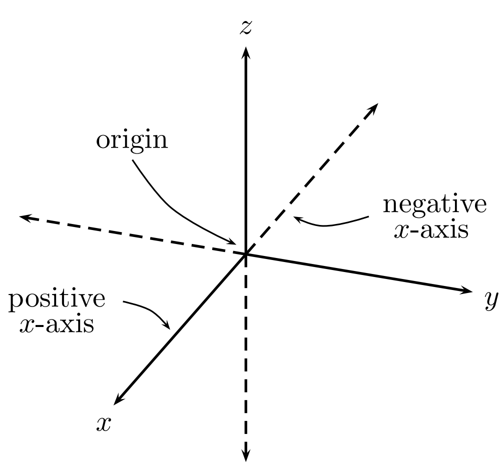
Performance Criteria:

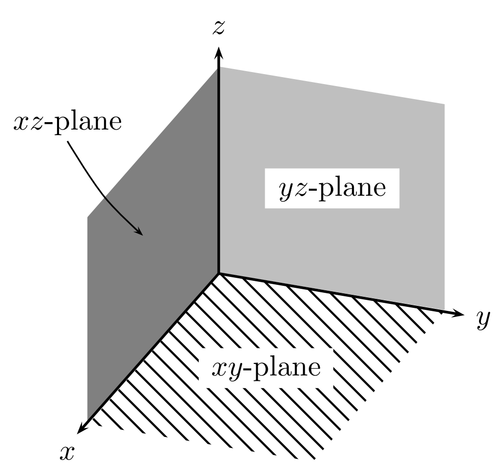
3. (a) Recognize the equation of a plane in $\mathbb{R}^3$ and determine where the plane intersects each of the three axes. Give the equation of any one of the three coordinate planes or any plane parallel to one of the coordinate planes.
   (b) Find the distance between two points in $\mathbb{R}^n$.

It is often taken for granted that everyone knows what we mean by the **real numbers**. To actually define the real numbers precisely is a bit of a chore and very technical. Suffice it to say that the real numbers include all numbers other than complex numbers (numbers containing $\sqrt{-1} = i$ or, for electrical engineers, j) that a scientist or engineer is likely to run into. The numbers 5, -31.2, π, $\sqrt{2}$, $\frac{2}{7}$, and e are all real numbers. We denote the set of all real numbers with the symbol $\mathbb{R}$, and the geometric representation of the real numbers is the familiar **real number line**, a horizontal line on which every real number has a place. This is possible because the real numbers are ordered: given any two real numbers, either they are equal to each other, one is less than the other, or vice-versa.

As mentioned previously, the set $\mathbb{R}^2$ is the set of all ordered pairs of real numbers. Geometrically, every such pair corresponds to a point in the **Cartesian plane**, which is the familiar xy-plane. $\mathbb{R}^3$ is the set of all ordered triples, each of which represents a point in three-dimensional space. We can continue on - $\mathbb{R}^4$ is the set of all ordered "4-tuples", and can be thought of geometrically as four dimensional space. Continuing further, an "n-tuple" is n real numbers, in a specific order; each n-tuple can be thought of as representing a point in n-dimensional space. These spaces are sometimes called "two-space," "three-space" and "n-space" for short.

Two-space is fairly simple, with the only major features being the two axes and the four quadrants that the axes divide the space into. Three-space is a bit more complicated. Obviously there are three coordinate axes instead of two. In addition to those axes, there are also three coordinate planes as well, the xy-plane, the xz-plane and the yz-plane. Finally the three coordinate planes divide the space into eight **octants**. The pictures below illustrate the coordinate axes and planes. The first octant is the one we are looking into, where all three coordinates are positive. It is not important that we know the numbering of the other octants.





Every plane in $\mathbb{R}^3$ (we will be discussing only $\mathbb{R}^3$ for now) consists of a set of points that behave in an orderly mathematical manner, described here:

Equation of a Plane in $\mathbb{R}^3$: The ordered triples corresponding to all the points in a plane satisfy an equation of the form

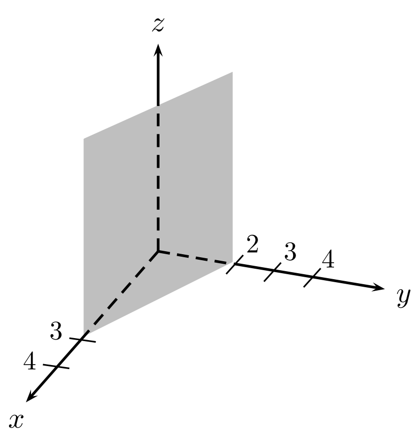
$$ax + by + cz = d,$$

where a, b, c and d are constants, and not all of a, b and c are zero.

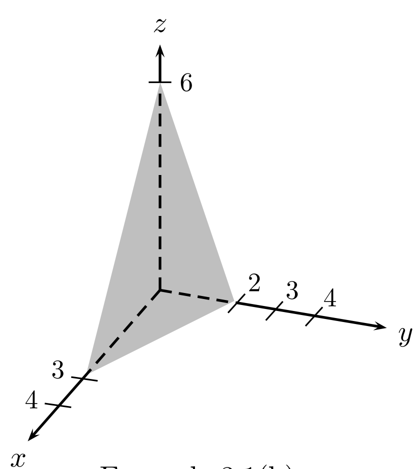
The xy-plane is the plane containing the x and y-axes. The only condition on points in that plane is that $z = 0$, so that is the equation of that plane. (Here the constants a, b and d are all zero, and $c = 1$.) The plane $z = 5$ is a horizontal plane that is five units above the xy-plane.

◇ **Example 3.1(a):** Graph the equation $2x + 3y = 6$ in the first octant. Indicate clearly where it intersects each of the coordinate axes, if it does.

Some points that satisfy the equation are $(3, 0, 0)$, $(6, -2, 5)$, and so on. Since z is not included in the equation, there are no restrictions on z; it can take any value. If we were to fix z at zero and plot all points that satisfy the equation, we would get a line in the xy-plane through the two points $(3, 0, 0)$ and $(0, 2, 0)$. These points are obtained by first letting y and z be zero, then by letting x and z be zero. Since z can be anything, the set of points satisfying $2x + 3y = 6$ is a vertical plane intersecting the xy-plane in that line. The plane is shown below and to the left.



Example 3.1(a)



Example 3.1(b)

◇ **Example 3.1(b):** Graph the equation $2x + 3y + z = 6$ in the first octant. Indicate clearly where it intersects each of the coordinate axes, if it does.

The set of points satisfying this equation is also a plane, but z is no longer free to take any value. The simplest way to "get a handle on" such a plane is to find where it intercepts the three axes. Note that every point on the x-axis has y- and z-coordinates of zero. So to find where the plane intersects the x-axis we put zero into the equation for y and z, then solve for x, getting $x = 3$. The plane then intersects the x-axis at $(3, 0, 0)$. A similar process gives us that the plane intersects the y and z axes at $(0, 2, 0)$ and $(0, 0, 6)$. The graph of the plane is shown in the drawing above and to the right.

Consider now a system of equations like

$$\begin{aligned} x + 3y - 2z &= -4 \\ 3x + 7y + z &= 4 \\ -2x + y + 7z &= 7 \end{aligned},$$

which has solution $(3, -1, 2)$. Each of the three equations represents a plane in $\mathbb{R}^3$, and the point $(3, -1, 2)$ is where the three planes intersect. This is completely analogous to the interpretation of the solution of a system of two linear equations in two unknowns as the point where the two lines representing the equations cross. This is the first of three interpretations we'll have for the solution to a system of equations.

The only other basic geometric fact we need about three-space is this:

Distance Between Points: The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbb{R}^n$ is given by

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

This is simply a three-dimensional version of the Pythagorean Theorem. This is in fact used in even higher dimensional spaces; even though we cannot visualize the distance geometrically, this idea is both mathematically valid and useful.

◇ **Example 3.1(c):** Find the distance in $\mathbb{R}^3$ between the points $(-4, 7, 1)$ and $(13, 0, -6)$.

Using the above formula we get

$$d = \sqrt{(-4 - 13)^2 + (7 - 0)^2 + (1 - (-6))^2} = \sqrt{(-17)^2 + 7^2 + 7^2} = \sqrt{387} \approx 19.7$$

We will now move on to our main tool for working in Euclidean space, vectors.

Section 3.1 Exercises

1. Determine whether each of the equations given describes a plane in $\mathbb{R}^3$. If not, say so. If it does describe a plane, give the points where it intersects each axis. If it doesn't intersect an axis, say so.

   (a) $-2x - y + 3z = -6$ (b) $x + 3z = 6$ (c) $y = -6$

   (d) $x + 3z^2 = 12$ (e) $x - 2y + 3z = -6$

2. Give the equation of a plane in $\mathbb{R}^3$ that does not intersect the y-axis but does intersect the other two axes. Give the points at which it intersects the x- and z-axes.

3. Give the equation of the plane that intersects the y-axis at 4 and does not intersect either of the other two axes.

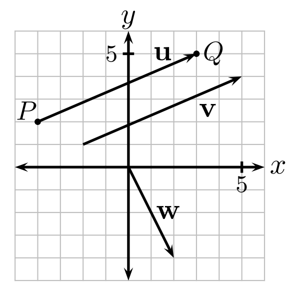
3.2 Introduction to Vectors

Performance Criteria:

3. (c) Find the vector from one point to another in $\mathbb{R}^n$. Find the length of a vector in $\mathbb{R}^n$.

There are a number of different ways of thinking about **vectors**; it is likely that you think of them as "arrows" in two- or three-dimensional space, which is the first concept of a vector that most people have. Each such arrow has a **length** (sometimes called **norm** or **magnitude**) and a direction. Physically, then, vectors represent quantities having both amount and direction. Examples would be things like force or velocity. Quantities having only amount, like temperature or pressure, are referred to as **scalar** quantities. We will represent scalar quantities with lower case italicized letters like $a, b, c, ..., x, y, z$. Vectors are denoted by lower case boldface letters when typeset, like $\mathbf{u}, \mathbf{v}, \mathbf{x}$, and so on. When written by hand, scalars and vectors are both lower case letters, but we put a small arrow pointing to the right over any letter denoting a vector.

Consider a vector represented by an arrow in $\mathbb{R}^2$. We will call the end with the arrowhead the **tip** of the vector, and the other end we'll call the **tail**. (The more formal terminology is **terminal point** and **initial point**.) The picture to the right shows three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^2$. It should be clear that a vector can be moved around in $\mathbb{R}^2$ in such a way that the direction and magnitude remain unchanged. Sometimes we say that two vectors related to each other this way are equivalent, but in this class we will simply say that they are the same vector. The vectors $\mathbf{u}$ and $\mathbf{v}$ are the same vector, just in different positions.



It is sometimes convenient to denote points with letters, and we use italicized capital letters for this. We commonly use P (for point!) Q and R, and the origin is denoted by O. (That's capital "oh," not zero.) Sometimes we follow the point immediately by its coordinates, like $P(-4, 3)$. The notation $\overrightarrow{PQ}$ denotes the vector that goes from point P to point Q, which in this case is vector $\mathbf{v}$. Any vector $\overrightarrow{OP}$ with its tail at the origin is said to be in **standard position**, and is called a **position vector**; $\mathbf{w}$ above is an example of such a vector. Note that for any point in $\mathbb{R}^2$ (or in $\mathbb{R}^n$), there is a corresponding vector that goes from the origin to that point. *In linear algebra we think of a point and its position vector as interchangeable.* In the next section you will see the advantage of thinking of a point as a position vector.

We will describe vectors with numbers; in $\mathbb{R}^2$ we give a vector as two numbers, the first telling how far to the right (positive) or left (negative) one must go to get from the tail to the tip of the vector, and the second telling how far up (positive) or down (negative) from tail to tip. These numbers are generally arranged in one of two ways. The first way is like an ordered pair, but with "square brackets" instead of parentheses. The vector $\mathbf{u}$ above is then $\mathbf{u} = [7, 3]$, and $\mathbf{w} = [2, -4]$. The second way to write a vector is as a **column vector**: $\mathbf{u} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$. This is, in fact, the form we will use more often.

The two numbers quantifying a vector in $\mathbb{R}^2$ are called the **components** of the vector. We generally use the same letter to denote the components of a vector as the one used to name the vector, but we distinguish them with subscripts. Of course the components are scalars, so we use italic letters for them. So we would have $\mathbf{u} = [u_1, u_2]$ and $\mathbf{v} = [v_1, v_2]$. The direction of a vector in $\mathbb{R}^2$ is given, in some sense, by the combination of the two components. The length is found using the Pythagorean theorem. For a vector $\mathbf{v} = [v_1, v_2]$ we denote and define the length of the vector by $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$. Of course everything we have done so far applies to vectors in higher dimensions. A vector $\mathbf{x}$ in $\mathbb{R}^n$ would be denoted by $\mathbf{x} = [x_1, x_2, ..., x_n]$. This shows that, in some sense, a **vector** is just an ordered list of numbers, like an n-tuple but with differences you will see in the next section. The length of a vector in $\mathbb{R}^n$ is found as follows.

DEFINITION 3.2.1: **Magnitude in $\mathbb{R}^n$**

The **magnitude**, or **length** of a vector $\mathbf{x} = [x_1, x_2, ..., x_n]$ in $\mathbb{R}^n$ is given by

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

◇ **Example 3.2(a):** Find the vector $\mathbf{x} = \overrightarrow{PQ}$ in $\mathbb{R}^3$ from the point $P(5, -3, 7)$ to $Q(-2, 6, 1)$, and find the length of the vector.

The components of $\mathbf{x}$ are obtained by simply subtracting each coordinate of P from each coordinate of Q:

$$\mathbf{x} = \overrightarrow{PQ} = [-2 - 5, 6 - (-3), 1 - 7] = [-7, 9, -6]$$

The length of $\mathbf{x}$ is

$$\|\mathbf{x}\| = \sqrt{(-7)^2 + 9^2 + (-6)^2} = \sqrt{166} \approx 12.9$$

There will be times when we need a vector with length zero; this is the special vector we will call (surprise, surprise!) the **zero vector**. It is denoted by a boldface zero, $\mathbf{0}$, to distinguish it from the scalar zero. This vector has no direction.

Let's finish with the following important note about how we will work with vectors in this class:

In this course, when working with vectors geometrically, we will almost always be thinking of them as position vectors. When working with vectors algebraically, we will always consider them to be column vectors.

Section 3.2 Exercises

1. For each pair of points P and Q, find the vector $\overrightarrow{PQ}$ in the appropriate space. Then find the length of the vector.

   (a) $P(-4, 11, 7)$, $Q(13, 5, -8)$

   (b) $P(-5, 1)$, $Q(7, -2)$

   (c) $P(-3, 0, 6, 1)$, $Q(7, -1, -1, 10)$

3.3 Addition and Scalar Multiplication of Vectors, Linear Combinations

Performance Criteria:

3. (d) Multiply vectors by scalars and add vectors, algebraically. Find linear combinations of vectors algebraically.
   (e) Illustrate the parallelogram method and tip-to-tail method for finding a linear combination of two vectors.
   (f) Find a linear combination of vectors equalling a given vector.

In the previous section a vector $\mathbf{x} = [x_1, x_2, ..., x_n]$ in n dimensions was starting to look suspiciously like an n-tuple $(x_1, x_2, ..., x_n)$ and we established a correspondence between any point and the position vector with its tip at that point. One might wonder why we bother then with vectors at all! The reason is that we can perform algebraic operations with vectors that make sense physically, and such operations make no sense with n-tuples. The two most basic things we can do with vectors are add two of them or multiply one by a scalar, and both are done component-wise:

DEFINITION 3.3.1: **Addition and Scalar Multiplication of Vectors**

Let $\mathbf{u} = [u_1, u_2, ..., u_n]$ and $\mathbf{v} = [v_1, v_2, ..., n_n]$, and let c be a scalar. Then we define the vectors $\mathbf{u}+\mathbf{v}$ and $c\mathbf{u}$ by

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix} \quad \text{and} \quad c\mathbf{u} = c\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$$

Note that result of adding two vectors or multiplying a vector by a scalar is also a vector. It clearly follows from these that we can get subtraction of vectors by first multiplying the second vector by the scalar -1, then adding the vectors. With just a little thought you will recognize that this is the same as just subtracting the corresponding components.

◇ **Example 3.3(a):** For $\mathbf{u} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 9 \\ 6 \end{bmatrix}$, find $\mathbf{u}+\mathbf{v}$, $3\mathbf{u}$ and $\mathbf{u}-\mathbf{v}$.

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 5+(-4) \\ -1+9 \\ 2+6 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 8 \end{bmatrix}$$

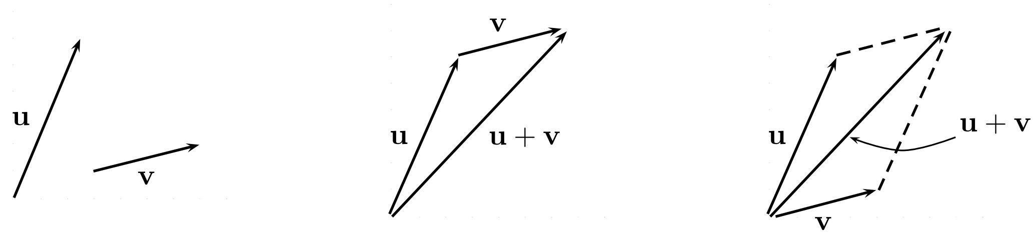
$$3\mathbf{u} = 3\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3(5) \\ 3(-1) \\ 3(2) \end{bmatrix} = \begin{bmatrix} 15 \\ -3 \\ 6 \end{bmatrix}$$

$$\mathbf{u}-\mathbf{v}=\begin{bmatrix}5\\-1\\2\end{bmatrix}-\begin{bmatrix}-4\\9\\6\end{bmatrix}=\begin{bmatrix}5-(-4)\\-1-\ 9\\2\ -\ 6\end{bmatrix}=\begin{bmatrix}9\\-10\\-4\end{bmatrix}$$

Addition of vectors can be thought of geometrically in two ways, both of which are useful. The first way is what we will call the **tip-to-tail method**, and the second method is called the **parallelogram method**. You should become very familiar with both of these methods, as they each have their advantages; they are illustrated below.

◇ **Example 3.3(b):** Add the two vectors $\mathbf{u}$ and $\mathbf{v}$ shown below and to the left, first by the tip-to-tail method, and second by the parallelogram method.

To add using the tip-to-tail method move the second vector so that its tail is at the tip of the first. (Be sure that its length and direction remain the same!) The vector $\mathbf{u}+\mathbf{v}$ goes from the tail of $\mathbf{u}$ to the tip of $\mathbf{v}$. See in the middle below.



tip-to-tail method   parallelogram method

To add using the parallelogram method, put the vectors $\mathbf{u}$ and $\mathbf{v}$ together at their tails (again being sure to preserve their lengths and directions). Draw a dashed line from the tip of $\mathbf{u}$, parallel to $\mathbf{v}$, and draw another dashed line from the tip of $\mathbf{v}$, parallel to $\mathbf{u}$. $\mathbf{u}+\mathbf{v}$ goes from the tails of $\mathbf{u}$ and $\mathbf{v}$ to the point where the two dashed lines cross. See to the right above. The reason for the name of this method is that the two vectors and the dashed lines create a parallelogram.

Each of these two methods has a natural physical interpretation. For the tip-to-tail method, imagine an object that gets *displaced* by the direction and amount shown by the vector $\mathbf{u}$. Then suppose that it gets displaced by the direction and amount given by $\mathbf{v}$ after that. Then the vector $\mathbf{u}+\mathbf{v}$ gives the *net* (total) displacement of the object. Now look at that picture for the parallelogram method, and imagine that there is an object at the tails of the two vectors. If we were then to have two forces acting on the object, one in the direction of $\mathbf{u}$ and with an amount (magnitude) indicated by the length of $\mathbf{u}$, and another with amount and direction indicated by $\mathbf{v}$, then $\mathbf{u}+\mathbf{v}$ would represent the net force. (In a statics or physics course you might call this the **resultant** force.)

A very important concept in linear algebra is that of a **linear combination**. Let me say it again:

Linear combinations are one of the most important concepts in linear algebra! You need to recognize them when you see them and learn how to create them. They will be central to almost everything that we will do from here on.

A linear combination of a set of vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ (note that the subscripts now distinguish different *vectors*, not the components of a single vector) is obtained when each of the vectors is multiplied by a scalar, and the resulting vectors are added up. So if $c_1, c_2, ..., c_n$ are the scalars that $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ are multiplied by, the resulting linear combination is the *single vector* $\mathbf{v}$ given by

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_n\mathbf{v}_n.$$

Emphasizing again the importance of this concept, let's provide a slightly more concise and formal definition:

> DEFINITION 3.3.2: **Linear Combination**
>
> A **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$, all in $\mathbb{R}^n$, is any vector of the form
>
> $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_n\mathbf{v}_n,$$
>
> where $c_1, c_2, ..., c_n$ are scalars.

Note that when we create a linear combination of a set of vectors we are doing virtually everything possible algebraically with those vectors, which is just addition and scalar multiplication!

You have seen this idea before; every polynomial like $5x^3 - 7x^2 + \frac{1}{2}x - 1$ is a linear combination of $1, x, x^2, x^3, ...$. Those of you who have had a differential equations class have seen things like $ds\frac{dy}{dt} + 3\frac{dy}{dt} + 2y$, which is a linear combination of the second, first and "zeroth" derivatives of a function $y = y(t)$. Here is why linear combinations are so important: In many applications we seek to have a basic set of objects (vectors) from which all other objects can be built as linear combinations of objects from our basic set. A large part of our study will be centered around this idea. This may not make any sense to you now, but hopefully it will by the end of the course.

◇ **Example 3.3(c):** For the vectors $\mathbf{v}_1 = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 9 \\ 6 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 8 \end{bmatrix}$, give the linear combination $2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$ as one vector.

$$2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3 = 2\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} - 3\begin{bmatrix} -4 \\ 9 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ 4 \end{bmatrix} - \begin{bmatrix} -12 \\ 27 \\ 18 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 8 \end{bmatrix} = \begin{bmatrix} -2 \\ -26 \\ 30 \end{bmatrix}$$

◇ **Example 3.3(d):** For the same vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ as in the previous exercise and scalars c_1, c_2 and c_3, give the linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ as one vector.

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = c_1\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -4 \\ 9 \\ 6 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 3 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5c_1 \\ -1c_1 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} -4c_2 \\ 9c_2 \\ 6c_2 \end{bmatrix} + \begin{bmatrix} 0c_3 \\ 3c_3 \\ 8c_3 \end{bmatrix}$$

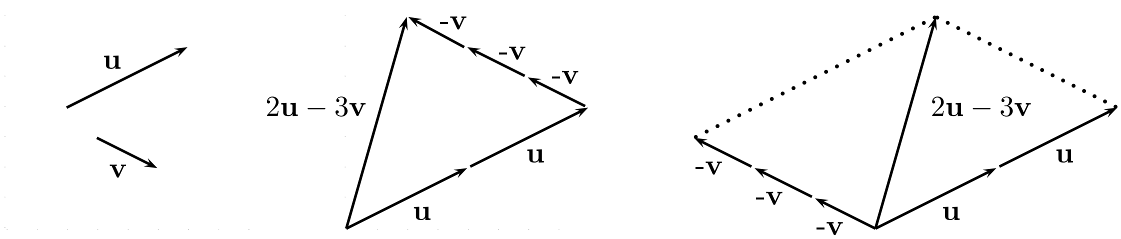
$$= \begin{bmatrix} 5c_1 - 4c_2 + 0c_3 \\ -1c_1 + 9c_2 + 3c_3 \\ 2c_1 + 6c_2 + 8c_3 \end{bmatrix}$$

Note that the final result is a single vector with three components that look suspiciously like the left sides of a system of three equations in three unknowns!

In the previous two examples we found linear combinations algebraically; in the next example we find a linear combination geometrically.

◇ **Example 3.3(e):** In the space below and to the right, sketch the vector $2\mathbf{u} - 3\mathbf{v}$ for the vectors $\mathbf{u}$ and $\mathbf{v}$ shown below and to the left.

In the center below the linear combination is obtained by the tip-to-tail method, and to the right below it is obtained by the parallelogram method.



The last example is probably the most important in this section.

◇ **Example 3.3(f):** Find a linear combination of the vectors $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$ that equals the vector $\mathbf{w} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$.

We are looking for two scalars c_1 and c_2 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{w}$. By the method of Example 3.3(d) we have

$$c_1 \begin{bmatrix} 3 \\ -4 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$$

$$\begin{bmatrix} 3c_1 \\ -4c_1 \end{bmatrix} + \begin{bmatrix} 7c_2 \\ -3c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$$

$$\begin{bmatrix} 3c_1 + 7c_2 \\ -4c_1 - 3c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$$

In the last line above we have two vectors that are equal. It should be intuitively obvious that this can only happen if the individual components of the two vectors are equal. This results in the system $\begin{array}{rcl} 3c_1 + 7c_2 & = & 1 \\ -4c_1 - 3c_2 & = & -14 \end{array}$ of two equations in the two unknowns c_1 and c_2. Solving, we arrive at $c_1 = 5,\ c_2 = -2$. It is easily verified that these are correct:

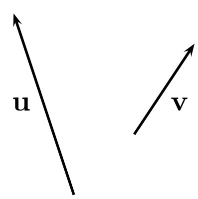
$$5\begin{bmatrix} 3 \\ -4 \end{bmatrix} - 2\begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 15 \\ -20 \end{bmatrix} - \begin{bmatrix} 14 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$$

We now conclude with an important observation. Suppose that we consider all possible linear combinations of a single vector $\mathbf{v}$. That is then the set of all vectors of the form $c\mathbf{v}$ for some scalar c, which is just all scalar multiples of $\mathbf{v}$. At the risk of being redundant, the set of all linear combinations of a single vector is all scalar multiples of that vector.

Section 3.3 Exercises

1. For the two vectors **u** and **v** shown below and to the left, illustrate the tip-to-tail and parallelogram methods for finding $-\mathbf{u} + 2\mathbf{v}$ in the spaces indicated.

Tip-to-tail: Parallelogram:



2. For the vectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$ and $\mathbf{v}_4 = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$, give the linear combination $5\mathbf{v}_1 + 2\mathbf{v}_2 - 7\mathbf{v}_3 + \mathbf{v}_4$ as one vector.

3. For the vectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 3 \\ -6 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -8 \\ 1 \\ 4 \end{bmatrix}$, give the linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ as one vector.

4. Give a linear combination of $\mathbf{u} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$ that equals $\begin{bmatrix} 17 \\ -4 \\ -9 \end{bmatrix}$.
   Demonstrate that your answer is correct by filling in the blanks:

$$\underline{\quad}\begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} + \underline{\quad}\begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} + \underline{\quad}\begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \end{bmatrix} + \begin{bmatrix} \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \end{bmatrix} + \begin{bmatrix} \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \end{bmatrix} = \begin{bmatrix} 17 \\ -4 \\ -9 \end{bmatrix}$$

5. For each of the following, find a linear combination of the vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ that equals $\mathbf{v}$. Conclude by giving the actual linear combination, not just some scalars.

(a) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(b) $\mathbf{u}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 8 \\ -6 \end{bmatrix}$

6. (a) Consider the vectors $\mathbf{u}_1 = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -2 \\ 6 \\ 5 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$.
If possible, find scalars a_1, a_2 and a_3 such that $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 = \mathbf{w}$.

(b) Consider the vectors $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -7 \\ 2 \\ 5 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$.
If possible, find scalars b_1, b_2 and b_3 such that $b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 = \mathbf{w}$.

(c) To do each of parts (a) and (b) you should have solved a system of equations. Let A be the coefficient matrix for the system in (a) and let B be the coefficient matrix for the system in part (b). Use your calculator to find $\det(A)$ and $\det(B)$, the determinants of matrices A and B. You will probably find the command for the determinant in the same menu as *rref*.

3.4 The Dot Product of Vectors, Projections

Performance Criteria:

3. (g) Find the dot product of two vectors, determine the length of a single vector.
   (h) Determine whether two vectors are orthogonal (perpendicular).
   (i) Find the projection of one vector onto another, graphically or algebraically.

The Dot Product and Orthogonality

There are two ways to "multiply" vectors, both of which you have likely seen before. One is called the **cross product**, and only applies to vectors in $\mathbb{R}^3$. It is quite useful and meaningful in certain physical situations, but it will be of no use to us here. More useful is the other method, called the **dot product**, which is valid in all dimensions.

DEFINITION 3.4.1: **Dot Product**

Let $\mathbf{u} = [u_1, u_2, ..., u_n]$ and $\mathbf{v} = [v_1, v_2, ..., v_n]$. The **dot product** of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \cdot \mathbf{v}$, is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 + \cdots + u_nv_n$$

The dot product is useful for a variety of things. Recall that the length of a vector $\mathbf{v} = [v_1, v_2, ..., v_n]$ is given by $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. Note also that $v_1^2 + v_2^2 + \cdots + v_n^2 = \mathbf{v} \cdot \mathbf{v}$, which implies that $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. Perhaps the most important thing about the dot product is that the dot product of two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ is zero if, and only if, the two vectors are perpendicular. In general, we make the following definition.

DEFINITION 3.4.2: **Orthogonal Vectors**

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are said to be **orthogonal** if, and only if, $\mathbf{u} \cdot \mathbf{v} = 0$.

◇ **Example 3.4(a):** For the three vectors $\mathbf{u} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$, find $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \cdot \mathbf{w}$ and $\mathbf{v} \cdot \mathbf{w}$. Are any of the vectors orthogonal to each other?

We find that

$$\mathbf{u} \cdot \mathbf{v} = (5)(-1) + (-1)(3) + (2)(4) = -5 + (-3) + 8 = 0,$$

$$\mathbf{u} \cdot \mathbf{w} = (5)(2) + (-1)(-1) + (2)(-3) = 10 + 1 + (-6) = 5,$$

$$\mathbf{v} \cdot \mathbf{w} = (-1)(2) + (3)(-1) + (4)(-3) = -2 + (-3) + (-12) = -17$$

From the first computation we can see that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

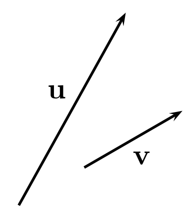
Projections

Given two vectors $\mathbf{u}$ and $\mathbf{v}$, we can create a new vector $\mathbf{w}$ called the **projection of $\mathbf{u}$ onto $\mathbf{v}$**, denoted by $\text{proj}_{\mathbf{v}}\mathbf{u}$. This is a very useful idea, in many ways. Geometrically, we can find $\text{proj}_{\mathbf{v}}\mathbf{u}$ as follows:

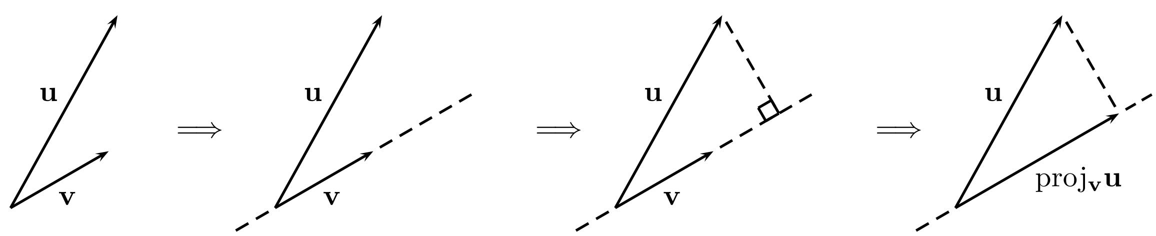
- Bring $\mathbf{u}$ and $\mathbf{v}$ together tail-to-tail.
- Sketch in the line containing $\mathbf{v}$, as a dashed line.
- Sketch in a dashed line segment from the tip of $\mathbf{u}$ to the dashed line containing $\mathbf{v}$, *perpendicular to that line.*
- Draw the vector $\text{proj}_{\mathbf{v}}\mathbf{u}$ from the point at the tails of $\mathbf{u}$ and $\mathbf{v}$ to the point where the dashed line segment meets $\mathbf{v}$ or the dashed line containing $\mathbf{v}$.

Note that $\text{proj}_{\mathbf{v}}\mathbf{u}$ is parallel to $\mathbf{v}$; if we were to find $\text{proj}_{\mathbf{u}}\mathbf{v}$ instead, the result would be parallel to $\mathbf{u}$ in that case. The above steps are illustrated in the following example.

◇ **Example 3.4(b):** For the vectors $\mathbf{u}$ and $\mathbf{v}$ shown to the right, find the projection $\text{proj}_{\mathbf{v}}\mathbf{u}$.

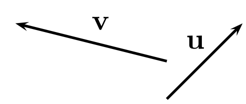


We follow the above steps:

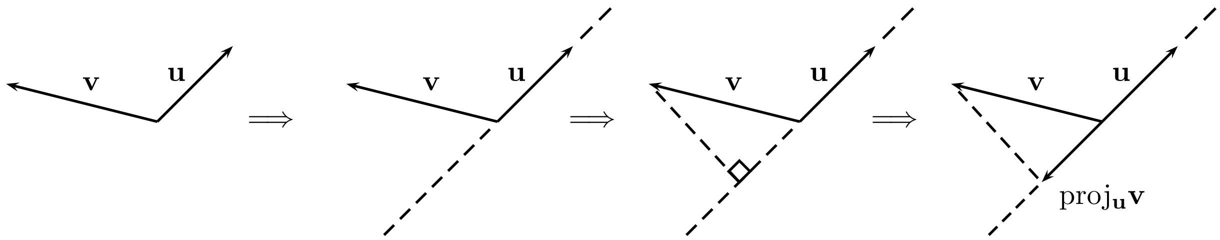


Projections are a bit less intuitive when the angle between the two vectors is obtuse, as seen in the next example.

◇ **Example 3.4(c):** For the vectors $\mathbf{u}$ and $\mathbf{v}$ shown to the right, find the projection $\text{proj}_{\mathbf{u}}\mathbf{v}$.



We follow the steps again, noting that this time we are projecting $\mathbf{v}$ onto $\mathbf{u}$:



Here we see that $\text{proj}_{\mathbf{u}}\mathbf{v}$ is in the direction opposite $\mathbf{u}$.

We will also want to know how to find projections algebraically:

<u>Definition 3.4.3</u>: **The Projection of One Vector on Another**

For two vectors $\mathbf{u}$ and $\mathbf{v}$, the vector $\text{proj}_{\mathbf{v}}\mathbf{u}$ is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}$$

Note that since both $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{v}\cdot\mathbf{v}$ are scalars, so is $\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}$. That scalar is then multiplied times $\mathbf{v}$, resulting in a vector parallel $\mathbf{v}$. if the scalar is positive the projection is in the direction of $\mathbf{v}$, as shown in Example 4.2(b); when the scalar is negative the projection is in the direction opposite the vector being projected onto, as shown in Example 4.3(c).

◇ **Example 3.4(d):** For the vectors $\mathbf{u} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$, find $\text{proj}_{\mathbf{u}}\mathbf{v}$.

Note that here we are projecting $\mathbf{v}$ onto $\mathbf{u}$. first we find

$$\mathbf{v}\cdot\mathbf{u} = (2)(5) + (-1)(-1) + (-3)(2) = 5 \qquad \text{and} \qquad \mathbf{u}\cdot\mathbf{u} = 5^2 + (-1)^2 + 2^2 = 30$$

Then

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{5}{30}\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ -\frac{1}{6} \\ \frac{1}{3} \end{bmatrix}$$

As stated before, the idea of projection is extremely important in mathematics, and arises in situations that do not appear to have anything to do with geometry and vectors as we are thinking of them now. You will see a clever geometric use of vectors in one of the exercises.

Section 3.4 Exercises

1. Consider the vectors $\mathbf{v} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

   (a) Draw a *neat and accurate* graph of $\mathbf{v}$ and $\mathbf{b}$, with their tails at the origin, labeling each.

   (b) Use the formula to find $\text{proj}_{\mathbf{b}}\mathbf{v}$, with its components rounded to the nearest tenth.

   (c) Add $\text{proj}_{\mathbf{b}}\mathbf{v}$ to your graph. Does it look correct?

2. For each pair of vectors $\mathbf{v}$ and $\mathbf{b}$ below, do each of the following

   i) Sketch $\mathbf{v}$ and $\mathbf{b}$ with the same initial point.

   ii) Find $\text{proj}_{\mathbf{b}}\mathbf{v}$ algebraically, using the formula for projections.

   iii) On the same diagram, sketch the $\text{proj}_{\mathbf{b}}\mathbf{v}$ you obtained in part (ii). If it does not look the way it should, find your error.

   iv) Find $\text{proj}_{\mathbf{b}}\mathbf{v}$, and sketch it as a new sketch. Compare with your previous sketch.

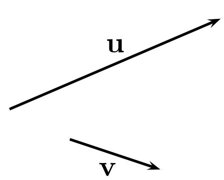
   (a) $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

   (b) $\mathbf{v} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

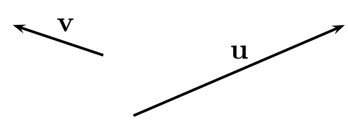
   (c) $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$

3. For each pair of vectors $\mathbf{u}$ and $\mathbf{v}$, sketch $\text{proj}_{\mathbf{v}}\mathbf{u}$. Indicate any right angles with the standard symbol.

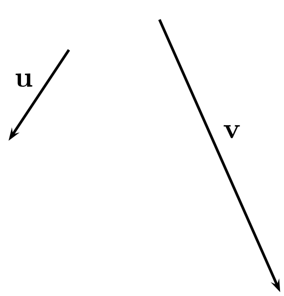
   (a)



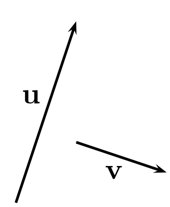
   (b)



   (c)



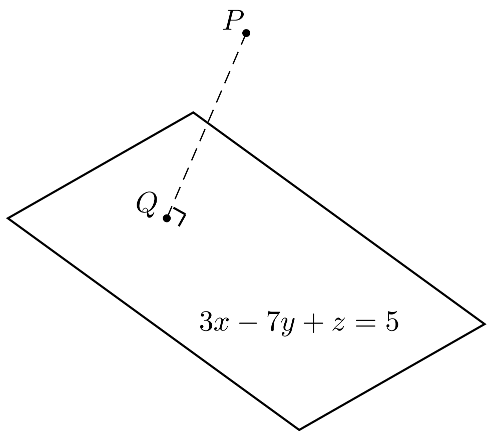
   (d)

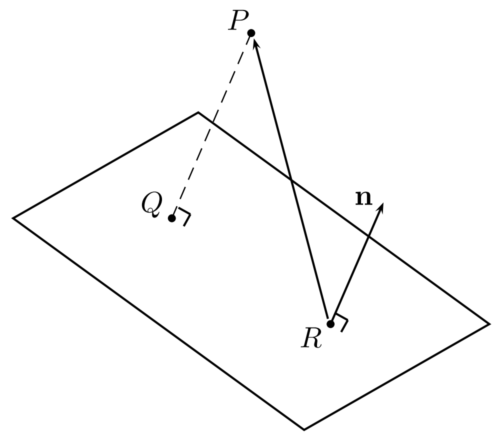


($\mathbf{u}$ and $\mathbf{v}$ are orthogonal)

3.5 Chapter 3 Exercises

1. The purpose of this exercise is to find the distance from the point $P(1,-1,2)$ to the plane with equation $3x-7y+z=5$. Of course this might seem ambiguous, since the distance depends on where on the plane we are talking about. In order to eliminate this uncertainty, we say that *the distance from a point to a plane is DEFINED to be the smallest of the distances between the point and all point on the plane.* It should be intuitively clear to you that the point Q on the plane where this smallest distance occurs is the one where a line though the point and perpendicular to the plane intersects the plane. See the picture below and to the left.





We will solve this problem using a projection. The key idea can be seen in the picture above and to the right. We find *any* point R on the plane and construct the vector $\overrightarrow{RP}$. If we then project $\overrightarrow{RP}$ onto the normal vector $\mathbf{n}$ (recall from Math 254N that the normal vector is $\mathbf{n}=[3,-7,1]$) of the plane, the length of the projection vector will be the same as the distance from P to Q, which is the distance from P to the plane. OK, now do it!

4 Vectors and Systems of Equations

Outcome:

4. Understand the relationship of vectors to systems of equations. Understand the dot product of two vectors and use it to determine whether vectors are orthogonal and to project one vector onto another.

   Performance Criteria:

   (a) Give the linear combination form of a system of equations.

   (b) Sketch a picture illustrating the linear combination form of a system of equations of two equations in two unknowns.

   (c) Give an algebraic description of a set of a set of vectors that has been described geometrically, and vice-versa.

   (d) Determine whether a set of vectors is closed under vector addition; determine whether a set of vectors is closed under scalar multiplication. If it is, prove that it is; if it is not, give a counterexample.

   (e) Give the vector equation of a line through two points in $\mathbb{R}^2$ or $\mathbb{R}^3$ or the vector equation of a plane through three points in $\mathbb{R}^3$.

   (f) Write the solution to a system of equations in vector form and determine the geometric nature of the solution.

In the first two chapters of this book we began our study of linear algebra with solving systems of linear equations; the notions and methods presented there were purely algebraic in nature. In the previous chapter we introduced some fundamentals of vectors. Now we will use vectors to look at systems of equations and their solutions from a geometric perspective.

4.1 Linear Combination Form of a System

Performance Criterion:

4. (a) Give the linear combination form of a system of equations.
   (b) Sketch a picture illustrating the linear combination form of a system of equations of two equations in two unknowns.

It should be clear that two vectors are equal if and only if their corresponding components are equal. That is,

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad \text{if, and only if,} \qquad \begin{array}{c} u_1 = v_1, \\ u_2 = v_2, \\ \vdots \\ u_n = v_n \end{array}$$

The words "if, and only if" mean that the above works "both ways" in the following sense:

- If we have two vectors of length n that are equal, then their corresponding entries are all equal, resulting in n equations.
- If we have a set of n equations, we can create a two vectors, one of whose components are all the left hand sides of the equations and the other whose components are all the right hand sides of the equations, and the two vectors created this way are equal.

Using the second bullet above, we can take the system of equations below and to the left and turn them into the single vector equation shown below and to the right:

$$\begin{array}{rcr} x_1 + 3x_2 - 2x_3 & = & -4 \\ 3x_1 + 7x_2 + \ x_3 & = & 4 \\ -2x_1 + \ x_2 + 7x_3 & = & 7 \end{array} \qquad \Longrightarrow \qquad \begin{bmatrix} x_1 + 3x_2 - 2x_3 \\ 3x_1 + 7x_2 + \ x_3 \\ -2x_1 + \ x_2 + 7x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix}$$

Considering the vector equation above and to the right, we can take the vector on the left side of the equation and break it into three vectors to get

$$\begin{bmatrix} x_1 \\ 3x_1 \\ -2x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 7x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} -2x_3 \\ x_3 \\ 7x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix}$$

and then we can factor the scalar unknown out of each vector to get the equation

$$x_1 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix}$$

Definition 4.1.1 **Linear Combination Form of a System**

A system of m linear equations in n unknowns can be written as a linear combination as follows:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

can be written as a linear combination of vectors equalling another vector:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

We will refer to this as the **linear combination form of the system of equations**.

Thus the system of equations below and to the left can be rewritten in the linear combination form shown below and to the right.

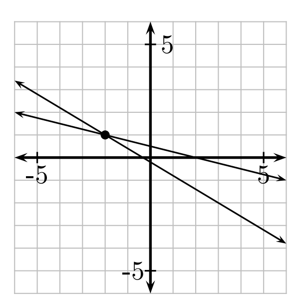
$$\begin{aligned} x_1 + 3x_2 - 2x_3 &= -4 \\ 3x_1 + 7x_2 + x_3 &= 4 \\ -2x_1 + x_2 + 7x_3 &= 7 \end{aligned} \qquad x_1 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix}$$

The question we originally asked for the system of linear equations was "Are there numbers x_1, x_2 and x_3 that make all three equations true?" Now we can see this is equivalent to a different question, "Is there a linear combination of the vectors $[1, 3, -2]$, $[3, 7, 1]$ and $[-2, 1, 7]$ that equals the vector $[-4, 4, 7]$?"

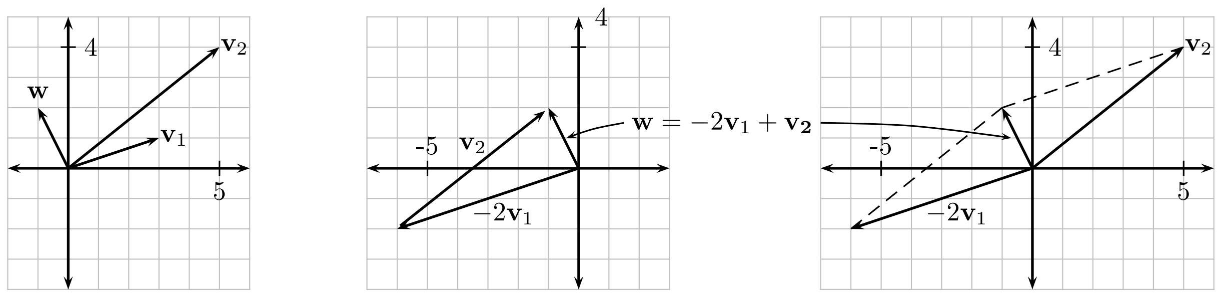
◇ **Example 4.1(a):** Give the linear combination form of the system $\begin{aligned} 3x_1 + 5x_2 &= -1 \\ x_1 + 4x_2 &= 2 \end{aligned}$ of linear equations.

The linear combination form of the system is $x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Let's consider the system from this last example a bit more. The goal is to solve the system of equations $\begin{aligned} 3x_1 + 5x_2 &= -1 \\ x_1 + 4x_2 &= 2 \end{aligned}$. In the past our geometric interpretation has been this: The set of solutions to the first equation is a line in $\mathbb{R}^2$, and the set of solutions to the second equation is another line. The solution to the system happens to be $x_1 = -2$, $x_2 = 1$, and the point $(-2, 1)$ in $\mathbb{R}^2$ is the point where the two lines cross. This is shown in the picture to the right.



Now consider the linear combination form $x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ of the system. Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. These vectors are shown in the diagram below and to the left. The solution $x_1 = -2,\ x_2 = 1$ to the system is the scalars that we can use for a linear combination of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ to get the vector $\mathbf{w}$. This is shown in the middle diagram below by the tip-to-tail method, and in the diagram below and to the right by the parallelogram method.



tip-to-tail method

parallelogram method

Section 4.1 Exercises

1. Give the linear combination form of each system:

(a) $$\begin{array}{rcr} x + y - 3z & = & 1 \\ -3x + 2y - z & = & 7 \\ 2x + y - 4z & = & 0 \end{array}$$

(b) $$\begin{array}{rcr} 5x_1 + x_3 & = & -1 \\ 2x_2 + 3x_3 & = & 0 \\ 2x_1 + x_2 - 4x_3 & = & 2 \end{array}$$

2. The system of equations $\begin{array}{rcr} 2x - 3y & = & -6 \\ 3x - y & = & 5 \end{array}$ has solution $x = 3,\ y = 4$. Write the system in linear combination form, then replace x and y with their values. Finally, sketch a picture illustrating the resulting vector equation. See the explanation after Example 4.1(a) if you have no idea what I am talking about.

4.2 Sets of Vectors

Performance Criterion:

4. (c) Give an algebraic description of a set of a set of vectors that has been described geometrically, and vice-versa.

   (d) Determine whether a set of vectors is closed under vector addition; determine whether a set of vectors is closed under scalar multiplication. If it is, prove that it is; if it is not, give a counterexample.

One of the most fundamental concepts of mathematics is that of **sets**, collections of objects called **elements**. You have probably encountered various sets of numbers, like the **whole numbers**

$$\{0, 1, 2, 3, ...\}$$

and the integers

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}.$$

As shown above, when we describe a set by listing all or some of its elements, we enclose them with "curly brackets." Sets are usually named by upper case letters. The above two sets are **infinite sets**.

Another kind of infinite set is an interval of the number line, like all real numbers between 1 and 5, including 1 and 5. You have likely seen the interval notation $[1, 5]$ for such sets. This set is also infinite, but in a different sense than the whole numbers and integers. That difference is not of concern to us here, but some of you may encounter that idea again. Of course, there are also finite sets like $\{2, 4, 6, 8\}$.

There will be a time soon when we will be very interested in sets of vectors, both finite and infinite. For example we might be interested in the finite set

$$A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\},$$

or the infinite set of all vectors of the form $\begin{bmatrix} a \\ 2a \end{bmatrix}$, where a is any real number. Let's examine this set a bit more.

◇ **Example 4.2(a):** Let B be the set of all vectors of the form $\begin{bmatrix} a \\ 2a \end{bmatrix}$, where a is any real number. Are the vectors $\mathbf{u} = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$ in B?

Because $2(3) \neq 10$, $\mathbf{u}$ is not in B. But $2(-2) = -4$, so $\mathbf{v}$ is in B.

◇ **Example 4.2(b):** Let C be the set of all vectors of the form $\begin{bmatrix} a \\ a+1 \end{bmatrix}$, where a is any real number. Are the vectors $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ in C?

$3 + 1 = 4$ and $-2 + 1 = -1$, so both $\mathbf{u}$ and $\mathbf{v}$ are in C.

In the future we will also be very interested in this question: Given an infinite set of vectors, is the sum of any two vectors a vector that is also in the set? When faced with such a question we should do one of two things:

- Give two *specific* vectors that are in the set, and show that their sum is not.
- Give two *arbitray* (general) vectors in the set and show that their sum is also in the set.

The following Examples illustrate these.

◇ **Example 4.2(c):** Let B be the set of all vectors of the form $\begin{bmatrix} a \\ 2a \end{bmatrix}$, where a is any real number. Determine whether the sum of any two vectors in B is also in B.

The vectors $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -4 \end{bmatrix}$ are both in B, and their sum $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is as well. We may have just gotten lucky, though, and maybe the sum of *any* two vectors in B is not necessarily in B. Let's see if the sum of two *arbitrary* vectors in B is in B:

$$\begin{bmatrix} a \\ 2a \end{bmatrix} + \begin{bmatrix} b \\ 2b \end{bmatrix} = \begin{bmatrix} a+b \\ 2a+2b \end{bmatrix} = \begin{bmatrix} a+b \\ 2(a+b) \end{bmatrix} = \begin{bmatrix} c \\ 2c \end{bmatrix},$$

where $c = a + b$. Therefore the sum of *any* two vectors in B is a vector in B.

◇ **Example 4.2(d):** Let C be the set of all vectors of the form $\begin{bmatrix} a \\ a+1 \end{bmatrix}$, where a is any real number. Is the sum of any two vectors in C also a vector in C?

The vectors $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$ are both in C, as shown in Example 4.2(b). Their sum is the vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, which is not in C because $1 + 1 \neq 3$. Thus the sum of any two vectors in C is not necessarily another vector in C.

◇ **Example 4.2(e):** Let D be the set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x \geq 0$ and $y \geq 0$. Is the sum of any two vectors in D also a vector in D?

Suppose that $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ are both in D, so all of x_1, y_1, x_2, y_2 are greater than or equal to zero. Clearly $x_1 + x_2 \geq 0$ and $y_1 + y_2 \geq 0$, so their sum $\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$ is in D.

Given a set of vectors, we are also interested in whether a scalar multiple of a vector in the set is also in the set. In the next example we determine whether that is the case for the set D from the previous example.

4.3 Vector Equations of Lines and Planes

Performance Criterion:

4. (e) Give the vector equation of a line through two points in $\mathbb{R}^2$ or $\mathbb{R}^3$ or the vector equation of a plane through three points in $\mathbb{R}^3$.

The idea of a linear combination does more for us than just give another way to interpret a system of equations. The set of points in $\mathbb{R}^2$ satisfying an equation of the form $y = mx + b$ is a line; any such equation can be rearranged into the form $ax + by = c$. (The values of b in the two equations are clearly not the same.) But if we add one more term to get $ax + by + cz = d$, with the (x, y, z) representing the coordinates of a point in $\mathbb{R}^3$, *we get the equation of a plane, not a line!* In fact, we cannot represent a line in $\mathbb{R}^3$ with a single *scalar* equation. The object of this section is to show how we can represent lines, planes and higher dimensional objects called *hyperplanes* using linear combinations of vectors.

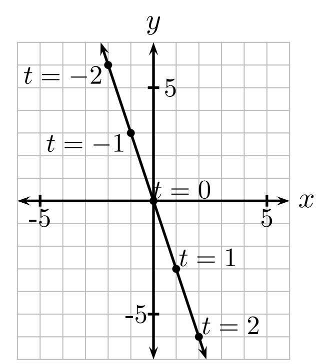
For the bulk of this course, we will think of most vectors as position vectors. (Remember, this means their tails are at the origin.) We will also think of each position vector as corresponding to the point at its tip, so the coordinates of the point will be the same as the components of the vector. Thus, for example, in $\mathbb{R}^2$ the vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ corresponds to the ordered pair $(x_1, x_2) = (1, -3)$.

◇ **Example 4.3(a):** Graph the set of points corresponding to all vectors $\mathbf{x}$ of the form $\mathbf{x} = t\begin{bmatrix} 1 \\ -3 \end{bmatrix}$, where t represents any real number.

We already know that when $t = 1$ the the vector x corresponds to the point $(1, -3)$. We then let $t = -2, -1, 0, 2$ and determine the corresponding vectors $\mathbf{x}$:

$$t = -2 \Rightarrow x = \begin{bmatrix} -2 \\ 6 \end{bmatrix}, \qquad t = -1 \Rightarrow x = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$t = 0 \Rightarrow x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad t = 2 \Rightarrow x = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$



These vectors correspond to the points with ordered pairs $(-2, 6)$, $(-1, 3)$, $(0, 0)$ and $(2, -6)$. When we plot those points and the point $(1, -3)$ that we already had, we get the line shown above and to the right.

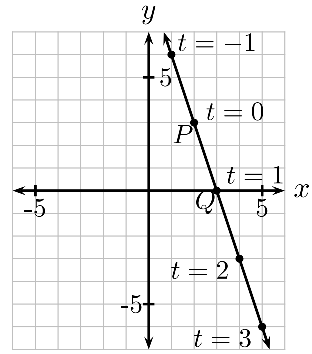
It should be clear from the above example that we could create a line through the origin in any direction by simply replacing the vector $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ with a vector in the direction of the desired line. The next question is, "how do we get a line that is *not* through the origin?" The next example illustrates how this is done.

◇ **Example 4.3(b):** Graph the set of points corresponding to all vectors $\mathbf{x}$ of the form $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, where t represents any real number.

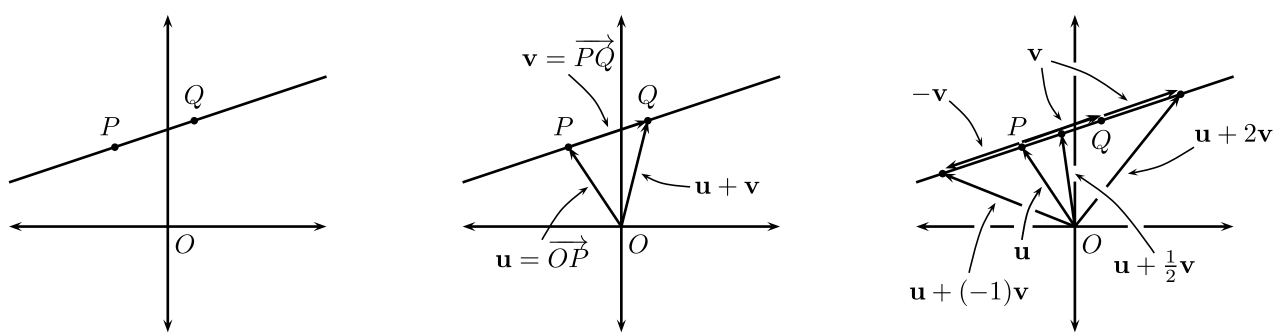
Performing the scalar multiplication by t and adding the two vectors, we get

$$\mathbf{x} = \begin{bmatrix} 2 - 3t \\ 3 + \ t \end{bmatrix}.$$

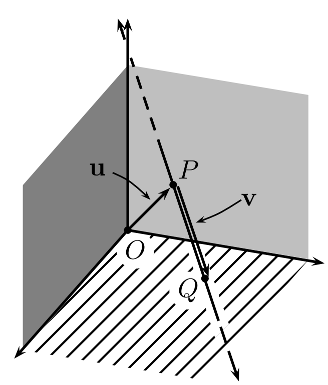
These vectors then correspond to all points of the form $(2 - 3t, 3 + t)$. When $t = 0$ this is the point $(2, 3)$ so our line clearly passes through that point. Plotting the points obtained when we let $t = -1,\ 1,\ 2$ and 3, we see that we will get the line shown to the right.



Now let's make two observations about the set of points represented by the set of all vectors $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, where t again represents any real number. These vectors correspond to the ordered pairs of the form $(4 - 3t, -2 + t)$. Plotting these results in the line through the point $(2, 3)$ and in the direction of the vector $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$. This is not a coincidence. Consider the line shown below and to the left, containing the points P and Q. If we let $\mathbf{u} = \overrightarrow{OP}$ and $\mathbf{v} = \overrightarrow{PQ}$, then the points P and Q correspond to the vectors $\mathbf{u}$ and $\mathbf{u} + \mathbf{v}$ (in standard position, which you should assume we mean from here on), as shown in the second picture. From this you should be able to see that if we consider all the vectors $\mathbf{x}$ defined by $\mathbf{x} = \mathbf{u} + t\mathbf{v}$ as t ranges over all real numbers, the resulting set of points is our line! This is shown in the third picture, where t is given the values -1, $\frac{1}{2}$ and 2.



Now this may seem like an overly complicated way to describe a line, but with a little thought you should see that the idea translates directly to three (and more!) dimensions, as shown in the picture to the right. This is all summarized below:



Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$

The **vector equation of a line** through two points P and Q in $\mathbb{R}^2$ and $\mathbb{R}^3$ (and even higher dimensions) is

$$\mathbf{x} = \overrightarrow{OP} + t\,\overrightarrow{PQ}.$$

By this we mean that the line consists of all the points corresponding to the position vectors $\mathbf{x}$ as t varies over all real numbers. The vector $\overrightarrow{PQ}$ is called the **direction vector** of the line.

◇ **Example 4.3(c):** Give the vector equation of the line in $\mathbb{R}^2$ through the points $P(-4,1)$ and $Q(5,3)$.

We need two vectors, one from the origin out to the line, and one in the direction of the line. For the first we will use $\overrightarrow{OP}$, and for the second we will use $\overrightarrow{PQ} = [9,2]$. We then have

$$\mathbf{x} = \overrightarrow{OP} + t\overrightarrow{PQ} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} + t\begin{bmatrix} 9 \\ 2 \end{bmatrix},$$

where $\mathbf{x} = [x_1, x_2]$ is the position vector corresponding to any point (x_1, x_2) on the line.

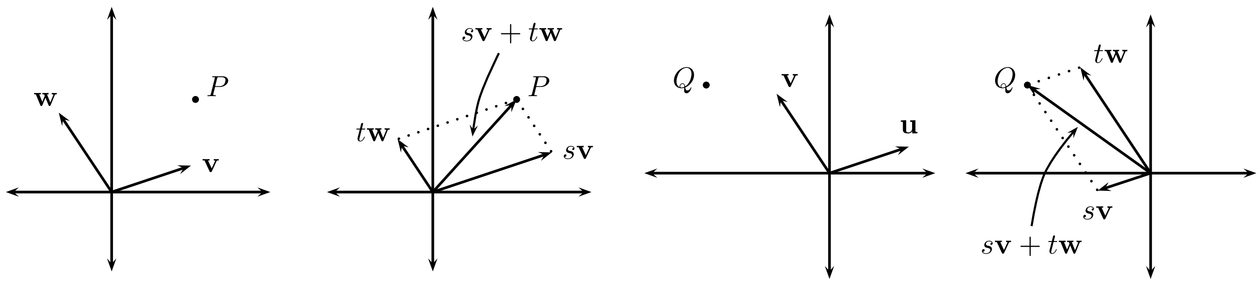
◇ **Example 4.3(d):** Give a vector equation of the line in $\mathbb{R}^3$ through the points $(-5,1,2)$ and $(4,6,-3)$.

Letting P be the point $(-5,1,2)$ and Q be the point $(4,6,-3)$, we get $\overrightarrow{PQ} = \langle 9,5,-5\rangle$. The vector equation of the line is then

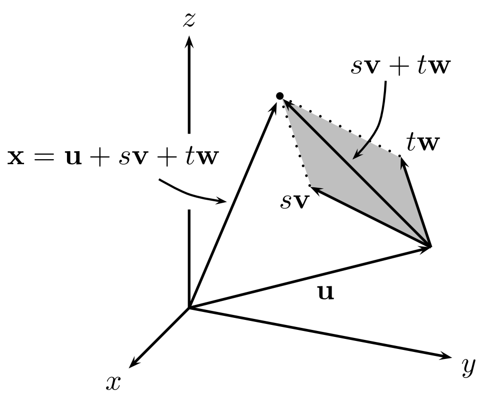
$$\mathbf{x} = \overrightarrow{OP} + t\overrightarrow{PQ} = \begin{bmatrix} -5 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 9 \\ 5 \\ -5 \end{bmatrix},$$

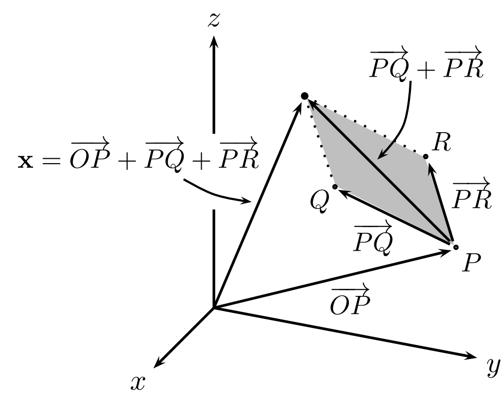
where $\mathbf{x} = [x_1, x_2, x_3]$ is the position vector corresponding to any point (x_1, x_2, x_3) on the line. The first vector can be any point on the line, so it could be the vector $[4,6,-3]$ instead of $[-5,1,2]$, and the second vector is a direction vector, so can be any scalar multiple of $\mathbf{d} = [9,5,-5]$.

The same general idea can be used to describe a plane in $\mathbb{R}^3$. Before seeing how that works, let's define something and look at a situation in $\mathbb{R}^2$. We say that two vectors are **parallel** if one is a scalar multiple of the other. Now suppose that $\mathbf{v}$ and $\mathbf{w}$ are two vectors in $\mathbb{R}^2$ that are *not* parallel (and neither is the zero vector either), as shown in the picture to the left below, and let P be the randomly chosen point in $\mathbb{R}^2$ shown in the same picture. The next picture shows that a linear combination of $\mathbf{v}$ and $\mathbf{w}$ can be formed that gives us a vector $s\mathbf{v} + t\mathbf{w}$ corresponding to the point P. In this case the scalar s is positive and less than one, and t is positive and greater than one. The third and fourth pictures show the same thing for another point Q, with both s being negative and t positive in that case. It should now be clear that *any* point in $\mathbb{R}^2$ can be obtained in this manner.



Now let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be three vectors in $\mathbb{R}^3$, and consider the vector $\mathbf{x} = \mathbf{u} + s\mathbf{v} + t\mathbf{w}$, where s and t are scalars that are allowed to take all real numbers as values. The vectors $s\mathbf{v} + t\mathbf{w}$ all lie in the plane containing $\mathbf{v}$ and $\mathbf{w}$. Adding $\mathbf{u}$ "moves the plane off the origin" to where it passes through the tip of $\mathbf{u}$ (again, in standard position). This is probably best visualized by thinking of adding $s\mathbf{v}$ and $t\mathbf{w}$ with the parallelogram method, then adding the result to $\mathbf{u}$ with the tip-to-tail method. I have attempted to illustrate this below and to the left , with the gray parallelogram being part of the plane created by all the points corresponding to the vectors $\mathbf{x}$.





The same diagram above and to the right shows how all of the previous discussion relates to the plane through three points P, Q and R in $\mathbb{R}^3$. This leads us to the description of a plane in $\mathbb{R}^3$ given at the top of the next page.

Planes in $\mathbb{R}^3$

The **vector equation of a plane** through three points P, Q and R in $\mathbb{R}^3$ (or higher dimensions) is

$$\mathbf{x} = \overrightarrow{OP} + s\,\overrightarrow{PQ} + t\,\overrightarrow{PR}.$$

By this we mean that the plane consists of all the points corresponding to the position vectors $\mathbf{x}$ as s and t vary over all real numbers.

◇ **Example 4.3(e):** Give a vector equation of the plane in $\mathbb{R}^3$ through the points $(2,-1,3)$, $(-5,1,2)$ and $(4,6,-3)$. What values of s and t give the point R?

Letting P be the point $(2,-1,3)$, Q be $(-5,1,2)$ and R be $(4,6,-3)$, we get $\overrightarrow{PQ} =$

$[-7, 2, -1]$ and $\overrightarrow{PR} = [2, 7, -6]$. The vector equation of the plane is then

$$\mathbf{x} = \overrightarrow{OP} + s\overrightarrow{PQ} + t\overrightarrow{PR} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + s \begin{bmatrix} -7 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 7 \\ -6 \end{bmatrix},$$

where $\mathbf{x} = [x_1, x_2, x_3]$ is the position vector corresponding to any point (x_1, x_2, x_3) on the plane. It should be clear that there are other possibilities for this. The first vector in the equation could be any of the three position vectors for P, Q or R. The other two vectors could be any two vectors from one of the points to another.

The vector corresponding to point R is $\overrightarrow{OR}$, which is equal to $\mathbf{x} = \overrightarrow{OP} + \overrightarrow{PR}$ (think about that), so $s = 0$ and $t = 1$.

We now summarize all of the ideas from this section.

Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$, Planes in $\mathbb{R}^3$

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ with $\mathbf{v} \neq \mathbf{0}$. Then the set of points corresponding to the vector $\mathbf{x} = \mathbf{u} + t\mathbf{v}$ as t ranges over all real numbers is a line through the point corresponding to $\mathbf{u}$ and in the direction of $\mathbf{v}$. (So if $\mathbf{u} = 0$ the line passes through the origin.)

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors $\mathbb{R}^3$, with $\mathbf{v}$ and $\mathbf{w}$ being nonzero and not parallel. (That is, not scalar multiples of each other.) Then the set of points corresponding to the vector $\mathbf{x} = \mathbf{u} + s\mathbf{v} + t\mathbf{w}$ as s and t range over all real numbers is a plane through the point corresponding to $\mathbf{u}$ and containing the vectors $\mathbf{v}$ and $\mathbf{w}$. (If $\mathbf{u} = 0$ the plane passes through the origin.)

Section 4.3 Exercises

1. For each of the following, give the vector equation of the line or plane described.

   (a) The line through the two points $P(3, -1, 4)$ and $Q(2, 6, 0)$ in $\mathbb{R}^3$.

   (b) The plane through the points $P(3, -1, 4)$, $Q(2, 6, 0)$ and $R(-1, 0, 3)$ in $\mathbb{R}^3$.

   (c) The line through the points $P(3, -1)$ and $Q(6, 0)$ in $\mathbb{R}^2$.

2. Find another point in the plane containing $P_1(-2, 1, 5)$, $P_2(3, 2, 1)$ and $P_3(4, -2, -3)$. **Show clearly how you do it**. (**Hint:** Find and use the vector equation of the plane.)

3. "Usually" a vector equation of the form $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$ gives the equation of a plane in $\mathbf{R}^3$.

   (a) Under what conditions on $\mathbf{p}$ and/or $\mathbf{u}$ and/or $\mathbf{v}$ would this be the equation of a line?

   (b) Under what conditions on $\mathbf{p}$ and/or $\mathbf{u}$ and/or $\mathbf{v}$ would this be the equation of a plane through the origin?

4.4 Interpreting Solutions to Systems of Linear Equations

Performance Criterion:

4. (f) Write the solution to a system of equations in vector form and determine the geometric nature of the solution.

We begin this section by considering the following two systems of equations.

$$\begin{aligned} 3x_1 - 3x_2 + 3x_3 &= 9 \\ 2x_1 - x_2 + 4x_3 &= 7 \\ 3x_1 - 5x_2 - x_3 &= -3 \end{aligned} \qquad \begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ -3x_1 + 3x_2 - 6x_3 &= -3 \\ 2x_1 - 2x_2 + 4x_3 &= 2 \end{aligned}$$

The augmented matrices for these two systems reduce to the following matrices, respectively.

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let's look at the first system. x_3 is a free variable, and x_1 and x_2 are leading variables. The general solution is $x_1 = -3t + 4$, $x_2 = -2t + 1$, $x_3 = t$. Algebraically, x_1, x_2 and x_3 are just numbers, but we can think of (x_1, x_2, x_3) as a point in $\mathbb{R}^3$. The corresponding position vector is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4-3t \\ 1-2t \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

We will call this the **vector form of the solution** to the system of equations. The beauty of expressing the solutions to a system of equations in vector form is that we can see what the set of all solutions looks like. In this case, the set of solutions is the set of all points in $\mathbb{R}^3$ on the line through $(4, 1, 0)$ and with direction vector $[-3, -2, 1]$.

◇ **Example 4.4(a):** The general solution to the second system of equations is $x_1 = 1 + s - 2t$, $x_2 = s$, $x_3 = t$. Express the solution on vector form and determine the geometric nature of the solution set in $\mathbb{R}^3$.

A process like the one just carried out leads to the general solution with position vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

(Check to make sure that you understand how this was arrived at.) Here the set of solutions is the set of all points in $\mathbb{R}^3$ on the plane through $(1, 0, 0)$ with direction vectors $[1, 1, 0]$ and $[-2, 0, 1]$.

Now recall that the three equations from this last example,

$$\begin{aligned} x_1 - \ \ x_2 + 2x_3 &= \ \ 1 \\ -3x_1 + 3x_2 - 6x_3 &= -3 \\ 2x_1 - 2x_2 + 4x_3 &= \ \ 2 \end{aligned}$$

represent three planes in $\mathbb{R}^3$, and when we solve the system we are looking for all points in $\mathbb{R}^3$ that are solutions to all three equations. Our results tell us that the set of solution points in this case is itself a plane, which can only happen if all three equations represent the same plane. If you look at them carefully you can see that the second and third equations are multiples of the first, so the points satisfying them also satisfy the first equation.

◇ **Example 4.4(b):** The general solution to the second system of equations is $x_1 = 1+s-2t$, $x_2 = s$, $x_3 = t$. Express the solution in vector form and determine the geometric nature of the solution set in $\mathbb{R}^3$.

A process like the one just carried out leads to the general solution with position vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

(Check to make sure that you understand how this was arrived at.) Here the set of solutions is the set of all points in $\mathbb{R}^3$ on the plane through $(1,0,0)$ with direction vectors $[1,1,0]$ and $[-2,0,1]$.

◇ **Example 4.4(c):** Give the vector form of the solution to the system

$$\begin{aligned} 3x_2 - \ \ 6x_3 - 4x_4 - 3x_5 &= -5 \\ x_1 - 3x_2 + 10x_3 + 4x_4 + 4x_5 &= \ \ 2 \\ 2x_1 - 6x_2 + 20x_3 + 2x_4 + 8x_5 &= -8 \end{aligned}$$

The augmented matrix of the system reduces to

$$\begin{bmatrix} 1 & 0 & 4 & 0 & 1 & -3 \\ 0 & 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{bmatrix}$$

We can see that x_3 and x_5 are free variables, and we can also see that $x_4 = 2$. Letting $x_5 = t$ and $x_3 = s$, $x_2 = 1 + 2s + t$ and $x_1 = -3 - 4s - t$. Therefore

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

How do we interpret this result geometrically? The set of points $(x_1, x_2, x_3, x_4, x_5)$ represents a two-dimensional plane in five-dimensional space. We could also have ended up with one, three or four dimensional "plane", often called a **hyperplane**, in five-dimensional space.

Section 4.4 Exercises

1. For each of the following, a student correctly finds the given the general solution (x_1, x_2, x_3) to a system of three equations in three unknowns. Give the vector form of the solution, then tell whether the set of all particular solutions is a point, line or plane.

   (a) $x_1 = s - t + 5, \quad x_2 = s, \quad x_3 = t$

   (b) $x_1 = 2t + 5, \quad x_2 = t, \quad x_3 = -1$

   (c) $x_1 = s - 2t + 5, \quad x_2 = s, \quad x_3 = t$

4.5 Chapter 4 Exercises

1. Consider the plane in $\mathbb{R}^3$ with vector equation $\mathbf{x} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 1 \\ 7 \\ -4 \end{bmatrix} + t \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix}$.

   The point $(18.9, 50.4, -28.1)$ lies on this plane - find the values of s and t that give this point. You may wish to use the following hints.

   - Where do the values 18.9, 50.4 and -28.1 go?
   - How many unknowns are you solving for? Based on this, how many columns should your augmented matrix have?

5 Matrices and Vectors

Outcome:

5. Understand matrices, their algebra, and their action on vectors. Use matrices to solve problems.

   Performance Criteria:

   (a) Give the dimensions of a matrix. Identify a given entry, row or column of a matrix.

   (b) Identify matrices as square, upper triangular, lower triangular, symmetric, diagonal. Give the transpose of a given matrix; know the notation for the transpose of a matrix.

   (c) Know when two matrices can be added or subtracted; add or subtract two matrices when possible.

   (d) Multiply a matrix times a vector.

   (e) Give the identity matrix (for a given dimensional space) and its effect when a vector is multiplied by it.

   (f) Determine whether a matrix is a projection matrix, rotation matrix, or neither, by its action on a few vectors.

   (g) Plot a discrete function. Give derivative and integral matrices, filter matrices.

5.1 Introduction

Performance Criteria:

5. (a) Give the dimensions of a matrix. Identify a given entry, row or column of a matrix.
   (b) Identify matrices as square, upper triangular, lower triangular, symmetric, diagonal. Give the transpose of a given matrix; know the notation for the transpose of a matrix.
   (c) Know when two matrices can be added or subtracted; add or subtract two matrices when possible.

A **matrix** is simply an array of numbers arranged in rows and columns. Here are some examples:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} -5 & 1 \\ 0 & 4 \\ 2 & -3 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & -2 & 1 \end{bmatrix}$$

We will always denote matrices with italicized capital letters. There should be no need to define the **rows** and **columns** of a matrix. The number of rows and number of columns of a matrix are called its **dimensions**. The second matrix above, B, has dimensions 3×2, which we read as "three by two." The numbers in a matrix are called its **entries**. Each entry of a matrix is identified by its row, then column. For example, the $(3, 2)$ entry of L is the entry in the 3rd row and second column, -2. In general, we will define the (i, j)th entry of a matrix to be the entry in the ith row and jth column.

There are a few special kinds of matrices that we will run into regularly:

- A matrix with the same number of rows and columns is called a **square matrix**. Matrices A, D and L above are square matrices. The entries that are in the same number row and column of a square matrix are called the **diagonal entries** of the matrix. For example, the diagonal entries of A are 1 and 3.

- A square matrix with zeros "above" the diagonal is called a **lower triangular matrix**; L is an example of a lower triangular matrix. Similarly, an **upper triangular matrix** is one whose entries below the diagonal are all zeros. (Note that the words "lower" and "upper" refer to the triangular parts of the matrices where the entries are *NOT* zero.)

- A square matrix all of whose entries above *AND* below the diagonal are zero is called a **diagonal matrix**. D is an example of a diagonal matrix.

- A diagonal matrix with only ones on the diagonal is called "the" **identity matrix**. We use the word "the" because in a given size there is only one identity matrix. We will soon see why it is called the "identity."

- Given a matrix, we can "flip the matrix over its diagonal," so that the rows of the original matrix become the columns of the new matrix, and vice versa. The new matrix is called the

transpose of the original. The transposes of the matrices B and L above are denoted by B^T and L^T. They are the matrices

$$B^T = \begin{bmatrix} -5 & 0 & 2 \\ 1 & 4 & -3 \end{bmatrix} \qquad L^T = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

- Notice that $A^T = A$. Such a matrix is called a **symmetric matrix**. One way of thinking of such a matrix is that the entries across the diagonal from each other are equal. Matrix D is also symmetric, as is the matrix

$$\begin{bmatrix} 1 & 5 & 0 & -2 \\ 5 & -4 & 7 & 3 \\ 0 & 7 & 0 & -6 \\ -2 & 3 & -6 & -3 \end{bmatrix}$$

When discussing an arbitrary matrix A with dimensions $m \times n$ we refer to each entry as a, but with a double subscript with each to indicate its position in the matrix. The first number in the subscript indicates the row of the entry and the second indicates the column of that entry:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2k} & \cdots & a_{2n} \\ \vdots & \vdots & & & \vdots & & \vdots \\ a_{j1} & a_{j2} & a_{j3} & \cdots & a_{jk} & \cdots & a_{jn} \\ \vdots & \vdots & & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mk} & \cdots & a_{mn} \end{bmatrix}$$

Under the right conditions it is possible to add, subtract and multiply two matrices. We'll save multiplication for a little, but we have the following:

DEFINITION 5.1.1: **Adding and Subtracting Matrices**

When two matrices have the same dimensions, they are added or subtracted by adding or subtracting the corresponding entries.

◇ **Example 5.1(a):** Determine which of the matrices below can be added, and add those that can be.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} -5 & 1 \\ 0 & 4 \\ 2 & -3 \end{bmatrix}, \qquad C = \begin{bmatrix} -7 & 4 \\ 1 & 5 \end{bmatrix}$$

B cannot be added to either A or C, but

$$A + C = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} -7 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 3 & 8 \end{bmatrix}$$

It should be clear that A and C could be subtracted, and that $A+C = C+A$ but $A-C \neq C-A$.

Section 5.1 Exercises

1. (a) Give the dimensions of matrices A, B and C in Exercise 3 below.

   (b) Give the entries b_{13} and c_{32} of the matrices B and C in Exercise 3 below.

2. Give examples of each of the following types of matrices.

   (a) lower triangular

   (b) diagonal

   (c) symmetric

   (d) identity

   (e) upper triangular but not diagonal

   (f) symmetric but without any zero entries

   (g) symmetric but not diagonal

   (h) diagonal but not a multiple of an identity

3. Give the transpose of each matrix. Use the correct notation to denote the transpose.

$$A = \begin{bmatrix} 1 & 0 & 5 \\ -3 & 1 & -2 \\ 4 & 7 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 4 & 7 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & -1 & 3 \\ -3 & 1 & 2 & 0 \\ 4 & 7 & 0 & -2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & -3 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}$$

4. Give all possible sums and differences of matrices from Exercise 3.

5.2 Multiplying a Matrix Times a Vector

Performance Criteria:

5. (d) Multiply a matrix times a vector.

   (e) Give the identity matrix (for a given dimensional space) and its effect when a vector is multiplied by it.

In Section 6.1 we will find out how to multiply a matrix times another matrix but, for now we'll multiply only matrices times vectors. This is not to say that doing so is a minor step on the way to learning to multiply matrices; multiplying a matrix times a vector is in some sense *THE* foundational operation of linear algebra.

Before getting into how to do this, we need to devise a useful notation. Consider the matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Each column of A, taken by itself, is a vector. We'll refer to the first column as the vector $\mathbf{a}_{*1}$, with the asterisk $*$ indicating that the row index will range through all values, and the 1 indicating that the values all come out of column one. Of course $\mathbf{a}_{*2}$ denotes the second column, and so on. Similarly, $\mathbf{a}_{1*}$ will denote the first row, $\mathbf{a}_{1*}$ the second row, etc. Technically speaking, the rows are not vectors, but we'll call them **row vectors** and we'll call the columns **column vectors**. If we use just the word *vector*, we will mean a column vector.

◇ **Example 5.2(a):** Give $\mathbf{a}_{2*}$ and $\mathbf{a}_{*3}$ for the matrix $A = \begin{bmatrix} -5 & 3 & 4 & -1 \\ 7 & 5 & 2 & 4 \\ 2 & -1 & -6 & 0 \end{bmatrix}$

$$\mathbf{a}_{2*} = \begin{bmatrix} 7 & 5 & 2 & 4 \end{bmatrix} \qquad \text{and} \qquad \mathbf{a}_{*3} = \begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix}$$

<u>Definition 5.2.1</u>: **Matrix Times a Vector**

An $m \times n$ matrix A can be multiplied times a vector $\mathbf{x}$ with n components. The result is a vector with m components, the ith component being the dot product of the ith row of A with $\mathbf{x}$, as shown below.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ a_{21}x_1 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{1*} \cdot \mathbf{x} \\ \mathbf{a}_{2*} \cdot \mathbf{x} \\ \vdots \\ \mathbf{a}_{m*} \cdot \mathbf{x} \end{bmatrix}$$

◇ **Example 5.2(b):** Multiply $\begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \\ 1 & -6 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -7 \end{bmatrix}$.

$$\begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \\ 1 & -6 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -7 \end{bmatrix} = \begin{bmatrix} (3)(2)+(0)(1)+(-1)(-7) \\ (-5)(2)+(2)(1)+(4)(-7) \\ (1)(2)+(-6)(1)+(0)(-7) \end{bmatrix} = \begin{bmatrix} 13 \\ -36 \\ -4 \end{bmatrix}$$

There is no need for the matrix multiplying a vector to be square, but when it is not, the resulting vector is not the same length as the original vector:

◇ **Example 5.2(c):** Multiply $\begin{bmatrix} 7 & -4 & 2 \\ -1 & 0 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 7 & -4 & 2 \\ -1 & 0 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} (7)(3)+(-4)(-5)+(2)(1) \\ (-1)(3)+(0)(-5)+(6)(1) \end{bmatrix} = \begin{bmatrix} 43 \\ 3 \end{bmatrix}$$

Multiplication of vectors by matrices has the following important properties, which are easily verified.

Theorem 5.2.2

Let A and B be matrices, $\mathbf{x}$ and $\mathbf{y}$ be vectors, and c be any scalar. Assuming that all the indicated operations below are defined (possible), then

(a) $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x}+A\mathbf{y}$ (b) $A(c\mathbf{x}) = c(A\mathbf{x})$

(b) $(A+B)\mathbf{x} = A\mathbf{x}+B\mathbf{x}$

We now come to a very important idea that depends on the first two properties above. When we act on a mathematical object with another object, the object doing the "acting on" is often called an **operator**. Some operators you are familiar with are the derivative operator and the antiderivative operator (indefinite integral), which act on functions to create other functions. Note that the derivative operator has the following two properties, for any functions f and g and real number c:

$$\frac{d}{dx}(f+g) = \frac{df}{dx}+\frac{dg}{dx}, \qquad \frac{d}{dx}(cf) = c\frac{df}{dx}$$

These are the same as the first two properties for multiplication of a vector by a matrix. A matrix can be thought of as an operator that operates on vectors by multiplying them. The first two properties of multiplication of a vector by a matrix, as well as the corresponding properties of the derivative, are called the **linearity properties**. Both the derivative operator and matrix multiplication operator are then called **linear operators**. This is why this subject is called *linear* algebra!

There is another way to compute a matrix times a vector. It is not as efficient to do by hand as what we have been doing so far, but it will be very important conceptually quite soon. Using our earlier definition of a matrix A times a vector $\mathbf{x}$, we see that

$$\begin{aligned} A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ a_{21}x_1 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix} &= \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{m1}x_1 \end{bmatrix} + \begin{bmatrix} a_{21}x_2 \\ a_{22}x_2 \\ \vdots \\ a_{m2}x_2 \end{bmatrix} + \cdots + \begin{bmatrix} a_{1n}x_n \\ a_{2n}x_n \\ \vdots \\ a_{mn}x_n \end{bmatrix} \\ &= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \\ &= x_1\mathbf{a}_{*1} + x_2\mathbf{a}_{*2} + \cdots + x_n\mathbf{a}_{*n} \end{aligned}$$

Let's think about what the above shows. It gives us the result below, which is illustrated in Examples 5.2(d) and (e).

Linear Combination Form of a Matrix Times a Vector

The product of a matrix A and a vector $\mathbf{x}$ is a linear combination of the columns of A, with the scalars being the corresponding components of $\mathbf{x}$.

◇ **Example 5.2(d):** Give the linear combination form of $\begin{bmatrix} 7 & -4 & 2 \\ -1 & 0 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 7 & -4 & 2 \\ -1 & 0 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 7 \\ -1 \end{bmatrix} - 5\begin{bmatrix} -4 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

◇ **Example 5.2(e):** Give the linear combination form of $\begin{bmatrix} 1 & 3 & -2 \\ 3 & 7 & 1 \\ -2 & 1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 3 & -2 \\ 3 & 7 & 1 \\ -2 & 1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$$

Section 5.2 Exercises

1. Multiply $\begin{bmatrix} 1 & 0 & -1 \\ -3 & 1 & 2 \\ 4 & 7 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 & -4 & 0 \\ -1 & 5 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$

2. Find a matrix A such that $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - 5x_2 \\ x_1 + x_2 \end{bmatrix}$.

3. Give the 3×3 identity matrix I. For any vector $\mathbf{x}$, $I\mathbf{x} =$ _______ .

5.3 Actions of Matrices on Vectors: Transformations in $\mathbb{R}^2$

In the previous section we defined a matrix times a vector in a purely computational sense. To put that operation to use we want to think of a matrix as *acting on a vector to create a new vector.* One might also think of this as a matrix *transforming* a vector into another vector.

In general, when a matrix acts on a vector the resulting vector will have a different direction and length than the original vector. There are a few notable exceptions to this:

- The matrix that acts on a vector without actually changing it at all is called the **identity matrix**. Clearly, then, when the identity matrix acts on a vector, neither the direction or magnitude is changed.
- A matrix that rotates every vector in $\mathbb{R}^2$ through a fixed angle θ is called a **rotation matrix**. In this case the direction changes, but not the magnitude. (Of course the direction doesn't change if $\theta = 0°$ and, in some sense, if $\theta = 180°$. In the second case, even though the direction is opposite, the resulting vector is still just a scalar multiple of the original.)
- For most matrices there are certain vectors, called **eigenvectors** whose directions don't change (other than perhaps reversing) when acted on by by the matrix under consideration. In those cases, the effect of multiplying such a vector by the matrix is the same as multiplying the vector by a scalar. This has very useful applications.

5.4 Chapter 5 Exercises

1. Let $A = \begin{bmatrix} 3 & -1 \\ -4 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

   (a) Find $A\mathbf{u}$, $A\mathbf{v}$ and $A\mathbf{w}$.

   (b) Plot and label $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $A\mathbf{u}$, $A\mathbf{v}$ and $A\mathbf{w}$ on *ONE* $\mathbb{R}^2$ coordinate grid.

   You should not see any special relationship between the vectors $\mathbf{x}$ and $A\mathbf{x}$ (where $\mathbf{x}$ is to represent any vector) here.

2. Let $B = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 3 \\ 4.5 \end{bmatrix}$.

   (a) Find $B\mathbf{u}$, $B\mathbf{v}$ and $B\mathbf{w}$, with their components as decimals to the nearest tenth.

   (b) Plot and label $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $B\mathbf{u}$, $B\mathbf{v}$ and $B\mathbf{w}$ on one $\mathbb{R}^2$ coordinate grid.

   (c) You should be able to see that B does not seem to change the length of a vector. To verify this, find $\|\mathbf{w}\|$ and $\|B\mathbf{w}\|$ to the nearest hundredth.

   (d) What does the matrix B seem to do to every vector?

   (e) The entries of B should look familiar to you. What is special about $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$?

3. Let $C = \begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 4.5 \\ 6 \end{bmatrix}$.

   (a) Find $C\mathbf{u}$, $C\mathbf{v}$ and $C\mathbf{w}$, with their components as decimals to the nearest tenth.

   (b) Plot and label $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $C\mathbf{u}$, $C\mathbf{v}$ and $C\mathbf{w}$ on one $\mathbb{R}^2$ coordinate grid.

   (c) What does the matrix C seem to do to every vector? (Does the magnitude change? Does the direction change?)

   (d) Can you see the role of the entries of the matrix here?

4. Again let $A = \begin{bmatrix} 3 & -1 \\ -4 & 0 \end{bmatrix}$, (see Exercise 8) but let $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

   (a) Find $A\mathbf{u}$, $A\mathbf{v}$ and $A\mathbf{w}$.

   (b) Plot and label $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $A\mathbf{u}$, $A\mathbf{v}$ and $A\mathbf{w}$ on one $\mathbb{R}^2$ coordinate grid.

   (c) For one of the vectors, there should be no apparent relationship between the vector and the result when it is multiplied by the matrix. Discuss what happened to the direction and magnitude of each of the other two vectors when the matrix acted on it.

   (d) Pick one of your two vectors for which something special happened and multiply it by three, and multiply the result by A; what is the effect of multiplying by A in this case?

   (e) Pick the other special vector, multiply it by five, then by A. What effect does multiplying by A have on the vector?

6 Matrix Multiplication

Outcome:

6. Understand the algebra of matrices, understand and compute the inverse of a matrix, use matrices to solve problems.

   Performance Criteria:

   (a) Know when two matrices can be multiplied, and know that matrix multiplication is not necessarily commutative.

   (b) Multiply two matrices "by hand."

   (c) Multiply two matrices "by hand" using all three of the linear combination of columns, outer product, and linear combination of rows methods.

   (d) Determine whether two matrices are inverses without finding the inverse of either.

   (e) Find the inverse of a 2×2 matrix using the formula.

   (f) Find the inverse of a matrix using the Gauss-Jordan method. Describe the Gauss-Jordan method for finding the inverse of a matrix.

   (g) Give the geometric or algebraic representations of the inverse or square of a rotation in $\mathbb{R}^2$. Demonstrate that the geometric and algebraic versions are the same.

   (h) Give the incidence matrix of a graph or digraph. Given the incidence matrix of a graph or digraph, identify the vertices and edges using correct notation, and draw the graph.

   (i) Determine the number of k-paths from one vertex of a graph to another. Solve problems using incidence matrices.

6.1 Multiplying Matrices

Performance Criteria:

6. (a) Know when two matrices can be multiplied, and know that matrix multiplication is not necessarily commutative.

   (b) Multiply two matrices "by hand."

When two matrices have appropriate sizes they can be multiplied by a process you are about to see. Although the most reliable way to multiply two matrices and get the correct result is with a calculator or computer software, *it is very important that you get quite comfortable with the way that matrices are multiplied.* That will allow you to better understand certain conceptual things you will encounter farther along.

The process of multiplying two matrices is a bit clumsy to describe, but I'll do my best here. First I will try to describe it informally, then I'll formalize it with a definition based on some special notation. To multiply two matrices we just dot each row of the first with each column of the second, with the results becoming the elements of the second matrix. Here is an informal description of the process:

(1) Dot the first row of the first matrix with the first column of the second. The result is the (1,1) entry (first row, first column) of the product matrix.

(2) Dot the first row of the first matrix with the second column of the second. The result is the (1,2) entry (first row, second column) of the product matrix.

(3) Continue dotting the first row of the first matrix with columns of the second to fill out the first row of the product matrix, stopping after dotting the first row of the first matrix with the last column of the second matrix.

(4) Begin filling out the second row of the product matrix by dotting the *second* row of the first matrix with the first column of the second matrix to get the (2,1) entry (second row, first column) of the product matrix.

(5) Continue dotting the second row of the first matrix with the columns of the second until the second row of the product matrix is filled out by dotting the second row of the first matrix with the last column of the second.

(6) Continue dotting each row of the first matrix with each column of the second until the last row of the first has been dotted with the last column of the second, at which point the product matrix will be complete.

Note that for this to work the number of columns of the first matrix must be equal the number of rows of the second matrix. Let's look at an example.

◇ **Example 6.1(a):** For $A = \begin{bmatrix} -5 & 1 \\ 0 & 4 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, find the product AB.

$$AB = \begin{bmatrix} -5 & 1 \\ 0 & 4 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -5(1)+1(2) & -5(2)+1(3) \\ 0(1)+4(2) & 0(2)+4(3) \\ 2(1)+(-3)(2) & 2(2)+(-3)(3) \end{bmatrix} = \begin{bmatrix} -3 & -7 \\ 8 & 12 \\ -4 & -5 \end{bmatrix}$$

In order to make a formal definition of matrix multiplication, we need to develop a bit of special notation. Given a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

we refer to, for example the third row as A_{3*}. Here the first subscript 3 indicates that we are considering the third row, and the $*$ indicates that we are taking the elements from the third row *in all columns*. Therefore A_{3*} refers to a $1 \times n$ matrix. Similarly, $\mathbf{a}_{*2}$ is the vector that is the second column of A. So we have

$$A_{3*} = \begin{bmatrix} a_{31} & a_{32} & \cdots & a_{3n} \end{bmatrix} \qquad \mathbf{a}_{*2} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{2m} \end{bmatrix}$$

A $1 \times n$ matrix like A_{3*} can be thought of like a vector; in fact, we sometimes call such a matrix a **row vector**. Note that the transpose of such a vector is a column vector. We then define a product like product $A_{i*}\mathbf{b}_{*j}$ by

$$A_{i*}\mathbf{b}_{*j} = A_{i*}^T \cdot \mathbf{b}_{*j} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

This is the basis for the following formal definition of the product of two matrices.

DEFINITION 6.1.1: **Matrix Multiplication**

Let A be an $m \times n$ matrix whose *rows* are the vectors $A_{1*}, A_{2*}, ..., A_{m*}$ and let B be an $n \times p$ matrix whose *columns* are the vectors $\mathbf{b}_{*1}, \mathbf{b}_{*2}, ..., \mathbf{b}_{*p}$. Then AB is the $m \times p$ matrix

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} A_{1*} \\ A_{2*} \\ \vdots \\ A_{m*} \end{bmatrix} \begin{bmatrix} \mathbf{b}_{*1} & \mathbf{b}_{*2} & \mathbf{b}_{*3} & \cdots & \mathbf{b}_{*p} \end{bmatrix}$$

$$= \begin{bmatrix} A_{1*}\mathbf{b}_{*1} & A_{1*}\mathbf{b}_{*2} & \cdots & A_{1*}\mathbf{b}_{*p} \\ A_{2*}\mathbf{b}_{*1} & A_{2*}\mathbf{b}_{*2} & \cdots & A_{2*}\mathbf{b}_{*p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m*}\mathbf{b}_{*1} & A_{m*}\mathbf{b}_{*2} & \cdots & A_{m*}\mathbf{b}_{*p} \end{bmatrix}$$

For the above computation to be possible, products in the last matrix.This implies that the number of columns of A must equal the number of rows of B.

◇ **Example 6.1(b):** For $C = \begin{bmatrix} -5 & 1 & -2 \\ 7 & 0 & 4 \\ 2 & -3 & 6 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -7 & 0 \\ 5 & 2 & 3 \end{bmatrix}$, find CD and DC.

$$CD = \begin{bmatrix} -5 & 1 & -2 \\ 7 & 0 & 4 \\ 2 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ -3 & -7 & 0 \\ 5 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -5-3-10 & -10-7-4 & 5+0-6 \\ 7+0+20 & 14+0+8 & -7+0+12 \\ 2+9+30 & 4+21+12 & -2+0+18 \end{bmatrix}$$

$$= \begin{bmatrix} -18 & -21 & -1 \\ 27 & 22 & 5 \\ 41 & 37 & 16 \end{bmatrix}$$

$$DC = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -7 & 0 \\ 5 & 2 & 3 \end{bmatrix} \begin{bmatrix} -5 & 1 & -2 \\ 7 & 0 & 4 \\ 2 & -3 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 4 & 0 \\ -34 & -3 & -22 \\ -5 & -4 & 16 \end{bmatrix}$$

We want to notice in the last example that $CD \neq DC$! This illustrates something very important:

Matrix multiplication is not necessarily commutative! That is, given two matrices A and B, it is not necessarily true that $AB = BA$. It is possible, but is not "usually" the case. In fact, one of AB and BA might exist and the other not.

This is not just a curiosity; the above fact will have important implications in how certain computations are done. The next example, along with Example 6.1(a), shows that one of the two products might exist and the other not.

◇ **Example 6.1(c):** For the same matrices $A = \begin{bmatrix} -5 & 1 \\ 0 & 4 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ from Example 6.1(a), find the product BA.

When we try to multiply $BA = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -5 & 1 \\ 0 & 4 \\ 2 & -3 \end{bmatrix}$ it is not even possible. We can't find the dot product of a row of B with a column of A because, as vectors, they don't have the same number of components. Therefore the product BA does not exist.

◇ **Example 6.1(d):** For $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, find the products I_2B, BI_2, CB and BC.

$$I_2B = BI_2 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \qquad CB = BC = \begin{bmatrix} 3 & 6 \\ 6 & 9 \end{bmatrix}$$

The notation I_2 here means the 2×2 identity matrix; note that when it is multiplied by another matrix A *on either side* the result is just the matrix A. The matrix C is really just $3I_2$, so it should be no surprise that $CA = 3A$ for any matrix A for which the multiplication can be carried out.

Let's take a minute to think a bit more about the idea of an "identity." In the real numbers we say zero is the **additive identity** because adding it to any real number a does not change the value of the number:

$$a + 0 = 0 + a = a$$

Similarly, the number one is the multiplicative identity:

$$a \cdot 1 = 1 \cdot a = a$$

Here the symbol $\cdot$ is just multiplication of real numbers, not the dot product of vectors. When we talk about an identity matrix, we are talking about a multiplicative identity, like the number one. We will come back to this analogy a couples times, very soon.

There are many other special and/or interesting things that can happen when multiplying two matrices. Here's an example that shows that we can take powers of a matrix if it is a square matrix.

⋄ **Example 6.1(e):** For the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 8 & 5 \\ 6 & -4 & -3 \end{bmatrix}$, find A^2 and A^3.

$$A^2 = AA = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 8 & 5 \\ 6 & -4 & -3 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ -2 & 8 & 5 \\ 6 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 15 & 5 \\ 8 & 42 & 27 \\ 8 & -14 & -17 \end{bmatrix}$$

$$A^3 = AA^2 = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 8 & 5 \\ 6 & -4 & -3 \end{bmatrix} \begin{bmatrix} 1 & 15 & 5 \\ 8 & 42 & 27 \\ 8 & -14 & -17 \end{bmatrix} = \begin{bmatrix} 3 & 101 & 59 \\ 102 & 236 & 121 \\ -50 & -36 & - \end{bmatrix}$$

There are a huge number of facts concerning multiplication of matrices, some perhaps more useful than others. I will limit us to two that will be important to us. Before giving the first, we must define define multiplication of a matrix by a scalar. To multiply a matrix A by a scalar c we simply multiply every entry of A by c.

⋄ **Example 6.1(f):** For the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 8 & 5 \\ 6 & -4 & -3 \end{bmatrix}$, find $3A$.

$$3A = 3\begin{bmatrix} 3 & 1 & -1 \\ -2 & 8 & 5 \\ 6 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 9 & 3 & -3 \\ -6 & 24 & 15 \\ 18 & -12 & -9 \end{bmatrix}$$

With a little thought the following should be clear:

THEOREM 6.1.2

Let A and B be matrices for which the product AB is defined, and let c be any scalar. Then
$$c(AB) = (cA)B = A(cB)$$

Note this carefully - when multiplying a product of two matrices by a scalar, we can instead multiply *one or the other, but NOT BOTH* of the two matrices by the scaler, then multiply the result with the remaining matrix.

Although one can do a great deal of study of matrices themselves, linear algebra is primarily concerned with the action of matrices on vectors. We will use the following in the future:

THEOREM 6.1.3

Let A and B be matrices and $\mathbf{x}$ a vector. Assuming that all the indicated operations below are defined (possible), then

$$(AB)\mathbf{x} = A(B\mathbf{x})$$

◇ **Example 6.1(g):** For the matrices $A = \begin{bmatrix} 1 & -1 \\ -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 \\ 7 & 0 \end{bmatrix}$ and the vector $\mathbf{x} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$, find $(AB)\mathbf{x}$ and $A(B\mathbf{x})$.

$$(AB)\mathbf{x} = \left(\begin{bmatrix} 1 & -1 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 7 & 0 \end{bmatrix}\right)\begin{bmatrix} 3 \\ -6 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 27 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 9 \\ 45 \end{bmatrix}$$

$$(AB)\mathbf{x} = \begin{bmatrix} 1 & -1 \\ -2 & 5 \end{bmatrix}\left(\begin{bmatrix} 4 & -3 \\ 7 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -6 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 30 \\ 21 \end{bmatrix} = \begin{bmatrix} 9 \\ 45 \end{bmatrix}$$

Section 6.1 Exercises

1. Multiply $\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 5 & -1 \end{bmatrix} =$

2. For the following matrices, there are *THIRTEEN* multiplications possible, including squaring some of the matrices. Find and do as many of them do as many of them as you can. When writing your answers, tell which matrices you multiplied to get any particular answer. For example, it *IS* possible to multiply A times B (how about B times A?), and you would then write
$$AB = \begin{bmatrix} -10 & 0 & 25 \\ -14 & 21 & -4 \end{bmatrix}$$

to give your answer. Now you have twelve left to find and do.

$$A = \begin{bmatrix} 0 & 5 \\ -3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -7 & 3 \\ -2 & 0 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} -5 \\ 4 \\ -7 \end{bmatrix}$$

$$D = \begin{bmatrix} 6 & 0 & 3 \\ -5 & 4 & 2 \\ 1 & 1 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 5 & -1 & 2 \end{bmatrix} \qquad F = \begin{bmatrix} 2 & -1 \\ 6 & 9 \end{bmatrix}$$

3. Fill in the blanks: $\begin{bmatrix} -5 & 1 & 3 \\ 2 & 4 & 0 \\ 1 & -1 & -6 \end{bmatrix} \begin{bmatrix} 6 & 0 & -1 \\ -5 & 7 & 2 \\ -4 & 1 & 3 \end{bmatrix} = \begin{bmatrix} * & * & * \\ \underline{\quad} & * & * \\ * & \underline{\quad} & * \end{bmatrix}$

4. Suppose that $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & & \\ a_{31} & & \ddots & \\ \vdots & & & \end{bmatrix}$ is a 5×5 matrix. Write an expression for the third row, second column entry of A^2.

6.2 More Multiplying Matrices

Performance Criteria:

6. (c) Multiply two matrices "by hand" using all three of the linear combination of columns, outer product, and linear combination of rows methods.

Recall the following notation from the previous section: Given a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

we refer to, for example the third row as A_{3*}. Here the first subscript 3 indicates that we are considering the third row, and the $*$ indicates that we are taking the elements from the third row *in all columns*. Therefore A_{3*} refers to a $1 \times n$ matrix. Similarly, $\mathbf{a}_{*2}$ is the vector that is the second column of A. So we have

$$A_{3*} = \begin{bmatrix} a_{31} & a_{32} & \cdots & a_{3n} \end{bmatrix} \qquad \mathbf{a}_{*2} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{2m} \end{bmatrix}$$

Using this notation, if we are multiplying the $m \times n$ matrix A times the $n \times p$ matrix B and $AB = C$, the c_{ij} entry of C is obtained by the product $A_{i*}b_{*j}$.

In some sense the product $A_{i*}\mathbf{b}_{*j}$ is performed as a dot product. Another name for the dot product is **inner product** and this method of multiplying two matrices we will call the inner product method. We will take it to be the the definition of the product of two matrices.

<u>Definition 6.2.1</u>: **Matrix Multiplication, Inner Product Method**

Let A and B be $m \times n$ and $n \times p$ matrices respectively. We define the product AB to be the $m \times p$ matrix C whose (i, j) entry is given by

$$c_{ij} = A_{i*}\mathbf{b}_{*j},$$

where A_{i*} and $\mathbf{b}_{*j}$ are as defined above. That is, each element c_{ij} of C is the product of the ith row of A times the jth column of B.

◇ **Example 6.2(a):** For the matrices $A = \begin{bmatrix} 3 & -1 \\ -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 4 \\ 7 & -2 \end{bmatrix}$, find $C = AB$ by the inner product method.

Here the matrix C will also be 2×2, with

$$c_{11} = \begin{bmatrix} 3 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} = 18 + (-7) = 11, \qquad c_{12} = \begin{bmatrix} 3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = 12 + 2 = 14,$$

$$c_{21} = \begin{bmatrix} -2 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} = -12+35 = 23, \qquad c_{22} = \begin{bmatrix} -2 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = -8+(-10) = -18,$$

so

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 11 & 14 \\ 23 & -18 \end{bmatrix}$$

We will now see three other methods for multiplying matrices. All three are perhaps more complicated than the above, but their value is not in computation of matrix products but rather in giving us conceptual tools that are useful when examining certain ideas in the subject of linear algebra. The first of these other methods uses the ever so important idea of linear combinations.

> THEOREM 6.2.2: **Matrix Multiplication, Linear Combination of Columns Method**
>
> Let A and B be $m \times n$ and $n \times p$ matrices respectively. The product $C = AB$ is the matrix for which
>
> $$\mathbf{c}_{*j} = b_{1j}\mathbf{a}_{*1} + b_{2j}\mathbf{a}_{*2} + b_{3j}\mathbf{a}_{*3} + \cdots + b_{nj}\mathbf{a}_{*n}$$
>
> That is, the jth column $\mathbf{c}_{*j}$ of C is the linear combination of all the columns of A, using the entries of the jth column of B as the scalars.

◇ **Example 6.2(b):** For the matrices $A = \begin{bmatrix} 3 & -1 \\ -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 4 \\ 7 & -2 \end{bmatrix}$, find $C = AB$ by the linear combination of columns method.

Again C will also be 2×2, with

$$\mathbf{c}_{*1} = 6\begin{bmatrix} 3 \\ -2 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ 23 \end{bmatrix} \qquad \mathbf{c}_{*2} = 4\begin{bmatrix} 3 \\ -2 \end{bmatrix} + (-2)\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 14 \\ -18 \end{bmatrix}$$

so

$$C = \begin{bmatrix} \mathbf{c}_{*1} & \mathbf{c}_{*2} \end{bmatrix} = \begin{bmatrix} 11 & 14 \\ 23 & -18 \end{bmatrix}$$

Suppose that we have two vectors $\mathbf{u} = \begin{bmatrix} -6 \\ 1 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}$. Now we see that $\mathbf{u}^T = \begin{bmatrix} -6 & 1 & 4 \end{bmatrix}$, which is a 1×3 matrix. Thinking of the vector $\mathbf{v}$ as a 3×1 matrix, we can use the inner product definition of matrix multiplication to get

$$\mathbf{u}^T\mathbf{v} = \begin{bmatrix} -6 & 1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix} = (-6)(3) + (1)(-5) + (4)(2) = -15 = \mathbf{u} \cdot \mathbf{v}.$$

As mentioned previously this is sometimes also called the **inner product** of $\mathbf{u}$ and $\mathbf{v}$.

We can consider instead $\mathbf{u}$ as a 3×1 matrix and $\mathbf{v}^T$ as a 1×3 matrix and look at the product $\mathbf{u}\mathbf{v}^T$. This is then a 3×3 matrix given by

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} -6 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & -5 & 2 \end{bmatrix} = \begin{bmatrix} (-6)(3) & (-6)(-5) & (-6)(2) \\ (1)(3) & (1)(-5) & (1)(2) \\ (4)(3) & (4)(-5) & (4)(2) \end{bmatrix} = \begin{bmatrix} -18 & 30 & -12 \\ 3 & -5 & 2 \\ 12 & -20 & 8 \end{bmatrix}$$

This last result is called the **outer product** of $\mathbf{u}$ and $\mathbf{v}$, and is used in our next method for multiplying two matrices.

<u>THEOREM 6.2.3</u>: **Matrix Multiplication, Outer Product Method**

Let A and B be $m \times n$ and $n \times p$ matrices respectively. The product $C = AB$ is the matrix

$$C = \mathbf{a}_{*1}B_{1*} + \mathbf{a}_{*2}B_{2*} + \mathbf{a}_{*3}B_{3*} + \cdots + \mathbf{a}_{*n}B_{n*}$$

That is, C is the $m \times p$ matrix given by the sum of all the $m \times p$ outer product matrices obtained from multiplying each column of A times the corresponding row of B.

◇ **Example 6.2(c):** For the matrices $A = \begin{bmatrix} 3 & -1 \\ -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 4 \\ 7 & -2 \end{bmatrix}$, find $C = AB$ by the outer product method.

$$C = \mathbf{a}_{*1}B_{1*} + \mathbf{a}_{*2}B_{2*} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \begin{bmatrix} 6 & 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \end{bmatrix} \begin{bmatrix} 7 & -2 \end{bmatrix} =$$

$$\begin{bmatrix} 18 & 12 \\ -12 & -8 \end{bmatrix} + \begin{bmatrix} -7 & 2 \\ 35 & -10 \end{bmatrix} = \begin{bmatrix} 11 & 14 \\ 23 & -18 \end{bmatrix}$$

<u>THEOREM 6.2.4</u>: **Matrix Multiplication, Linear Combination of Rows Method**

Let A and B be $m \times n$ and $n \times p$ matrices respectively. The product $C = AB$ is the matrix for which

$$C_{i*} = a_{i1}B_{1*} + a_{i2}B_{2*} + a_{i3}B_{3*} + \cdots + a_{in}B_{n*}.$$

That is, the ith row C_{i*} of C is the linear combination of all the rows of B, using the entries of the ith row of A as the scalars.

◇ **Example 6.2(d):** For the matrices $A = \begin{bmatrix} 3 & -1 \\ -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 4 \\ 7 & -2 \end{bmatrix}$, find $C = AB$ by the linear combination of rows method.

By the above theorem we have

$$C_{1*} = 3\begin{bmatrix} 6 & 4 \end{bmatrix} + (-1)\begin{bmatrix} 7 & -2 \end{bmatrix} = \begin{bmatrix} 18 & 12 \end{bmatrix} + \begin{bmatrix} -7 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 14 \end{bmatrix}$$

and

$$C_{2*} = -2\begin{bmatrix} 6 & 4 \end{bmatrix} + 5\begin{bmatrix} 7 & -2 \end{bmatrix} = \begin{bmatrix} -12 & -8 \end{bmatrix} + \begin{bmatrix} 35 & -10 \end{bmatrix} = \begin{bmatrix} 23 & -18 \end{bmatrix}$$

so

$$C = \begin{bmatrix} C_{1*} \\ C_{2*} \end{bmatrix} = \begin{bmatrix} 11 & 14 \\ 23 & -18 \end{bmatrix}$$

Section 6.2 Exercises

1. Let $A = \begin{bmatrix} -5 & 1 & -2 \\ 7 & 0 & 4 \\ 2 & -3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -7 & 0 \\ 5 & 2 & 3 \end{bmatrix}$.

   (a) Find the second column of $C = AB$, using the linear combination of columns method, showing clearly the linear combination used and the resulting column. Label the result using the correct notation for the column of a matrix, as described at the beginning of the section.

   (b) Find the product $C = AB$, showing the sum of outer products, the sum of resulting matrices, and the final result.

   (c) Find the third row of $C = AB$, using the linear combination of rows method, showing clearly the linear combination used and the resulting column. Label the result using the correct notation for the row of a matrix, as described at the beginning of the section.

2. Let $C = \begin{bmatrix} -5 & 1 \\ 0 & 4 \\ 2 & -3 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

   (a) Find the product $A = CD$, using the linear combination of rows method. Show clearly how each row of A is obtained, labeling each using the correct notation. Then give the final result A.

   (b) Find the product $A = CD$, using the linear combination of columns method. Show clearly how each column of A is obtained, labeling each using the correct notation. Then give the final result A.

   (c) Find the product $A = CD$ using the outer product method.

6.3 Inverse Matrices

Performance Criteria:

6. (c) Determine whether two matrices are inverses without finding the inverse of either.
   (d) Find the inverse of a 2×2 matrix using the formula.
   (e) Find the inverse of a matrix using the Gauss-Jordan method. Describe the Gauss-Jordan method for finding the inverse of a matrix.

Let's begin with an example!

◇ **Example 6.3(a):** Find AC and CA for the matrices $A = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$.

$$AC = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad CA = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We see that $AC = CA = I_2$.

Now let's remember that the identity matrix is like the number one for multiplication of numbers. Note that, for example, $\frac{1}{5} \cdot 5 = 5 \cdot \frac{1}{5} = 1$. This is exactly what we are seeing in the above example. We say the numbers 5 and $\frac{1}{5}$ are **multiplicative inverses**, and we say that the matrices A and C above are inverses of each other.

<u>DEFINITION 6.3.1</u> **Inverse Matrices**

Suppose that for matrices A and B we have $AB = BA = I$. Then we say that A and B are **inverse matrices**.

Notationally we write $B = A^{-1}$ or $A = B^{-1}$, and we will say that A and B are **invertible**. Note that in order for us to be able to do both multiplications AB and BA, both matrices must be square and of the same dimensions. It also turns out that that to test two square matrices to see if they are inverses we only need to multiply them in one order:

<u>THEOREM 6.3.2</u> **Test for Inverse Matrices**

To test two *square* matrices A and B to see if they are inverses, compute AB. If it is the identity, then the matrices are inverses.

Here a few notes about inverse matrices:

- Not every square matrix has an inverse, but "many" do. If a matrix does have an inverse, it is said to be **invertible**.
- The inverse of a matrix is unique, meaning there is only one.
- Matrix multiplication *IS* commutative for inverse matrices.

Two questions that should be occurring to you now are

1) How do we know whether a particular matrix has an inverse?

2) If a matrix does have an inverse, how do we find it?

There are a number of ways to answer the first question; here is one:

> <u>THEOREM 6.3.3</u> **Test for Invertibility of a Matrix**
>
> A square matrix A is invertible if, and only if, $rref\,(A) = I$.

Here is the answer to the second question in the case of a 2×2 matrix:

> <u>DEFINITION 6.3.4</u> **Inverse of a 2×2 Matrix**
>
> The inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

◇ **Example 6.3(b):** Find the inverse of $A = \begin{bmatrix} -2 & 7 \\ 1 & -5 \end{bmatrix}$.

$$A^{-1} = \frac{1}{(-2)(-5) - (1)(7)}\begin{bmatrix} -5 & -7 \\ -1 & -2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -5 & -7 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} & -\frac{7}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

Before showing how to find the inverse of a larger matrix we need to go over the idea of **augmenting** a matrix with a vector or another matrix. To augment a matrix A with a matrix B, both matrices must have the same number of rows. A new matrix, denoted $[A\,|\,B]$ is formed as follows: the first row of $[A\,|\,B]$ is the first row of A followed by the first row of B, and every other row in $[A\,|\,B]$ is formed the same way.

◇ **Example 6.3(c):** Let $A = \begin{bmatrix} -5 & 1 & -2 \\ 7 & 0 & 4 \\ 2 & -3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 9 & 1 \\ -1 & 8 \\ -6 & -3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} -7 \\ 10 \\ 4 \end{bmatrix}$.

Give the augmented matrices $[A \,|\, \mathbf{x}]$ and $[A \,|\, B]$.

$$[A \,|\, \mathbf{x}] = \begin{bmatrix} -5 & 1 & -2 & -7 \\ 7 & 0 & 4 & 10 \\ 2 & -3 & 6 & 4 \end{bmatrix}, \qquad [A \,|\, B] = \begin{bmatrix} -5 & 1 & -2 & 9 & 1 \\ 7 & 0 & 4 & -1 & 8 \\ 2 & -3 & 6 & 6 & -3 \end{bmatrix}$$

Gauss-Jordan Method for Finding Inverse Matrices

Let A be an $n \times n$ invertible matrix and I_n be the $n \times n$ identity matrix. Form the augmented matrix $[A \,|\, I_n]$ and find *rref* $([A \,|\, I_n]) = [I_n \,|\, B]$. (The result of row-reduction will have this form.) Then $B = A^{-1}$.

◇ **Example 6.3(d):** Find the inverse of $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{bmatrix}$, if it exists.

We begin by augmenting with the 3×3 identity: $[A \,|\, I_3] = \begin{bmatrix} 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$. Performing Gauss-Jordan elimination (row reducing) then gives $\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & -3 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & -4 & -6 & 7 \end{bmatrix}$, so $A^{-1} = \begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ -4 & -6 & 7 \end{bmatrix}$.

The above example is a bit unusual; the inverse of a randomly generated matrix will usually contain fractions.

◇ **Example 6.3(e):** Find the inverse of $B = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 0 & 2 & -2 \end{bmatrix}$, if it exists.

$[B \,|\, I_3] = \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 0 & 0 & 1 \end{bmatrix} \overset{rref}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & -1 & \frac{3}{2} \end{bmatrix}$. Because the left side of the reduced matrix is not the identity, the matrix B is not invertible.

◇ **Example 6.3(f):** Find a matrix such that $AB = C$, where $A = \begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -3 \\ -2 & 3 \end{bmatrix}$.

Note that if we multiply both sides of $AB = C$ on the left by A^{-1} we get $A^{-1}AB = A^{-1}C$. But $A^{-1}AB = B$, so we have

$$B = A^{-1}C = \begin{bmatrix} -1 & -1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & -3 \end{bmatrix}$$

Section 6.3 Exercises

1. Determine whether $A = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$ and $C = \begin{bmatrix} 8 & -4 \\ -3 & 2 \end{bmatrix}$ are inverses, *without actually finding the inverse of either*. Show clearly how you do this.

2. Consider the matrix $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$.

   (a) Apply row reduction (*"by hand"*) to $[\,A \mid I_2\,]$ until you obtain $[\,I_2 \mid B\,]$. That is, find the *reduced* row-echelon form of $[\,A \mid I_2\,]$.

   (b) Find AB and BA.

   (c) What does this illustrate?

6.4 Applications of Matrices II: Rotations and Projections, Graph Theory

Performance Criteria:

6. (f) Give the geometric or algebraic representations of the inverse or square of a rotation. Demonstrate that the geometric and algebraic versions are the same

   (g) Give the incidence matrix of a graph or digraph. Given the incidence matrix of a graph or digraph, identify the vertices and edges using correct notation, and draw the graph.

   (h) Determine the number of k-paths from one vertex of a graph to another. Solve problems using incidence matrices.

Rotation and Projection Matrices

In the Chapter 5 Exercises you should have encountered a matrix that rotated every vector in $\mathbb{R}^2$ thirty degrees counterclockwise, and another matrix that projected every vector in $\mathbb{R}^2$ onto the line $y = \frac{3}{4}x$. Here are the general formulas for rotation and projection matrices in $\mathbb{R}^2$:

Rotation Matrix in $\mathbb{R}^2$

For the matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and any position vector $\mathbf{x}$ in $\mathbb{R}^2$, the product

$A\mathbf{x}$ is the vector resulting when $\mathbf{x}$ is rotated counterclockwise around the origin by the angle θ.

Projection Matrix in $\mathbb{R}^2$

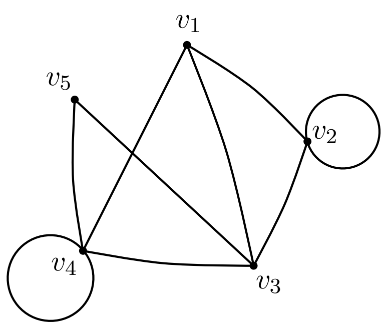
For the matrix $B = \begin{bmatrix} \dfrac{a^2}{a^2+b^2} & \dfrac{ab}{a^2+b^2} \\ \dfrac{ab}{a^2+b^2} & \dfrac{b^2}{a^2+b^2} \end{bmatrix}$ and any position vector $\mathbf{x}$ in $\mathbb{R}^2$, the

product $B\mathbf{x}$ is the vector resulting when $\mathbf{x}$ is projected onto the line containing the origin and the point (a, b).

In the Chapter 6 Exercises you will investigate multiplication of matrices of rotation and projection matrices.

Graphs and Digraphs

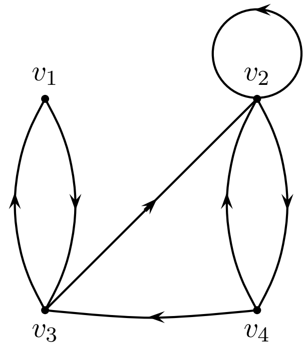
A **graph** is a set of dots, called **vertices**, connected by segments of lines or curves, called **edges**. An example is shown at the top left of the next page; note that a vertex can be connected to itself. We will often label each of the vertices with a subscripted v, as shown. We can then create a matrix, called an **incidence matrix** to show which pairs of vertices are connected (and which are not). The (i, j) entry of the matrix is a one if v_i and v_j are connected by a single edge and a zero if they are not. If $i = j$ the entry is a one if that vertex is connected to itself by an edge, and zero if it is not. You should be able to see this by comparing the graph and and corresponding incidence matrix below. *Note that the incidence matrix is symmetric; that is the case for all incidence matrices of graphs.*



$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Even though there is no edge from vertex one to vertex five for the graph shown, we can get from vertex one to vertex five via vertex three or vertex four. We call such a "route" a **path**, and we denote the paths by the sequence of vertices, like $v_1v_3v_5$ or $v_1v_4v_5$. These paths, in particular, are called **2-paths**, since they consist of two edges. There are other paths from vertex one to vertex five, like the 3-path $v_1v_2v_3v_5$ and the 4-path $v_1v_2v_2v_3v_5$.

In some cases we want the edges of a graph to be "one-way." We indicate this by placing an arrow on each edge, indicating the direction it goes. *We will not put two arrowheads on one edge; if we can travel both ways between two vertices, we will show that by drawing TWO edges between them.* Such a graph is called a **directed graph**, or **digraph** for short. Below is a digraph and its incidence matrix. The (i, j) entry of the incidence matrix is one only if there is a directed edge from v_i to v_j. Of course *the incidence matrix for a digraph need not be symmetric*, since there may be an edge going one way between two vertices but not the other way. Digraphs have incidence matrices as well. Below is a digraph and its incidence matrix.



$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Both the graph and the digraph above are what we call **connected graphs**, meaning that every two vertices are connected by some path. All graphs that we will consider will be connected; we will leave further discussion/investigation of graphs and incidence matrices to the exercises.

Section 6.4 Exercises

1. Sketch a graph with four vertices (labeled v_1 - v_4) and five edges (with no more than one edge from a given vertex to another, but you can have an edge from a vertex to itself) for which the second row, fourth column of A^k is zero for $k = 1, 2, 3, 4, ...$

6.5 Chapter 6 Exercises

1. Let $A = \begin{bmatrix} -5 & 1 \\ 0 & 4 \\ 2 & -3 \end{bmatrix}$.

   (a) Give A^T, the transpose of A.

   (b) Find A^TA and AA^T. Are they the same (equal)?

   (c) Your answers to (b) are special in two ways. What are they? (What I'm looking for here is two of the special types of matrices described in Section 5.1.)

2. For the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ -3 & 1 & 2 \end{bmatrix}$, give a matrix B such that AB *DOES NOT* exist (cannot be found) and a matrix C such that AC *DOES* exist.

3. Explain how to determine whether two matrices A and B are inverses of each other. *Be sure to tell what will happen if they ARE inverses.*

4. Consider the matrices $A = \begin{bmatrix} 3 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 3 \end{bmatrix}$.

   (a) Tell why a person might think that A and B are inverses. *You should be able to do this in a single sentence or mathematical statement.*

   (b) *Tell* why A and B are not inverses. *You should be able to do this in a single sentence.*

5. Show how to find the inverse of the matrix $\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$ in the same way that you would find the inverse of a 10×10 matrix.

Some Trig Identities

You might find the following useful in understanding/answering some of the remaining exercises for this section.

$$\sin^2\theta + \cos^2\theta = 1 \qquad \cos(-\theta) = \cos\theta \qquad \sin(-\theta) = -\sin\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta$$

6. Consider the general rotation matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

   (a) Suppose that we were to apply A to a vector $\mathbf{x}$, then apply A again, to the result. Thinking only geometrically (don't do any calculations), give a single matrix B that should have the same effect.

   (b) Find the matrix A^2 algebraically, by multiplying A by itself.

(c) Use some of the trigonometric facts above to continue your calculations from part (b) until you arrive at matrix B. This of course shows that that $A^2 = B$.

7. Consider again the general rotation matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

(a) Give a matrix C that should "undo" what A does. Do this thinking only geometrically.

(b) Find the matrix A^{-1} algebraically, using the formula for the inverse of a 2×2 matrix..

(c) Use some of the trigonometric facts above to show that $C = A^{-1}$. Do this by starting with C, then modifying it a step at a time to get to A^{-1}.

(d) Give the transpose matrix A^T. It should look familiar - tell how.

7 Matrices and Systems of Equations

Outcome:

7. Understand the relationship of matrices with systems of equations.

   Performance Criteria:

   (a) Express a system of equations as a coefficient matrix times a vector equalling another vector.

   (b) Use LU-factorization to solve a system of equations, given the LU-factorization of its coefficient matrix.

   (c) Solve a system of equations using an inverse matrix. Describe how to use an inverse matrix to solve a system of equations.

   (d) Find the determinant of a 2×2 or 3×3 matrix by hand. Use a calculator to find the determinant of an $n \times n$ matrix.

   (e) Use the determinant to determine whether a system of equations has a unique solution.

   (f) Know the nature of the solution to a homogenous system.

   (g) Determine whether a homogeneous system has more than one solution.

7.1 Matrix Equation Form of a System

Performance Criteria:

7. (a) Express a system of equations as a coefficient matrix times a vector equalling another vector.

DEFINITION 7.1.1 **Matrix Equation Form of a System**

A system of m linear equations in n unknowns (note that m and n need not be equal) can be written as $A\mathbf{x} = \mathbf{b}$ where A is the $m \times n$ coefficient matrix of the system, $\mathbf{x}$ is the vector consisting of the n unknowns and $\mathbf{b}$ is the vector consisting of the m right-hand sides of the equations, as shown below.

$$\begin{array}{rcl} a_{11}x_1 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & & \\ a_{m1}x_1 + \cdots + a_{mn}x_n & = & b_m \end{array} \iff \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

We will refer to this as the **matrix form of a system of equations**.

This form of a system of equations can be used, as you will soon see, in another method (besides row-reduction) for solving a system of equations. That method is occasionally useful, though not generally used in practice due to algorithmic inefficiency. The main benefit of this idea is that it allows us to write a system of equations in the very compact form $A\mathbf{x} = \mathbf{b}$ that allows us to discuss both concepts and practical methods in a way that is much less cumbersome than the systems themselves.

◇ **Example 7.1(a):** Give the matrix form of the system

$$\begin{array}{rcr} x_1 + 3x_2 - 2x_3 & = & -4 \\ 3x_1 + 7x_2 + x_3 & = & 4 \\ -2x_1 + x_2 + 7x_3 & = & 7 \end{array}$$

The matrix form of the system is $\begin{bmatrix} 1 & 3 & -2 \\ 3 & 7 & 1 \\ -2 & 1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix}$

We now have three interpretations of the solution (x_1, x_2, x_3) to a system $A\mathbf{x} = \mathbf{b}$ of three equations in three unknowns, like the one above, assuming that we have a unique solution:

1) (x_1, x_2, x_3) is the point where the planes with the three equations intersect.

2) x_1, x_2 and x_3 are the three scalars for a linear combination of the columns of A that equals the vector $\mathbf{b}$.

3) $\mathbf{x} = [x_1, x_2, x_3]$ is the vector that A transforms into the vector $\mathbf{b}$.

Section 7.1 Exercises

1. Multiply $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

2. Give the matrix form $A\mathbf{x} = \mathbf{b}$ of each system of equations.

(a) $\begin{aligned} x + y - 3z &= 1 \\ -3x + 2y - z &= 7 \\ 2x + y - 4z &= 0 \end{aligned}$

(b) $\begin{aligned} 5x - 3y + z &= -4 \\ x + y - 7z &= 2 \end{aligned}$

7.2 Solving a System With An LU-Factorization

Performance Criterion:

7. (b) Use LU-factorization to solve a system of equations, given the LU-factorization of its coefficient matrix.

In many cases a square matrix A can be "factored" into a product of a lower triangular matrix and an upper triangular matrix, in that order. That is, $A = LU$ where L is lower triangular and U is upper triangular. The product LU is called the **LU-factorization** of the matrix. In that case, for a system $A\mathbf{x} = \mathbf{b}$ that we are trying to solve for $\mathbf{x}$ we have

$$A\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad (LU)\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad L(U\mathbf{x}) = \mathbf{b}$$

Note that $U\mathbf{x}$ is simply a vector; let's call it $\mathbf{y}$. We then have two systems, $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$. To solve the system $A\mathbf{x} = \mathbf{b}$ we first solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$. Once we know $\mathbf{y}$ we can then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, which was our original goal. Here is an example:

◇ **Example 7.2(a):** Solve the system of equations $\begin{array}{rcl} 7x_1 - \ 2x_2 + \ \ x_3 & = & 12 \\ 14x_1 - \ 7x_2 - \ 3x_3 & = & 17 \\ -7x_1 + 11x_2 + 18x_3 & = & 5 \end{array}$, given that the coefficient matrix factors as

$$\begin{bmatrix} 7 & -2 & 1 \\ 14 & -7 & -3 \\ -7 & 11 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 7 & -2 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

Because of the above factorization we can write the system in matrix form as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 7 & -2 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 17 \\ 5 \end{bmatrix}$$

We now let $\begin{bmatrix} 7 & -2 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ $(*)$ and the above system becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 17 \\ 5 \end{bmatrix} \quad (**)$$

The system $(**)$ is easily solved for the vector $\mathbf{y} = [y_1, y_2, y_3]$ by **forward-substitution**. From the first row we see that $y_1 = 12$; from that it follows that $y_2 = 17 - 2y_1 = 17 - 24 = -7$. Finally, $y_3 = 5 + y_1 + 3y_2 = -4$.

Now that we know $\mathbf{y}$, the system $(*)$ becomes

$$\begin{bmatrix} 7 & -2 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ -7 \\ -4 \end{bmatrix}$$

This is now solved by back-substitution. We can see that $x_3 = -1$, so

$$-3x_2 - 5x_3 = -7 \quad \Longrightarrow \quad -3x_2 + 5 = -7 \quad \Longrightarrow \quad x_2 = 4$$

Finally,

$$7x_1 - 2x_2 + x_3 = 12 \quad \Longrightarrow \quad 7x_1 - 9 = 12 \quad \Longrightarrow \quad x_1 = 3$$

The solution to the original system of equations is $(3, 4, -1)$.

This may seem overly complicated, but the factorization of A into LU is done by row reducing, so this method is no more costly than row-reduction in terms of operations used. An added benefit is that if we wish to find $\mathbf{x}$ for various vectors $\mathbf{b}$, we do not have to row-reduce all over again each time. Here are a few additional comments about this method:

- We will see how the LU-factorization is obtained through a series of exercises.
- The LU-factorization of a matrix is not unique; that is, there are different ways to factor a given matrix.
- LU-factorization can be done with non-square matrices, but we are not concerned with that idea.

Section 7.2 Exercises

1. In this exercise you will be working again with the system $\begin{array}{rcr} x_1 + 3x_2 - 2x_3 & = & -4 \\ 3x_1 + 7x_2 + x_3 & = & 4 \\ -2x_1 + x_2 + 7x_3 & = & 7 \end{array}$.

   For the purposes of the exercise we will let

   $$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & -\frac{7}{2} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 3 & -2 \\ 0 & -2 & 7 \\ 0 & 0 & \frac{55}{2} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix}$$

   (a) Write the system $L\mathbf{y} = \mathbf{b}$ as a system of three equations in the three unknowns y_1, y_2, y_3. Then solve the system by hand, showing clearly how it is done. In the end, give the vector $\mathbf{y}$.

   (b) Write the system $U\mathbf{x} = \mathbf{y}$ as a system of three equations in the three unknowns x_1, x_2, x_3. Then solve the system by hand, showing clearly how it is done. In the end, give the vector $\mathbf{x}$.

   (c) Use the linear combination of vectors interpretation of the system to show that the x_1, x_2, x_3 you found in part (b) is a solution to the system of equations. Show the scalar multiplication and vector addition as two separate steps.

(d) Multiply L times U, in that order. What do you notice about the result? If you don't see something, you may have gone astray somewhere!

2. Let A be the coefficient matrix for the system from the previous exercise.

   (a) Give the matrix E_1 be the matrix for which E_1A is the result of the first row operation used to reduce A to U. Give the matrix E_1A.

   (b) Give the matrix E_2 such that $E_2(E_1A)$ is the result after the second row operation used to reduce A to U. Give the matrix E_2E_1A.

   (c) Give the matrix E_3 such that $E_3(E_2E_1A)$ is U.

   (d) Find the matrix $B = E_3E_2E_1$, then use your calculator to find B^{-1}. What is it? If you don't recognize it, you are asleep or you did something wrong!

3. (a) Fill in the blanks of the second matrix below with the entries from E_1. Then, without using your calculator, fill in the blanks in the first matrix so that the product of the first two matrices is the 3×3 identity, as shown.

$$\begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix} \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   Call the matrix you found F_1. Do the same thing with E_2 and E_3 to find matrices F_2 and F_3.

   (b) Find the product $F_1F_2F_3$, in that order. Again, you should recognize the result.

7.3 Inverse Matrices and Systems

Performance Criterion:

7. (c) Solve a system of equations using an inverse matrix. Describe how to use an inverse matrix to solve a system of equations.

Let's consider a simple algebraic equation of the form $ax = b$, where a and b are just constants. If we multiply both sides on the left by $\frac{1}{a}$, the multiplicative inverse of a, we get $x = \frac{1}{a} \cdot b$. for example,

$$\begin{aligned} 3x &= 5 \\ \tfrac{1}{3}(3x) &= \tfrac{1}{3} \cdot 5 \\ \left(\tfrac{1}{3} \cdot 3\right) x &= \tfrac{5}{3} \\ 1x &= \tfrac{5}{3} \\ x &= \tfrac{5}{3} \end{aligned}$$

The following shows how an inverse matrix can be used to solve a system of equations by exactly the same idea:

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^{-1}(A\mathbf{x}) &= A^{-1}\mathbf{b} \\ (A^{-1}A)\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

Note that this only "works" if A is invertible! The upshot of all this is that when A is invertible the solution to the system $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x} = A^{-1}\mathbf{b}$. The above sequence of steps shows the details of why this is. Although this may seem more straightforward than row reduction, it is more costly in terms of computer time than row reduction or LU-factorization. Therefore it is not used in practice.

◇ **Example 7.3(a):** Solve the system of equations $\begin{aligned} 5x_1 + 4x_2 &= 25 \\ -2x_1 - 2x_2 &= -12 \end{aligned}$ using an inverse matrix, showing all steps given above.

The matrix form of the system is $\begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 25 \\ -12 \end{bmatrix}$, and $A^{-1} = -\frac{1}{2}\begin{bmatrix} -2 & -4 \\ 2 & -5 \end{bmatrix}$. A^{-1} can now be used to solve the system:

$$\begin{aligned} \begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 25 \\ -12 \end{bmatrix} \\ -\frac{1}{2}\begin{bmatrix} -2 & -4 \\ 2 & -5 \end{bmatrix} \left(\begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) &= -\frac{1}{2}\begin{bmatrix} -2 & -4 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} 25 \\ -12 \end{bmatrix} \end{aligned}$$

$$\left(-\frac{1}{2}\begin{bmatrix} -2 & -4 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -2 \\ -10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

The solution to the system is $(1, 5)$.

Section 7.3 Exercises

1. Assume that you have a system of equations $A\mathbf{x} = \mathbf{b}$ for some invertible matrix A. Show how the inverse matrix is used to solve the system, showing all steps in the process clearly. Check your answer against what is shown in Section 7.3.

2. Consider the system of equations $\begin{aligned} 2x_1 - 3x_2 &= 4 \\ 4x_1 + 5x_2 &= 3 \end{aligned}$.

   (a) Write the system in matrix times a vector form $A\mathbf{x} = \mathbf{b}$.

   (b) Apply the formula in Section 6.2 to obtain the inverse matrix A^{-1}. **Show a step or two in how you do this.**

   (c) **Demonstrate** that your answer to (b) really is the inverse of A.

   (d) Use the inverse matrix to solve the system. **Show ALL steps outlined in Section 7.3, and give your answer in exact form.**

   (e) Apply row reduction (**"by hand"**) to $[\,A \mid I_2\,]$ until you obtain $[\,I_2 \mid B\,]$. That is, find the *reduced* row-echelon form of $[\,A \mid I_2\,]$. What do you notice about B?

3. Consider the system of equations $\begin{aligned} 5x + 7y &= -1 \\ 2x + 3y &= 4 \end{aligned}$

   (a) Write the system in $A\mathbf{x} = \mathbf{b}$ form.

   (b) Use the formula for the inverse of a 2×2 matrix to find A^{-1}.

   (c) Give the matrix that is to be row reduced to find A^{-1} by the Gauss-Jordan method. Then give the reduced row-echelon form obtained using your calculator.

   (d) Repeat *EVERY* step of the process for solving $A\mathbf{x} = \mathbf{b}$ using the inverse matrix. See Section 7.3 for all the steps.

7.4 Determinants and Matrix Form

Performance Criterion:

7. (d) Find the determinant of a 2×2 or 3×3 matrix by hand. Use a calculator to find the determinant of an $n \times n$ matrix.

   (e) Use the determinant to determine whether a system of equations has a unique solution.

Associated with every *square* matrix is a scalar that is called the **determinant** of the matrix, and determinants have numerous conceptual and practical uses. For a square matrix A, the determinant is denoted by $\det(A)$. This notation implies that the determinant is a function that takes a matrix returns a scalar. Another notation is that the determinant of a specific matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

There is a simple formula for finding the determinant of a 2×2 matrix:

DEFINITION 7.4.1: **Determinant of a 2×2 Matrix**

The determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A) = ad - bc$.

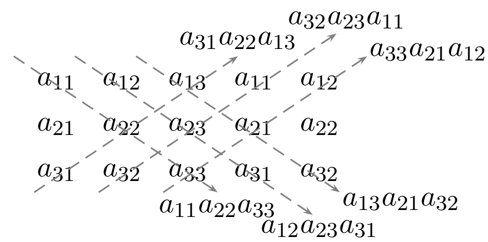
◇ **Example 7.4(a):** Find the determinant of $A = \begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix}$

$$\det(A) = (5)(-2) - (-2)(4) = -10 + 8 = -2$$

There is a fairly involved method of breaking the determinant of a larger matrix down to where it is a linear combination of determinants of 2×2 matrices, but we will not go into that here. It is called the **cofactor expansion** of the determinant, and can be found in most any other linear algebra book, or online. Of course your calculator will find determinants of matrices whose entries are numbers,as will online matrix calculators and various software like *MATLAB*.

Later we will need to be able to find determinants of matrices containing an unknown parameter, and it will be necessary to find determinants of 3×3 matrices. For that reason, we now show a relatively simple method for finding the determinant of a 3×3 matrix. (This will not look simple here, but it is once you are familiar with it.) *This method only works for* 3×3 *matrices.*

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Longrightarrow$$

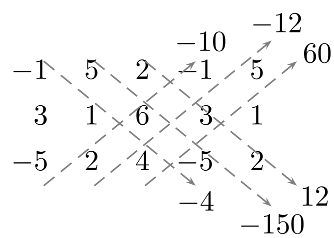


We get the determinant by adding up each of the results of the downward multiplications and then subtracting each of the results of the upward multiplications. This is shown below.

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

◇ **Example 7.4(b):** Find the determinant of $A = \begin{bmatrix} -1 & 5 & 2 \\ 3 & 1 & 6 \\ -5 & 2 & 4 \end{bmatrix}$.

$$\det\begin{bmatrix} -1 & 5 & 2 \\ 3 & 1 & 6 \\ -5 & 2 & 4 \end{bmatrix} \Longrightarrow$$



$$\det(A) = (-4) + (-150) + 12 - (-10) - (-12) - 60 = -4 - 150 + 12 + 10 + 12 - 60 = -180$$

In the future we will need to compute determinants like the following.

◇ **Example 7.4(c):** Find the determinant of $B = \begin{bmatrix} 1-\lambda & 0 & 3 \\ 1 & -1-\lambda & 2 \\ -1 & 1 & -2-\lambda \end{bmatrix}$.

$$\begin{aligned} \det(B) &= (1-\lambda)(-1-\lambda)(-2-\lambda) + (0)(2)(-1) + (3)(1)(1) \\ &\quad - (-1)(-1-\lambda)(3) - (1)(2)(1-\lambda) - (-2-\lambda)(1)(0) \\ &= (-1+\lambda^2)(-2-\lambda) + 3 - 3 - 3\lambda - 2 + 2\lambda \\ &= 2 + \lambda - 2\lambda^2 - \lambda^3 - \lambda - 2 \\ &= -\lambda^3 - 2\lambda^2 \end{aligned}$$

Here is why we care about determinants right now:

> <u>THEOREM 7.4.2</u>: **Determinants and Invertibility, Systems**
>
> Let A be a square matrix.
>
> (a) A is invertible if, and only if, $\det(A) \neq 0$.
>
> (b) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution if, and only if, A is invertible.
>
> (c) If A is not invertible, the system $A\mathbf{x} = \mathbf{b}$ will have either no solution or infinitely many solutions.

Recall that when things are "nice" the system $A\mathbf{x} = \mathbf{b}$ can be solved as follows:

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^{-1}(A\mathbf{x}) &= A^{-1}\mathbf{b} \\ (A^{-1}A)\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

In this case the system will have the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$. (When we say unique, we mean only one.) *If A is not invertible, the above process cannot be carried out, and the system will not have a single unique solution.* In that case there will either be no solution or infinitely many solutions.

We previously discussed the fact that the above computation is analogous to the following ones involving simple numbers and an unknown number x:

$$\begin{aligned} 3x &= 5 \\ \tfrac{1}{3}(3x) &= \tfrac{1}{3}\cdot 5 \\ \left(\tfrac{1}{3}\cdot 3\right)x &= \tfrac{5}{3} \\ 1x &= \tfrac{5}{3} \\ x &= \tfrac{5}{3} \end{aligned}$$

Now let's consider the following two equations, of the same form $ax = b$ but for which $a = 0$:

$$0x = 5 \qquad\qquad 0x = 0$$

We first recognize that we can't do as before and multiply both sides of each by $\frac{1}{0}$, since that is undefined. The first equation has no solution, since there is no number x that can be multiplied by zero and result in five! In the second case, every number is a solution, so the system has infinitely many solutions. *These equations are analogous to $A\mathbf{x} = \mathbf{b}$ when $\det(A) = 0$. The one difference is that $A\mathbf{x} = \mathbf{b}$ can have infinitely many solutions even when $\mathbf{b}$ is NOT the zero vector.*

Section 7.4 Exercises

1. Explain/show how to use the determinant to determine whether
$$\begin{aligned} x + 3y - 3z &= -5 \\ 2x - y + z &= -3 \\ -6x + 3y - 3z &= 4 \end{aligned}$$
has a unique solution. **You may use your calculator for finding determinants - be sure to conclude by saying whether or not this particular system has a solution!**

2. Suppose that you hope to solve a system $A\mathbf{x} = \mathbf{b}$ of n equations in n unknowns.

   (a) If the determinant of A is zero, what does it tell you about the nature of the solution? (By "the nature of the solution" I mean no solution, a unique solution or infinitely many solutions.)

   (b) If the determinant of A is *NOT* zero, what does it tell you about the nature of the solution?

3. Suppose that you hope to solve a system $A\mathbf{x} = \mathbf{0}$ of n equations in n unknowns.

   (a) If the determinant of A is zero, what does it tell you about the nature of the solution? (By "the nature of the solution" I mean no solution, a unique solution or infinitely many solutions.)

   (b) If the determinant of A is *NOT* zero, what does it tell you about the nature of the solution?

7.5 Homogeneous Systems

Performance Criterion:

7. (f) Know the nature of the solution to a homogenous system.
   (g) Determine whether a homogeneous system has more than one solution.

Homogenous systems are important and will come up in a couple places in the future, but there is not a whole lot that can be said about them! A **homogenous system** is one of the form $A\mathbf{x} = \mathbf{0}$. With a tiny bit of thought this should be clear: *Every homogenous system has at least one solution - the zero vector!* Given the results from the previous section, if A is invertible (so $\det(A) \neq 0$), that is the only solution. If A is not invertible there will be infinitely many solutions, the zero vector being just one of them.

7.6 Chapter 7 Exercises

1. Consider the system $\begin{array}{rcr} 3x+2y & = & -1 \\ 4x+5y & = & 1 \end{array}$. The inverse of $\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$ is $\frac{1}{7}\begin{bmatrix} 5 & -2 \\ -4 & 3 \end{bmatrix}$. Use the inverse to solve the system of equations, showing all steps in the process clearly.

2. Suppose that we are trying to solve the system of equations $\begin{array}{rcr} x_1+3x_2 & = & -4 \\ 5x_1-2x_2 & = & 5 \end{array}$. One geometric interpretation of this problem is "Find the point in $\mathbb{R}^2$ where the two lines $x_1+3x_2=-4$ and $5x_1-2x_2=5$ intersect." Two other interpretations follow; fill the empty spaces in each **with specific vectors or matrices**.

   (a) Find a linear combination of the two vectors   and   that equals   .

   (b) Find a vector   that when multiplied by the matrix   results in   .

8 Vector Spaces and Subspaces

Outcome:

8. Understand subspaces of $\mathbb{R}^n$. Find a least squares solution to an inconsistent system of equations.

   Performance Criteria:

   (a) Describe the span of a set of vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ as a line or plane containing a given set of points.

   (b) Determine whether a vector $\mathbf{w}$ is in the span of a set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ of vectors. If it is, write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$.

   (c) Determine whether a set is closed under an operation. If it is, prove that it is; if it is not, give a counterexample.

   (d) Determine whether a subset of $\mathbb{R}^n$ is a subspace. If so, prove it; if not, give an appropriate counterexample.

   (e) Determine whether a vector is in the column space or null space of a matrix, based only on the definitions of those spaces.

   (f) Find the least-squares approximation to the solution of an inconsistent system of equations. Solve a problem using least-squares approximation.

   (g) Give the least squares error and least squares error vector for a least squares approximation to a solution to a system of equations.

8.1 Span of a Set of Vectors

Performance Criteria:

8\. (a) Describe the span of a set of vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ as a line or plane containing a given set of points.

(b) Determine whether a vector $\mathbf{w}$ is in the span of a set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ of vectors. If it is, write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$.

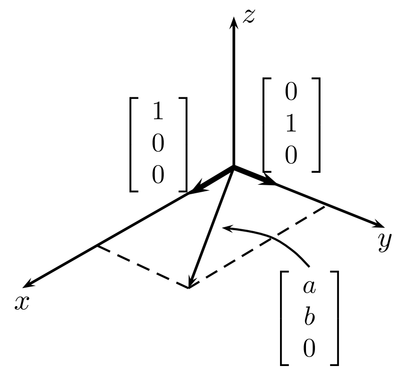
DEFINITION 8.1.1: The **span of a set $\mathcal{S}$ of vectors**, denoted $\text{span}(\mathcal{S})$ is the set of all linear combinations of those vectors.

◇ **Example 8.1(a):** Describe the span of the set $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^3$.

Note that ANY vector with a zero third component can be written as a linear combination of these two vectors:

$$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

All the vectors with $x_3 = 0$ (or $z = 0$) are the xy plane in $\mathbb{R}^3$, so the span of this set is the xy plane. Geometrically we can see the same thing in the picture to the right.



◇ **Example 8.1(b):** Describe $\text{span}\left(\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right)$.

By definition, the span of this set is all vectors $\mathbf{v}$ of the form

$$\mathbf{v} = c_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix},$$

which, because the two vectors are not scalar multiples of each other, we recognize as being a plane through the origin. It should be clear that all vectors created by such a linear combination will have a third component of zero, so the particular plane that is the span of the two vectors is the xy-plane. Algebraically we see that any vector $[a, b, 0]$ in the xy-plane can be created by

$$\left(\frac{a-3b}{7}\right)\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \left(\frac{2a+b}{7}\right)\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{a-3b}{7} \\ \frac{-2a+6b}{7} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{6a+3b}{7} \\ \frac{2a+b}{7} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{7a}{7} \\ \frac{7b}{7} \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

You might wonder how one would determine the scalars $\frac{a-3b}{7}$ and $\frac{2a+b}{7}$. You will see how this is done in the exercises!

At this point we should make a comment and a couple observations:

- First, some language: we can say that the span of the two vectors in Example 8.1(b) is the xy-plane, but we also say that the two vectors span the xy-plane. That is, the word span can be used as either a noun or a verb, depending on how it is used.
- Note that in the two examples above we considered two different sets of two vectors, but in each case the span was the same. This illustrates that *different sets of vectors can have the same span.*
- Consider also the fact that if we were to include in either of the two sets additional vectors that are also in the xy-plane, it would not change the span. However, if we were to add another vector not in the xy-plane, the span would increase to all of $\mathbb{R}^3$.
- In either of the preceding examples, removing either of the two given vectors would reduce the span to a linear combination of a single vector, which is a line rather than a plane. But in some cases, removing a vector from a set does not change its span.
- The last two bullet items tell us that *adding or removing vectors from a set of vectors may or may not change its span.* This is a somewhat undesirable situation that we will remedy in the next chapter.
- It may be obvious, but it is worth emphasizing that (in this course) we will consider spans of finite (and usually rather small) sets of vectors, but a span itself always contains infinitely many vectors (unless the set $\mathcal{S}$ consists of only the zero vector).

It is often of interest to know whether a particular vector is in the span of a certain set of vectors. The next examples show how we do this.

◇ **Example 8.1(c):** Is $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ -4 \\ 1 \end{bmatrix}$ in the span of $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$?

The question is, "can we find scalars c_1, c_2 and c_3 such that

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -4 \\ 1 \end{bmatrix} \text{?"} \qquad (1)$$

We should recognize this as the linear combination form of the system of equations below and to the left. The augmented matrix for the system row reduces to the matrix below and to the right.

$$\begin{aligned} c_1 + c_2 + 2c_3 &= 3 \\ 2c_1 - c_2 \qquad &= -2 \\ 3c_1 + c_2 - 3c_3 &= -4 \\ 4c_1 - c_2 + \ \ c_3 &= 1 \end{aligned} \qquad \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This tells us that the system above and to the left has no solution, so there are no scalars c_1, c_2 and c_3 for which equation (1) holds. Thus $\mathbf{v}$ is not in the span of $\mathcal{S}$.

◇ **Example 8.1(d):** Is $\mathbf{v} = \begin{bmatrix} 19 \\ 10 \\ -1 \end{bmatrix}$ in $\text{span}(\mathcal{S})$, where $\mathcal{S} = \left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -4 \end{bmatrix} \right\}$?

Here we are trying to find scalars c_1, c_2 and c_3 such that

$$c_1 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 19 \\ 10 \\ -1 \end{bmatrix} \qquad (2)$$

We should recognize this as the linear combination form of the system of equations below and to the left. The augmented matrix for the system row reduces to the matrix below and to the right.

$$\begin{aligned} 3c_1 - 5c_2 + \ \ c_3 &= 19 \\ -c_1 \qquad + 7c_3 &= 10 \\ 2c_1 + \ c_2 - \ 4c_3 &= -1 \end{aligned} \qquad \qquad \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

This tells us that (2) holds for $c_1 = 4$, $c_2 = -1$ and $c_3 = 2$, so $\mathbf{v}$ is in $\text{span}(\mathcal{S})$.

Sometimes, with a little thought, no computations are necessary to answer such questions, as the next examples show.

◇ **Example 8.1(e):** Is $\mathbf{v} = \begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix}$ in the span of $\mathcal{S} = \left\{ \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}$?

One can see that any linear combination of the two vectors in S will have zero as its second component:

$$c_1 \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3c_1 \\ 0 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} -5c_2 \\ 0 \\ 1c_2 \end{bmatrix} = \begin{bmatrix} -3c_1 - 5c_2 \\ 0 \\ 2c_1 + c_2 \end{bmatrix}$$

Since the second component of $\mathbf{v}$ is not zero, $\mathbf{v}$ is not in the span of the set $\mathcal{S}$.

◇ **Example 8.1(f):** Is $\mathbf{v} = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$ in $\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$?

Here we can see that if we multiply the three vectors in $\mathcal{S}$ by 4, 7 and -1, respectively, and add them, the result will be $\mathbf{v}$:

$$4\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$$

Therefore $\mathbf{v}$ is in $\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$.

Sometimes we will be given an infinite set of vectors, and we'll ask whether a particular finite set of vectors *spans the infinite set*. By this we are asking whether the span of the finite set is the infinite set. For example, we might ask whether the vector $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ spans $\mathbb{R}^2$. Because the span of the single vector $\mathbf{v}$ is just a line, $\mathbf{v}$ does not span $\mathbb{R}^2$. With the knowledge we have at this point, it can sometimes be difficult to tell whether a finite set of vectors spans a particular infinite set. The next chapter will give us a means for making such a judgement a bit easier.

We conclude with a few more observations. With a little thought, the following can be seen to be true. (Assume all vectors are non-zero.)

- The span of a single vector is all scalar multiples of that vector. In $\mathbb{R}^2$ or $\mathbb{R}^3$ the span of a single vector is a line through the origin.
- The span of a set of two non-parallel vectors in $\mathbb{R}^2$ is all of $\mathbb{R}^2$. In $\mathbb{R}^3$ it is a plane through the origin.
- The span of three vectors in $\mathbb{R}^3$ that do not lie in the same plane is all of $\mathbb{R}^3$.

Section 8.1 Exercises

1. Describe the span of each set of vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ by telling what it is geometrically and, if it is a standard set like one of the coordinate axes or planes, specifically what it is. If it is a line that is not one of the axes, give two points on the line. If it is a plane that is not one of the coordinate planes, give three points on the plane.

   (a) The vector $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ in $\mathbb{R}^2$.

   (b) The set of vectors $\left\{\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}\right\}$ in $\mathbb{R}^3$.

   (c) The vectors $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ in $\mathbb{R}^2$.

   (d) The set $\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$ in $\mathbb{R}^3$.

(e) The vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ in $\mathbb{R}^3$.

2. For each of the following, determine whether the vector $\mathbf{w}$ is in the span of the set S. If it is, write it as a linear combination of the vectors in S.

(a) $\mathbf{w} = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}, \quad S = \left\{ \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ -11 \\ -1 \end{bmatrix} \right\}$

(b) $\mathbf{w} = \begin{bmatrix} -5 \\ -23 \\ 12 \\ 8 \end{bmatrix}, \quad S = \left\{ \begin{bmatrix} 1 \\ -4 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -4 \\ 5 \end{bmatrix} \right\}$

(c) $\mathbf{w} = \begin{bmatrix} 8 \\ 38 \\ -14 \\ 11 \end{bmatrix}, \quad S = \left\{ \begin{bmatrix} 1 \\ -4 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -4 \\ 5 \end{bmatrix} \right\}$

(d) $\mathbf{w} = \begin{bmatrix} 3 \\ 7 \\ -4 \end{bmatrix}, \quad S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

8.2 Closure of a Set Under an Operation

Performance Criteria:

8. (c) Determine whether a set is closed under an operation. If it is, prove that it is; if it is not, give a counterexample.

Consider the set $\{0, 1, 2, 3, ...\}$, which are called the whole numbers. Notice that if we add or multiply any two whole numbers the result is also a whole number, but if we try subtracting two such numbers it is possible to get a number that is not in the set. We say that the whole numbers are **closed under addition and multiplication**, but the set of whole numbers is not closed under subtraction. If we enlarge our set to be the integers $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ we get a set that is closed under addition, subtraction and multiplication. These operations we are considering are called **binary operations** because they take two elements of the set and create a single new element. An operation that takes just one element of the set and gives another (possibly the same) element of the set is called a **unary operation**. An example would be absolute value; note that the set of integers is closed under absolute value.

DEFINITION 8.2.1: **Closed Under an Operation**

A set $\mathcal{S}$ is said to be closed under a binary operation $*$ if for every s and t in $\mathcal{S}$, $s * t$ is in $\mathcal{S}$. $\mathcal{S}$ is closed under a unary operation $\langle\ \rangle$ if for every s in $\mathcal{S}$, $\langle s \rangle$ is in $\mathcal{S}$.

Notice that the term "closed," as defined here, only makes sense in the context of a set with an operation. Notice also that *it is the set that is closed*, not the operation. The operation is important as well; as we have seen, a given set can be closed under one operation but not another.

When considering closure of a set $\mathcal{S}$ under a binary operation $*$, our considerations are as follows:

- We first wish to determine whether we think $\mathcal{S}$ *IS* closed under $*$.
- If we do think that $\mathcal{S}$ is closed under $*$, we then need to prove that it is. To do this, we need to take two general, or *arbitrary* elements x and y of $\mathcal{S}$ and show that $x * y$ is in $\mathcal{S}$.
- If we think that $\mathcal{S}$ is not closed under $*$, we need to take two *specific* elements x and y of $\mathcal{S}$ and show that $x * y$ is not in $\mathcal{S}$.

◇ **Example 8.2(a):** The odd integers are the numbers $..., -5, -3, -1, 1, 3, 5, ...$. Are the odd integers closed under addition? Multiplication?

We see that $3 + 5 = 8$. Because 3 and 5 are both odd but their sum isn't, the odd integers are not closed under addition. Let's try multiplying some odds:

$$3 \times 5 = 15 \qquad -7 \times 9 = -63 \qquad -1 \times -7 = 7$$

Based on these three examples, it appears that the odd integers are perhaps closed under multiplication. Let's attempt to prove it. First we observe that any number of the form $2n+1$, where n is any integer, is odd. (This is in fact the definition of an odd integer.) So if we have two *possibly different* odd integers, we can write them as $2m+1$ and $2n+1$, where m and n are not necessarily the same integers. Their product is

$$(2m+1)(2n+1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1.$$

Because the integers are closed under multiplication and addition, $2mn+m+n$ is an integer and the product of $2m+1$ and $2n+1$ is of the form two times an integer, plus one, so it is odd as well. Therefore the odd integers are closed under multiplication.

Closure of a set under an operation is a fairly general concept; let's narrow our focus to what is important to us in linear algebra.

◇ **Example 8.2(b):** Prove that then span of a set $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_k\}$ in $\mathbb{R}^n$ is closed under addition and scalar multiplication.

Suppose that $\mathbf{u}$ and $\mathbf{w}$ are in span($\mathcal{S}$). Then there are scalars $c_1, c_2, c_3, ..., c_k$ and $d_1, d_2, d_3, ..., d_k$ such that

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_k\mathbf{v}_k \quad \text{and} \quad \mathbf{w} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{v}_3 + \cdots + d_k\mathbf{v}_k.$$

Therefore

$$\begin{aligned} \mathbf{u}+\mathbf{w} &= (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_k\mathbf{v}_k) + (d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{v}_3 + \cdots + d_k\mathbf{v}_k) \\ &= (c_1+d_1)\mathbf{v}_1 + (c_2+d_2)\mathbf{v}_2 + (c_3+d_3)\mathbf{v}_3 + \cdots + (c_k+d_k)\mathbf{v}_k \end{aligned}$$

This last expression is a linear combination of the vectors in $\mathcal{S}$, so it is in span($\mathcal{S}$). Therefore span($\mathcal{S}$) is closed under addition. Now suppose that $\mathbf{u}$ is as above and a is any scalar. Then

$$\begin{aligned} a\mathbf{u} &= a(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_k\mathbf{v}_k) \\ &= ac_1\mathbf{v}_1 + ac_2\mathbf{v}_2 + ac_3\mathbf{v}_3 + \cdots + ac_k\mathbf{v}_k \end{aligned}$$

which is also a linear combination of the vectors in $\mathcal{S}$, so it is also in span($\mathcal{S}$). Thus span($\mathcal{S}$) is closed under multiplication by scalars.

The result of the above Example is that

> Theorem 8.2.2: The span of a set $\mathcal{S}$ of vectors is closed under vector addition and scalar multiplication.

This seemingly simple observation is the beginning of one of the most important stories in the subject of linear algebra. The remainder of this chapter and all of the next will fill out the rest of that story.

8.3 Vector Spaces and Subspaces

Performance Criterion:

8. (d) Determine whether a subset of $\mathbb{R}^n$ is a subspace. If so, prove it; if not, give an appropriate counterexample.

Vector Spaces

The term "space" in math simply means a set of objects with some additional special properties. There are metric spaces, function space, topological spaces, Banach spaces, and more. The vectors that we have been dealing with make up the **vector spaces** called $\mathbb{R}^2$, $\mathbb{R}^3$ and, for larger values, $\mathbb{R}^n$. In general, a vector space is simply a collection of objects called vectors (and a set of scalars) that satisfy certain properties.

<u>Definition 8.3.1</u>: **Vector Space**

A **vector space** is a set V of objects called **vectors** and a set of scalars (usually the real numbers $\mathbb{R}$), with the operations of vector addition and scalar multiplication, for which the following properties hold for all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and scalars c and d.

1. $\mathbf{u}+\mathbf{v}$ is in V
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
4. There exists a vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$. This vector is called the **zero vector**.
5. For every $\mathbf{u}$ in V there exists a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=0$.
6. $c\mathbf{u}$ is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$
9. $c(d\mathbf{u})=(cd)\mathbf{u}$
10. $1\mathbf{u}=\mathbf{u}$

Note that items 1 and 6 of the above definition say that the vector space V is closed under addition and scalar multiplication.

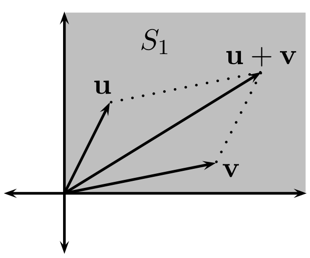
When working with vector spaces, we will be very interested in certain subsets of those vector spaces that are the span of a set of vectors. As you proceed, recall Example 8.2(b), where we showed that *the span of a set of vectors is closed under addition and scalar multiplication.*

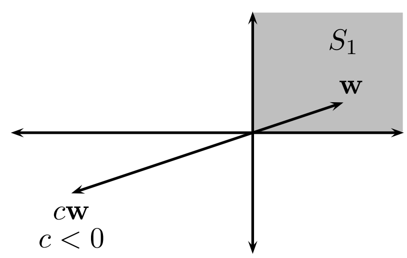
Subspaces of Vector Spaces

As you should know by now, the two main operations with vectors are multiplication by scalars and addition of vectors. (Note that these two combined give us linear combinations, the foundation of almost everything we've done.) A given vector space can have all sorts of subsets; consider the following subsets of $\mathbb{R}^2$.

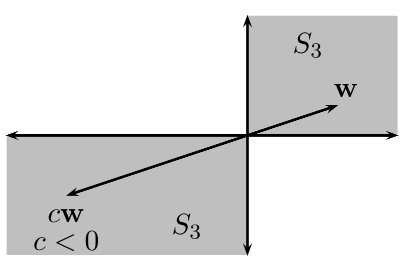
- The set S_1 consisting of the first quadrant and the nonnegative parts of the two axes, or all vectors of the form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $x_1 \geq 0$ and $x_2 \geq 0$.
- The set S_2 consisting of the line containing the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Algebraically this is all vectors of the form $t\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ where t ranges over all real numbers.
- The set S_3 consisting of the first and third quadrants and both axes. This can be described as the set of all vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with $x_1 x_2 \geq 0$.

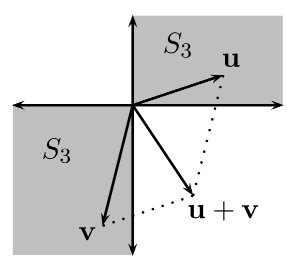
Our current concern is whether these subsets of $\mathbb{R}^2$ are closed under addition and scalar multiplication. With a bit of thought you should see that S_1 is closed under addition, but not scalar multiplication when the scalar is negative:





In some sense we can solve the problem of not being closed under scalar multiplication by including the third quadrant as well to get S_3, but then the set isn't closed under addition:





Finally, the set S_2 is closed under both addition and scalar multiplication. That is a bit messy to show with a diagram, but consider the following. S_2 is the span of the single vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, and we showed in the last section that the span of any set of vectors is closed under addition and scalar multiplication.

It turns out that when working with vector spaces the only subsets of any real interest are the ones that are closed under both addition and scalar multiplication. We give such subsets a name:

> <u>DEFINITION 8.3.2</u>: **Subspace of $\mathbb{R}^n$**
>
> A subset S of $\mathbb{R}^n$ is called a **subspace** of $\mathbb{R}^n$ if for every scalar c and any vectors $\mathbf{u}$ and $\mathbf{v}$ in S, $c\mathbf{u}$ and $\mathbf{u}+\mathbf{v}$ are also in S. That is, S is closed under scalar multiplication and addition.

You will be asked whether certain subsets of $\mathbb{R}^2$, $\mathbb{R}^3$ or $\mathbb{R}^n$ are subspaces, and it is your job to back your answers up with some reasoning. This is done as follows:

- When a subset *IS* a subspace a general proof is required. That is, we must show that the set is closed under scalar multiplication and addition, for *ALL* scalars and *ALL* vectors. We may have to do this outright, but if it is clear that the set of vectors is the span of some set of vectors, then we know from the argument presented in Example 8.2(b) that the set is closed under addition and scalar multiplication, so it is a subspace.
- When a subset *IS NOT* a subspace, we demonstrate that fact with a *SPECIFIC* example. Such an example is called a **counterexample**. Notice that all we need to do to show that a subset is not a subspace is to show that that it is not closed under scalar multiplication *OR* vector addition. If either is the case, then the set in question is not a subspace. *Even if both are the case, we need only show one.*

The following examples illustrate these things.

◇ **Example 8.3(a):** Show that the set S_1 consisting of all vectors of the form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $x_1 \geq 0$ and $x_2 \geq 0$ is not a subspace of $\mathbb{R}^2$.

As mentioned before, this set is not closed under multiplication by negative scalars, so we just need to give a specific example of this. Let $\mathbf{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $c = -2$. Clearly $\mathbf{u}$ is in S_1 and $c\mathbf{u} = (-2)\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -6 \\ -10 \end{bmatrix}$, which is not in S_1. Therefore S_1 is not closed under scalar multiplication so it is not a subspace of $\mathbb{R}^2$.

◇ **Example 8.3(b):** Show that the set S_2 consisting of all vectors of the form $t\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, where t ranges over all real numbers, is a subspace of $\mathbb{R}^2$.

Let c be any scalar and let $\mathbf{u} = s\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{v} = t\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Then $\mathbf{u}$ and $\mathbf{v}$ are both in S_2 and

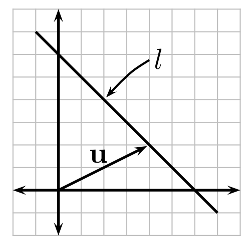
$$c\mathbf{u} = c\left(s\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = (cs)\begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{u}+\mathbf{v} = s\begin{bmatrix} 3 \\ 2 \end{bmatrix} + t\begin{bmatrix} 3 \\ 2 \end{bmatrix} = (s+t)\begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Because cs and $s+t$ are scalars, we can see that both $c\mathbf{u}$ and $\mathbf{u}+\mathbf{v}$ are in S_2, so S_2 is a subspace of $\mathbb{R}^2$.

This last example demonstrates the general method for showing that a set of vectors is closed under addition and scalar multiplication. That said, the given subspace could have been shown to be s subspace by simply observing that it is the span of the set consisting of the single vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. From Theorem 8.2.2 we know that the span of any set of vectors is a subspace, so the set described in the above example is a subspace of $\mathbb{R}^2$.

◇ **Example 8.3(c):** Determine whether the subset S of $\mathbb{R}^3$ consisting of all vectors of the form $\mathbf{x} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + t\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$ is a subspace. If it is, prove it. If it is not, provide a counterexample.

We recognize this as a line in $\mathbb{R}^3$ passing through the point $(2, 5, -1)$, and it is not hard to show that the line does not pass through the origin. Remember that what we mean by the line is really all position vectors (so with tails at the origin) whose tips are on the line. Considering a similar situation in $\mathbb{R}^2$, we see that $\mathbf{u}$ is such a vector for the line l shown. It should be clear that if we multiply $\mathbf{u}$ by any scalar other than one, the resulting vector's tip will not lie on the line. Thus we would guess that the set S, even though it is in $\mathbb{R}^3$, is probably not closed under scalar multiplication.



Now let's prove that it isn't. To do this we first let $\mathbf{u} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$, which is in S. Let $c = 2$, so $2\mathbf{u} = \begin{bmatrix} 12 \\ 8 \\ 4 \end{bmatrix}$. We need to show that this vector is not in S. If it were, there would have to be a scalar t such that $\begin{bmatrix} 12 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + t\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$. Subtracting $\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ we get $\begin{bmatrix} 10 \\ 3 \\ 5 \end{bmatrix} = t\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$. We can see that the value of t that would be needed to give the correct second component would be -3, but this would result in a third component of -9, which is not correct. Thus there is no such t and the vector $\begin{bmatrix} 12 \\ 8 \\ 4 \end{bmatrix}$ is not in S. Thus S is not closed under scalar multiplication, so it is not a subspace of $\mathbb{R}^3$.

We should compare the results of Examples 8.3(b) and 8.3(c). Note that both are lines in their respective $\mathbb{R}^n$'s, but the line in 8.3(b) passes through the origin, and the one in 8.3(c) does not. *It is no coincidence that the set in 8.3(b) is a subspace and the set in 8.3(c) is not.* If a set S is a subspace, being closed under scalar multiplication means that zero times any vector in the subspace must also be in the subspace. But zero times a vector is the zero vector $\mathbf{0}$. Therefore

> Theorem 8.3.3: **Subspaces Contain the Zero Vector**
>
> If a subset S of $\mathbb{R}^n$ is a subspace, then the zero vector is in S.

This type of a statement is called a **conditional statement**. Related to any conditional statement are two other statements called the **converse** and **contrapositive** of the conditional statement. In this case we have

- **Converse:** If the zero vector is in a subset S of $\mathbb{R}^n$, then S is a subspace.
- **Contrapositive:** If the zero vector is *not* in a subset S of $\mathbb{R}^n$, then S is *not* a subspace.

When a conditional statement is true, its converse may or may not be true. In this case the converse is not true. This is easily seen in Example 8.3(a), where the set contains the zero vector but is not a subspace. However, when a conditional statement is true, its contrapositive is true as well. Therefore the second statement above is the most useful of the three statements, since it gives us a quick way to rule out a set as a subspace. In Example 8.3(c) this would have saved us the trouble of providing a counterexample, although we'd still need to convincingly show that the zero vector is not in the set.

◇ **Example 8.3(d):** Determine whether the set of all vectors of the form $\mathbf{x} = \begin{bmatrix} a \\ a+b \\ b \\ a-b \end{bmatrix}$, for some real numbers a and b, is a subspace of $\mathbb{R}^4$.

We first note that a vector $\mathbf{x}$ of the given form will be the zero vector if $a = b = 0$. By the previous discussion we cannot then rule out the possibility that the given set is a subspace, but neither do we yet know it *IS* a subspace. But we observe that

$$\mathbf{x} = \begin{bmatrix} a \\ a+b \\ b \\ a-b \end{bmatrix} = \begin{bmatrix} a \\ a \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ b \\ -b \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

Thus the set of vectors under consideration is the span of $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$, so it is a subspace.

We conclude this section with an example that gives us the "largest" and "smallest" subspaces of $\mathbb{R}^n$.

◇ **Example 8.3(e):** Of course a scalar times any vector in $\mathbb{R}^n$ is also in $\mathbb{R}^n$, and the sum of any two vectors in $\mathbb{R}^n$ is in $\mathbb{R}^n$, so $\mathbb{R}^n$ is a subspace of itself. Also, the zero vector by itself is a subspace of $\mathbb{R}^n$ as well, often called the **trivial subspace**.

At this point we have seen a variety of subspaces, and some sets that are not subspaces as well. From Example 8.2(b) we know that the span of two linearly independent vectors in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$. But we know that the span of two linearly independent vectors in $\mathbb{R}^3$ is a plane through the origin. Note that we could impose a coordinate system on any plane to make it essentially $\mathbb{R}^2$, so we can think of this particular variety of subspace as just being a copy of $\mathbb{R}^2$ sitting inside $\mathbb{R}^3$. This illustrates what is in fact a general principle: *any subspace of* $\mathbb{R}^n$ *is essentially a copy of* $\mathbb{R}^m$*, for some* $m \leq n$*, sitting inside* $\mathbb{R}^n$ *with its origin at the origin of* $\mathbb{R}^n$*.* More formally we have the following:

Subspaces of $\mathbb{R}^n$

- The only non-trivial subspaces of $\mathbb{R}^2$ are lines through the origin and all of $\mathbb{R}^2$.
- The only non-trivial subspaces of $\mathbb{R}^3$ are lines through the origin, planes through the origin, and all of $\mathbb{R}^3$.
- The only non-trivial subspaces of $\mathbb{R}^n$ are hyperplanes (including lines) through the origin and all of $\mathbb{R}^n$.

Section 8.3 Exercises

1. For each of the following sub*sets* of $\mathbb{R}^3$, think of each point as a position vector; each set then becomes a set of vectors rather than points. For each,
   - determine whether the set is a sub*space* and
   - if it is *NOT* a subspace, give a reason why it isn't by doing one of the following:
     - ◇ stating that the set does not contain the zero vector
     - ◇ giving a vector that is in the set and a scalar multiple that isn't (show that it isn't)
     - ◇ giving two vectors that in the set and showing that their sum is not in the set

   (a) All points on the horizontal plane at $z = 3$.

   (b) All points on the xz-plane.

   (c) All points on the line containing $\mathbf{u} = [-3, 1, 4]$.

   (d) All points on the lines containing $\mathbf{u} = [-3, 1, 4]$ and $\mathbf{v} = [5, 0, 2]$.

   (e) All points for which $x \geq 0, y \geq 0$ and $z \geq 0$.

   (f) All points $\mathbf{x}$ given by $\mathbf{x} = \mathbf{w} + s\mathbf{u} + t\mathbf{v}$, where $\mathbf{w} = [1, 1, 1]$ and $\mathbf{u}$ and $\mathbf{v}$ are as in (d).

   (g) All points $\mathbf{x}$ given by $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are as in (d).

   (h) The vector $\mathbf{0}$.

   (i) All of $\mathbb{R}^3$.

2. Consider the vectors $\mathbf{u} = \begin{bmatrix} 8 \\ -2 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -16 \\ 4 \\ -8 \end{bmatrix}$.

(a) Is the set of all vectors $\mathbf{x} = \mathbf{u} + t\mathbf{v}$, where t ranges over all real numbers, a subspace of $\mathbb{R}^3$? If not, tell why not.

(b) Is the set of all vectors $\mathbf{x} = \mathbf{u} + t\mathbf{w}$, where t ranges over all real numbers, a subspace of $\mathbb{R}^3$? If not, tell why not.

8.4 Column Space and Null Space of a Matrix

Performance Criteria:

8. (e) Determine whether a vector is in the column space or null space of a matrix, based only on the definitions of those spaces.

In this section we will define two important subspaces associated with a matrix A, its **column space** and its **null space**.

DEFINITION 8.4.1: **Column Space of a Matrix**

The **column space** of an $m \times n$ matrix A is the span of the columns of A. It is a subspace of $\mathbb{R}^m$ and we denote it by $\text{col}(A)$.

◇ **Example 8.4(a):** Determine whether $\mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$ are in the column space of $A = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -7 & -5 \\ 3 & 4 & -2 \end{bmatrix}$.

We need to solve the two vector equations of the form

$$c_1 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ -7 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix} = \mathbf{b}, \tag{1}$$

with $\mathbf{b}$ first being $\mathbf{u}$, then $\mathbf{v}$. The respective reduced row-echelon forms of the augmented matrices corresponding to the two systems are

$$\begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore we can find scalars c_1, c_2 and c_3 for which (1) holds when $\mathbf{b} = \mathbf{u}$, but not when $\mathbf{b} = \mathbf{v}$. From this we deduce that $\mathbf{u}$ is in $\text{col}(A)$, but $\mathbf{v}$ is not.

Recall that the system $A\mathbf{x} = \mathbf{b}$ of m linear equations in n unknowns can be written in linear combination form:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Note that the left side of this equation is simply a linear combination of the columns of A, with the scalars being the components of $\mathbf{x}$. The system will have a solution if, and only if, $\mathbf{b}$ can be written as a linear combination of the columns of A. Stated another way, we have the following:

> <u>Theorem 8.4.2</u>: A system $A\mathbf{x} = \mathbf{b}$ has a solution (meaning *at least* one solution) if, and only if, $\mathbf{b}$ is in the column space of A.

Let's consider now only the case where $m = n$, so we have n linear equations in n unknowns. We have the following facts:

- If $\text{col}(A)$ is all of $\mathbb{R}^n$, then $A\mathbf{x} = \mathbf{b}$ will have a solution for any vector $\mathbf{b}$. What's more, *the solution will be unique.*
- If $\text{col}(A)$ is a proper subspace of $\mathbb{R}^n$ (that is, it is not all of $\mathbb{R}^n$), then the equation $A\mathbf{x} = \mathbf{b}$ will have a solution if, and only if, $\mathbf{b}$ is in $\text{col}(A)$. If $\mathbf{b}$ is in $\text{col}(A)$ the system will have infinitely many solutions.

Next we define the **null space** of a matrix.

> <u>Definition 8.4.3</u>: **Null Space of a Matrix**
>
> The **null space** of an $m \times n$ matrix A is the set of all solutions to $A\mathbf{x} = \mathbf{0}$. It is a subspace of $\mathbb{R}^n$ and is denoted by $\text{null}(A)$.

◇ **Example 8.4(b):** Determine whether $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ are in the null space of $A = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -7 & -5 \\ 3 & 4 & -2 \end{bmatrix}$.

A vector $\mathbf{x}$ is in the null space of a matrix A if $A\mathbf{x} = \mathbf{0}$. We see that

$$A\mathbf{u} = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -7 & -5 \\ 3 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -21 \\ 11 \end{bmatrix} \quad \text{and} \quad A\mathbf{v} = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -7 & -5 \\ 3 & 4 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so $\mathbf{v}$ is in the $\text{null}(A)$ and $\mathbf{u}$ is not.

Still considering only the case where $m = n$, we have the following fact about the null space:

- If $\text{null}(A)$ is just the zero vector, A is invertible and $A\mathbf{x} = \mathbf{b}$ has a unique solution for any vector $\mathbf{b}$.

We conclude by pointing out the important fact that for an $m \times n$ matrix A, the null space of A is a subspace of $\mathbb{R}^n$ and the column space of A is a subspace of .

Section 8.4 Exercises

1. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 2 \\ 9 \\ -17 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 15 \\ 2 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 8 \\ -8 \\ -4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}.$$

   (a) The column space of A is the set of all vectors that are linear combinations of the columns of A. Determine whether the vector $\mathbf{u}_1$ is in the column space of A by determining whether $\mathbf{u}_1$ is a linear combination of the columns of A. Give the vector equation that you are trying to solve, and your row reduced augmented matrix. **Be sure to tell whether $\mathbf{u}_1$ is in the column space of A or not! Do this with a brief sentence.**

   (b) If $\mathbf{u}_1$ *IS* in the column space of A, give a *specific* linear combination of the columns of A that equals $\mathbf{u}_1$.

   (c) Repeat parts (a) and (b) for the vector $\mathbf{u}_2$.

2. Again let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 2 \\ 9 \\ -17 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 15 \\ 2 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 8 \\ -8 \\ -4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}.$$

   The null space of A is all the vectors $\mathbf{x}$ for which $A\mathbf{x} = \mathbf{0}$, and it is denoted by null(A). This means that to check to see if a vector $\mathbf{x}$ is in the null space we need only to compute $A\mathbf{x}$ and see if it is the zero vector. Use this method to determine whether either of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is in null(A). Give your answer as a brief sentence.

8.5 Least Squares Solutions to Inconsistent Systems

Performance Criterion:

8. (f) Find the least-squares approximation to the solution of an inconsistent system of equations. Solve a problem using least-squares approximation.

   (g) Give the least squares error and least squares error vector for a least squares approximation to a solution to a system of equations.

Recall that an inconsistent system is one for which there is no solution. Often we wish to solve inconsistent systems and it is just not acceptable to have no solution. In those cases we can find some vector (whose components are the values we are trying to find when attempting to solve the system) that is "closer to being a solution" than all other vectors. The theory behind this process is part of the second term of this course, but we now have enough knowledge to find such a vector in a "cookbook" manner.

Suppose that we have a system of equations $A\mathbf{x} = \mathbf{b}$. Pause for a moment to reflect on what we know and what we are trying to find when solving such a system: We have a system of linear equations, and the entries of A are the coefficients of all the equations. The vector $\mathbf{b}$ is the vector whose components are the right sides of all the equations, and the vector $\mathbf{x}$ is the vector whose components are the unknown values of the variables we are trying to find. So we know A and $\mathbf{b}$ and we are trying to find $\mathbf{x}$. If A is invertible, the solution vector $\mathbf{x}$ is given by $\mathbf{x} = A^{-1}\mathbf{b}$. If A is not invertible there will be no solution vector $\mathbf{x}$, but we can usually find a vector $\bar{\mathbf{x}}$ (usually spoken as "ex-bar") that comes "closest" to being a solution. Here is the formula telling us how to find that $\bar{\mathbf{x}}$:

<u>Theorem 8.5.1</u>: **The Least Squares Theorem:** Let A be an $m \times n$ matrix and let $\mathbf{b}$ be in $\mathbb{R}^m$. If $A\mathbf{x} = \mathbf{b}$ has a **least squares solution** $\bar{\mathbf{x}}$, it is given by

$$\bar{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$$

◇ **Example 8.5(a):** Find the least squares solution to

$$\begin{aligned} 1.3x_1 + 0.6x_2 &= 3.3 \\ 4.7x_1 + 1.5x_2 &= 13.5 \\ 3.1x_1 + 5.2x_2 &= -0.1 \end{aligned}.$$

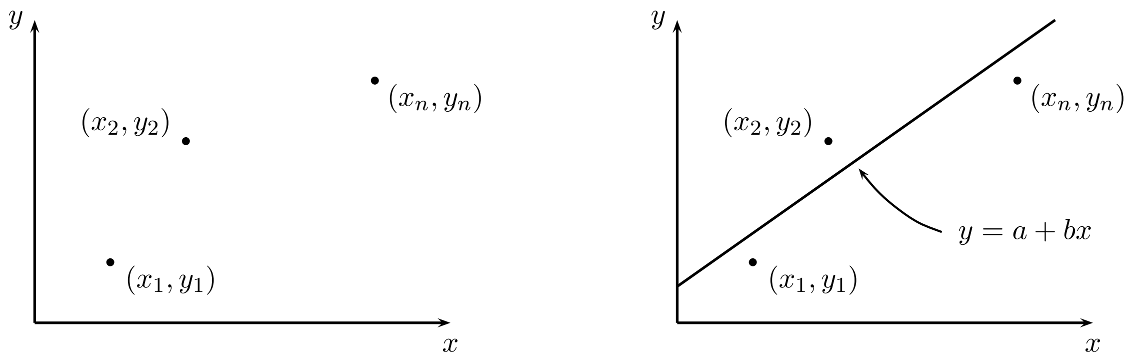
First we note that if we try to solve by row reduction we get no solution; this is an overdetermined system because there are more equations than unknowns. The matrix A and vector $\mathbf{b}$ are

$$A = \begin{bmatrix} 1.3 & 0.6 \\ 4.7 & 1.5 \\ 3.1 & 5.2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3.3 \\ 13.5 \\ -0.1 \end{bmatrix}$$

Using a calculator or $MATLAB$, we get

$$\bar{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} 3.5526 \\ -2.1374 \end{bmatrix}$$

A classic example of when we want to do something like this is when we have a bunch of (x, y) data pairs from some experiment, and when we graph all the pairs they describe a trend. We then want to find a simple function $y = f(x)$ that best models that data. In some cases that function might be a line, in other cases maybe it is a parabola, and in yet other cases it might be an exponential function. Let's try to make the connection between this and linear algebra. Suppose that we have the data points (x_1, y_1), (x_2, y_2), ..., (x_n, y_n), and when we graph these points they arrange themselves in roughly a line, as shown below and to the left. We then want to find an equation of the form $a + bx = y$ (note that this is just the familiar $y = mx + b$ rearranged and with different letters for the slope and y-intercept) such that $a + bx_i \approx y_i$ for $i = 1, 2, ..., n$, as shown below and to the right.

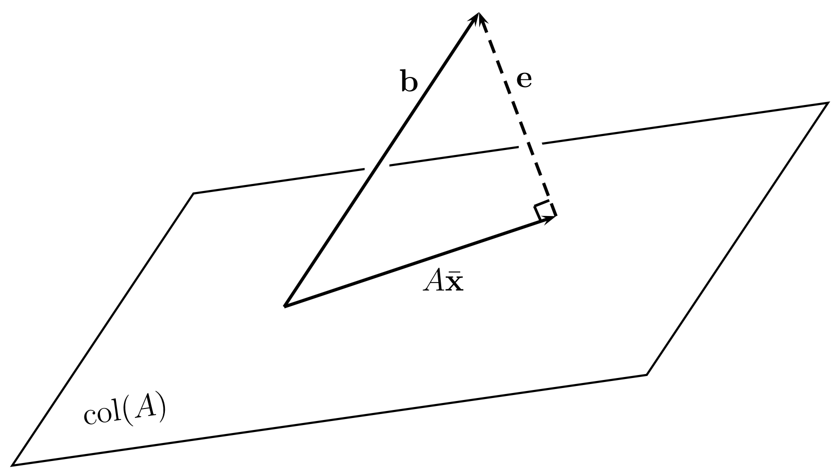


If we substitute each data pair into $a + bx = y$ we get a system of equations which can be thought of in several different ways. Remember that all the x_i and y_i are known values - the unknowns are a and b.

$$\begin{array}{c} a + x_1b = y_1 \\ a + x_2b = y_2 \\ \vdots \\ a + x_nb = y_n \end{array} \iff \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \\ 1 & x_n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \iff A\mathbf{x} = \mathbf{b}$$

Above we first see the system that results from putting each of the (x_i, y_i) pairs into the equation $a + bx = y$. After that we see the $A\mathbf{x} = \mathbf{b}$ form of the system. We must be careful of the notation here. A is the matrix whose columns are a vector in $\mathbb{R}^n$ consisting of all ones and a vector whose components are the x_i values. It would be logical to call this last vector $\mathbf{x}$, but instead $\mathbf{x}$ is the vector $\begin{bmatrix} a \\ b \end{bmatrix}$. $\mathbf{b}$ is the column vector whose components are the y_i values. Our task, as described by this interpretation, is to *find a vector* $\mathbf{x}$ *in* $\mathbb{R}^2$ *that* A *transforms into the vector* $\mathbf{b}$ *in* $\mathbb{R}^n$. Even if such a vector did exist, it couldn't be given as $\mathbf{x} = A^{-1}\mathbf{b}$ because A is not square, so can't be invertible. However, it is likely no such vector exists, but we *CAN* find the least-squares vector $\bar{\mathbf{x}} = \begin{bmatrix} a \\ b \end{bmatrix} = (A^TA)^{-1}A^T\mathbf{b}$. When we do, its components a and b are the intercept and slope of our line.

Theoretically, here is what is happening. Least squares is generally used in situations that are **over determined**. This means that there is too much information and it is bound to "disagree" with itself somehow. In terms of systems of equations, we are talking about cases where there are more equations than unknowns. Now the fact that the system $A\mathbf{x} = \mathbf{b}$ has no solution means that $\mathbf{b}$ is not in the column space of A. The least squares solution to $A\mathbf{x} = \mathbf{b}$ is simply the vector $\bar{\mathbf{x}}$ for which $A\bar{\mathbf{x}}$ is the projection of $\mathbf{b}$ onto the column space of A. This is shown simplistically below, for the situation where the column space is a plane in $\mathbb{R}^3$.



To recap a bit, suppose we have a system of equations $A\mathbf{x} = \mathbf{b}$ where there is no vector $\mathbf{x}$ for which $A\mathbf{x}$ equals $\mathbf{b}$. What the least squares approximation allows us to do is to find a vector $\bar{\mathbf{x}}$ for which $A\bar{\mathbf{x}}$ is as "close" to $\mathbf{b}$ as possible. We generally determine "closeness" of two objects by finding the difference between them. Because both $A\bar{\mathbf{x}}$ and $\mathbf{b}$ are both vectors of the same length, we can subtract them to get a vector $\mathbf{e}$ that we will call the **error vector**, shown above. The **least squares error** is then the magnitude of this vector:

DEFINITION 8.5.2: If $\bar{\mathbf{x}}$ is the least-squares solution to the system $A\mathbf{x} = \mathbf{b}$, the **least squares error vector** is

$$\vec{\varepsilon} = \mathbf{b} - A\bar{\mathbf{x}}$$

and the **least squares error** is the magnitude of $\vec{\varepsilon}$.

◇ **Example 8.5(b):** Find the least squares error vector and least squares error vector for the solution obtained in Example 8.5(a).

The least squares error vector is

$$\vec{\varepsilon} = \mathbf{b} - A\bar{\mathbf{x}} = \begin{bmatrix} 3.3 \\ 13.5 \\ -0.1 \end{bmatrix} - \begin{bmatrix} 1.3 & 0.6 \\ 4.7 & 1.5 \\ 3.1 & 5.2 \end{bmatrix} \begin{bmatrix} 3.5526 \\ -2.1374 \end{bmatrix} = \begin{bmatrix} -0.0359 \\ 0.0089 \\ 0.0016 \end{bmatrix}$$

The least squares error is $\|\vec{\varepsilon}\| = 0.0370$.

Section 8.5 Exercises

1. Find the least squares approximating parabola for the points $(1, 8)$, $(2, 7)$, $(3, 5)$, $(4, 2)$. **Give the system of equations to be solved (in any form), and give the equation of the parabola.**

8.6 Chapter 8 Exercises

1. (a) Give a set of three non-zero vectors in $\mathbb{R}^3$ whose span is a line.

   (b) Suppose that you have a set of two non-zero vectors in $\mathbb{R}^3$ that are not scalar multiples of each other. What is their span? How can you create a new vector that is not a scalar multiple of either of the other two vectors but, when added to the set, does not increase the span?

   (c) How many vectors need to be in a set for it to have a chance of spanning all of $\mathbb{R}^3$?

2. Give a set of nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^2$ that **DOES NOT** span $\mathbb{R}^2$. Then give a third vector $\mathbf{v}_3$ so that all three vectors **DO** span $\mathbb{R}^2$.

3. Give a set of three vectors, with no one being a scalar multiple of just one other, that span the xy-plane in $\mathbb{R}^3$.

3. The things in a set are called *elements*. The union of two sets A and B is a new set C consisting of every element of A along with every element of B and nothing else. (If something is an element of both A and B, it is only included in C once.) Every subspace of an $\mathbb{R}^n$ is a subset of that $\mathbb{R}^n$ that possesses some additional special properties. Show that the union of two subspaces is not generally a subspace by giving a specific $\mathbb{R}^n$ and two specific subspaces, then showing that the union is not a subspace.

9 Bases of Subspaces

Outcome:

9. Understand bases of vector spaces and subspaces.

   Performance Criteria:

   (a) Determine whether a set $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$ of vectors is a linearly independent or linearly dependent. If the vectors are linearly dependent, (1) give a linear combination of them that equals the zero vector, (2) give any one (that is possible) as a linear combination of the others.

   (b) Determine whether a given set of vectors is a basis for a given subspace. Give a basis and the dimension of a subspace.

   (c) Find the dimensions of and bases for the column space and null space of a given matrix.

   (d) Given the dimension of the column space and/or null space of the coefficient matrix for a system of equations, say as much as you can about how many solutions the system has.

   (e) Determine, from given information about the coefficient matrix A and vector $\mathbf{b}$ of a system $A\mathbf{x} = \mathbf{b}$, whether the system has any solutions and, if it does, whether there is more than one solution.

A very important concept in linear algebra is that all vectors of interest in a given situation can be constructed out of a small set of vectors, using linear combinations. That is the key idea that we will explore in this chapter. This will seem to take us farther from some of the more concrete ideas that we have used in applications, but these ideas have huge value in a practical sense as well.

9.1 Linear Independence

Performance Criterion:

9. (a) Determine whether a set $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$ of vectors is a linearly independent or linearly dependent. If the vectors are linearly dependent, (1) give a non-trivial linear combination of them that equals the zero vector, (2) give any one as a linear combination of the others, when possible.

Suppose that we are trying to create a set $\mathcal{S}$ of vectors that spans $\mathbb{R}^3$. We might begin with one vector, say $\mathbf{u}_1 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$, in $\mathcal{S}$. We know by now that the span of this single vector is all scalar multiples of it, which is a line in $\mathbb{R}^3$. If we wish to increase the span, we would add another vector to $\mathcal{S}$. If we were to add a vector like $\begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix}$ to $\mathcal{S}$, we would not increase the span, because this new vector is a scalar multiple of $\mathbf{u}_1$, so it is on the line we already have and would contribute nothing new to the span of $\mathcal{S}$. To increase the span, we need to add to $\mathcal{S}$ a second vector $\mathbf{u}_2$ that is not a scalar multiple of the vector $\mathbf{u}_1$ that we already have. It should be clear that the vector $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not a scalar multiple of $\mathbf{u}_1$, so adding it to $\mathcal{S}$ would increase it's span.

The span of $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2\}$ is a plane. When $\mathcal{S}$ included only a single vector, it was relatively easy to determine a second vector that, when added to $\mathcal{S}$, would increase it's span. Now we wish to add a third vector to $\mathcal{S}$ to further increase its span. Geometrically it is clear that we need a third vector that is *not in the plane spanned by* $\{\mathbf{u}_1, \mathbf{u}_2\}$. Probabilistically, just about any vector in $\mathbb{R}^3$ would do, but what we would like to do here is create an algebraic condition that needs to be met by a third vector so that adding it to $\mathcal{S}$ will increase the span of $\mathcal{S}$.

Let's begin with what we *DON'T* want: we don't want the new vector to be in the plane spanned by $\{\mathbf{u}_1, \mathbf{u}_2\}$. Now every vector $\mathbf{v}$ in that plane is of the form $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$ for some scalars c_1 and c_2. We say the vector $\mathbf{v}$ created this way is "dependent" on $\mathbf{u}_1$ and $\mathbf{u}_2$, and that is what causes it to not be helpful in increasing the span of a set that already contains those two vectors. Assuming that neither of c_1 and c_2 is zero, we could also write

$$\mathbf{u}_1 = \frac{c_2}{c_1}\mathbf{u}_2 - \frac{1}{c_1}\mathbf{v} \qquad \text{and} \qquad \mathbf{u}_2 = \frac{c_1}{c_2}\mathbf{u}_1 - \frac{1}{c_2}\mathbf{v},$$

showing that $\mathbf{u}_1$ is "dependent" on $\mathbf{u}_2$ and $\mathbf{v}$, and $\mathbf{u}_2$ is "dependent" on $\mathbf{u}_1$ and $\mathbf{v}$. So whatever "dependent" means (we'll define it more formally soon) all three vectors are dependent on each other. We can create another equation that is equivalent to all three of the ones given so far, and that does not "favor" any particular one of the three vectors:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{v} = \mathbf{0},$$

where $c_3 = -1$.

Of course, if we want a third vector $\mathbf{u}_3$ to add to $\{\mathbf{u}_1, \mathbf{u}_2\}$ to increase its span, we would not want to choose $\mathbf{u}_3 = \mathbf{v}$; instead we would want a third vector that is "independent" of the two we already have. Based on what we have been doing, we would suspect that we would want

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 \neq \mathbf{0}. \tag{1}$$

Of course even if $\mathbf{u}_3$ was not in the plane spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$, (1) would be true if $c_1 = c_2 = c_3 = 0$, but we want that to be the only choice of scalars that makes (1) true.

We now make the following definition, based on our discussion:

<u>Definition 9.1.1</u>: **Linear Dependence and Independence**

A set $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ of vectors is **linearly dependent** if there exist scalars $c_1, c_2, ..., c_k$, *not all equal to zero* such that

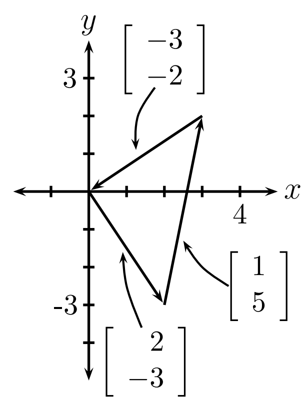
$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}. \tag{2}$$

If (2) only holds for $c_1 = c_2 = \cdots = c_k = 0$ the set $\mathcal{S}$ is **linearly independent**.

We can state linear dependence (independence) in either of two ways. We can say that the set is linearly dependent, or the vectors are linearly dependent. Either way is acceptable. Often we will get lazy and leave off the "linear" of linear dependence or linear independence. This does no harm, as there is no other kind of dependence/independence that we will be interested in.

Let's explore the idea of linearly dependent vectors a bit more by first looking at a specific example; consider the following sum of vectors in $\mathbb{R}^2$:

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3}$$

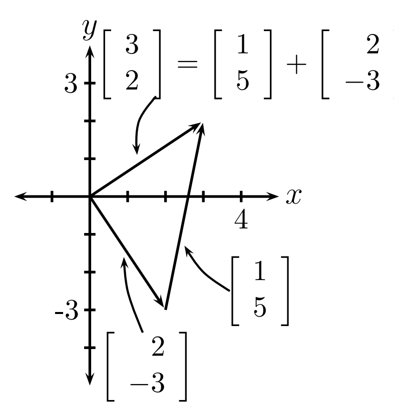


The picture to the right gives us some idea of what is going on here. Recall that when adding two vectors by the tip-to-tail method, the sum is the vector from the tail of the first vector to the tip of the second. We can add three vectors in the same way, putting the tail of the second at the tip of the first, and the tail of the third at the tip of the second. The sum is then the vector from the tail of the first vector to the tip of the third; in this case it is the zero vector since both the tail of the first vector and the tip of the third are at the origin.

Letting $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, equation (3) above becomes

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = 0,$$

where $c_1 = c_2 = c_3 = 1$. Therefore the three vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ are linearly dependent.



Now if we add the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ to both sides of equation (3) we obtain the equation

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The geometry of this equation can be seen in the second picture on the previous page. We have basically "reversed" the vector $\begin{bmatrix} -3 \\ -2 \end{bmatrix}$, and we can now see that the "reversed" vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is a linear combination of the two vectors $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$. This indicates that if three vectors are linearly dependent, then one of them can be written as a linear combination of the others.

Let's consider the more general case of a set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ of linearly dependent vectors in $\mathbb{R}^n$. By definitions, there are scalars $c_1, c_2, ..., c_k$, not all equal to zero, such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}$$

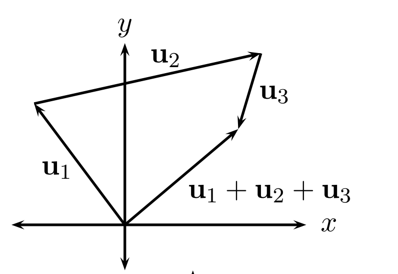
Let c_j, for some j between 1 and k, be one of the non-zero scalars. (By definition there has to be at least one such scalar.) Then we can do the following:

$$\begin{aligned}
c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_j\mathbf{u}_j + \cdots + c_k\mathbf{u}_k &= \mathbf{0} \\
c_j\mathbf{u}_j &= -c_1\mathbf{u}_1 - c_2\mathbf{u}_2 - \cdots - c_k\mathbf{u}_k \\
\mathbf{u}_j &= -\tfrac{c_1}{c_j}\mathbf{u}_1 - \tfrac{c_2}{c_j}\mathbf{u}_2 - \cdots - \tfrac{c_k}{c_j}\mathbf{u}_k \\
\mathbf{u}_j &= d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + \cdots + d_k\mathbf{u}_k
\end{aligned}$$

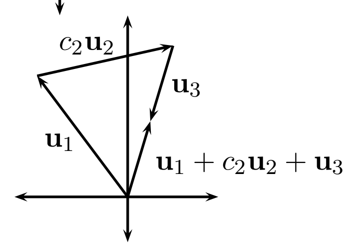
This, along with the previous specific example in $\mathbb{R}^2$, gives us the following:

> <u>THEOREM 9.1.2</u>: If a set $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is linearly dependent, then at least one of these vectors can be written as a linear combination of the remaining vectors.

The importance of this, which we'll reiterate again later, is that *if we have a set of linearly dependent vectors with a certain span, we can eliminate at least one vector from our original set without reducing the span of the set.* If, on the other hand, we have a set of linearly *independent* vectors, eliminating any vector from the set will reduce the span of the set.

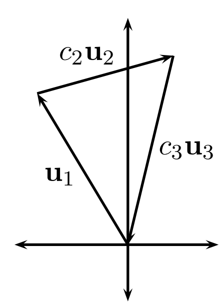


We now consider three vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ in $\mathbb{R}^2$ whose sum is not the zero vector, and for which no two of the vectors are parallel. I have arranged these to show the tip-to-tail sum in the top diagram to the right; clearly their sum is not the zero vector.



At this point if we were to multiply $\mathbf{u}_2$ by some scalar c_2 less than one we could shorten it to the point that after adding it to $\mathbf{u}_1$ the tip of $c_2\mathbf{u}_2$ would be in such a position as to line up $\mathbf{u}_3$ with the origin. This is shown in the bottom diagram to the right.

Finally, we could then multiply $\mathbf{u}_3$ by a scalar c_3 greater than one to lengthen it to the point of putting its tip at the origin. We would then have $\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$. You should play around with a few pictures to convince yourself that this can always be done with three vectors in $\mathbb{R}^2$, as long as none of them are parallel (scalar multiples of each other). This shows us that *any three vectors in $\mathbb{R}^2$ are always linearly dependent.* In fact, we can say even more:



$\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$

> Theorem 9.1.3: Any set of more than n vectors in $\mathbb{R}^n$ must be linearly dependent.

Let's start looking at some specific examples now.

◇ **Example 9.1(a):** Determine whether the vectors $\begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix}$ are linearly dependent, or linearly independent. If they are dependent, give a non-trivial linear combination of them that equals the zero vector. (Non-trivial means that not all of the scalars are zero!)

To make such a determination we always begin with the vector equation from the definition:

$$c_1 \begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

We recognize this as the linear combination form of a system of equations that has the augmented matrix shown below and to the left, which reduces to the matrix shown below and to the right.

$$\begin{bmatrix} -1 & 1 & 7 & 0 \\ -7 & -3 & -1 & 0 \\ 3 & 2 & 4 & 0 \\ 11 & 5 & 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we see that there are infinitely many solutions, so there are certainly values of c_1, c_2 and c_3, not all zero, that make (4) true, so the set of vectors is linearly dependent. To find a non-trivial linear combination of the vectors that equals the zero vector we let the free variable c_3 *be any value other than zero.* (You should try letting it be zero to see what happens.) If we take c_3 to be one, then $c_2 = -5$ and $c_1 = 2$. Then

$$2 \begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix} - 5 \begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -14 \\ 6 \\ 22 \end{bmatrix} + \begin{bmatrix} -5 \\ 15 \\ -10 \\ -25 \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

◇ **Example 9.1(b):** Determine whether the vectors $\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 5 \\ -1 \end{bmatrix}$ are linearly dependent, or linearly independent. If they are dependent, give a non-trivial linear combination of them that equals the zero vector. (Non-trivial means that not all of the scalars are zero!)

To make such a determination we always begin with the vector equation from the definition:

$$c_1 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We recognize this as the linear combination form of a system of equations that has the augmented matrix shown below and to the left, which reduces to the matrix shown below and to the right.

$$\begin{bmatrix} 3 & 4 & -2 & 0 \\ -1 & 7 & 5 & 0 \\ 2 & 0 & -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We see that the only solution to the system is $c_1 = c_2 = c_3 = 0$, so the vectors are linearly independent.

A comment is in order at this point. The system $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}$ is homogeneous, so it will always have at least the zero vector as a solution. It is precisely *when the only solution is the zero vector that the vectors are linearly independent*.

Here's an example demonstrating the fact that if a set of vectors is linearly dependent, at least one of them can be written as a linear combination of the others:

◇ **Example 9.1(c):** In Example 9.1(a) we determined that the vectors $\begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix}$ are linearly dependent. Give one of them as a linear combination of the others.

When testing for linear dependence we found that we could write

$$2\begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix} - 5\begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

The easiest vector to solve for is the third, by simply subtracting the other two and their scalars from both sides of this equation:

$$\begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix} = -2\begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix} + 5\begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix}$$

However, going back to (5) we could have instead subtracted the second and third vectors and their scalars from both sides, then multiplied both sides by $\frac{1}{2}$ to get

$$\begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix} = \frac{5}{2}\begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix}$$

Of course we could also solve for the second vector in a similar manner!

Section 9.1 Exercises

1. Consider the vectors $\mathbf{u}_1 = \begin{bmatrix} -5 \\ 9 \\ 4 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix}$.

   (a) Give the *VECTOR* equation that we must consider in order to determine whether the three vectors are linearly independent.

   (b) Your equation has one solution for sure. What is it? What does it mean (in terms of linear dependence or independence) if that is the *ONLY* solution?

   (c) Write your equation from (a) as a system of linear equations. Then give the augmented matrix for the system.

   (d) Does the system have more solutions than the one you gave in (b)? If so, find one of them. (By "one" I mean one ordered triple of three numbers.)

   (e) Find each of the three vectors as a linear combination of the other two.

2. Show that the vectors $\mathbf{u} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 5 \\ 1 \\ -6 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -4 \\ 4 \\ 9 \end{bmatrix}$ are linearly dependent. Then give one of the vectors as a linear combination of the others.

3. For the following, use the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$.

   (a) Determine whether $\mathbf{u} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 \\ 17 \\ -17 \end{bmatrix}$ are in span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

   (b) Show that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent by the definition of linearly dependent. In other words, produce scalars c_1, c_2 and c_3 and demonstrate that they and the vectors satisfy the equation given in the definition.

   (c) Since the vectors are linearly dependent, at least one the vectors can be expressed as a linear combination of the other two. Express $\mathbf{v}_1$ as a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$.

9.2 Bases of Subspaces, Dimension

Performance Criterion:

9. (b) Determine whether a given set of vectors is a basis for a given subspace. Give a basis and the dimension of a subspace.

We have seen that the span of any set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$. In a sense, the vectors whose span is being considered are the "building blocks" of the subspace. That is, every vector in the subspace is some linear combination of those vectors. Now, recall that if a set of vectors is linearly dependent, one of the vectors can be represented as a linear combination of the others. So if we are considering the span of a set of dependent vectors, we can throw out the one that can be obtained as a linear combination without affecting the span of the set of vectors.

So given a subspace, it is desirable to find what we might call a *minimal spanning set*, the smallest set of vectors whose linear combinations gives the entire subspace. Such a set is called a **basis**.

DEFINITION 9.2.1: **Basis of a Subspace**

For a subspace S, a **basis** is a set $\mathcal{B}$ of vectors such that

- the span of $\mathcal{B}$ is S,
- the vectors in $\mathcal{B}$ are linearly independent

The plural of basis is *bases* (pronounced "base-eez"). With a little thought, you should believe that *every subspace has infinitely many bases.* (This is a tiny lie - the trivial subspace consisting of just the zero has no basis vectors, which is a funny consequence of logic.)

◇ **Example 9.2(a):** Is the set $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

Clearly for any vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ we have $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, so the span of $\mathcal{B}$ is all of $\mathbb{R}^3$. The augmented matrix for testing for linear independence is simply the identity augmented with the zero vector, giving only the solution where all the scalars are zero, so the vectors are linearly independent. Therefore the set $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

The basis just given is called the **standard basis** for $\mathbb{R}^3$, and its vectors are often denoted by $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. There is a standard basis for every $\mathbb{R}^n$, and $\mathbf{e}_1$ is always the vector whose first component is one and all others are zero, $\mathbf{e}_2$ is the vector whose second component is one and all others are zero, and so on.

◇ **Example 9.2(b):** Let $S_1 = \left\{ \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ -4 \end{bmatrix} \right\}$. In the previous section we saw that $\text{span}(S_1)$ is a subspace of $\mathbb{R}^3$. Is S_1 a basis for $\text{span}(S_1)$?

Clearly S_1 meets the first condition for being a basis and, since we can see that neither of these vectors is a scalar multiple of the other, they are linearly independent. Therefore they are a basis for span(S_1).

◇ **Example 9.2(c):** Let $S_2 = \left\{ \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ -9 \end{bmatrix} \right\}$. Span$(S_2)$ is a subspace of $\mathbb{R}^3$; Is S_2 a basis for span(S_2)?

Once again this set meets the first condition of being a subspace. We can also see that $(-1)\begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -9 \end{bmatrix}$, so the set S_2 is linearly dependent. Therefore it is NOT a basis for span(S).

◇ **Example 9.2(d):** The yz-plane in $\mathbb{R}^3$ is a subspace. Give a basis for this subspace.

We know that a set of two linearly independent vectors will span a plane, so we simply need two vectors in the yz-plane that are not scalar multiples of each other. The simplest choices are the two vectors $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, so they are a basis for the yz-plane.

Considering this last example, it is not hard to show that the set $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is also a basis for the yz-plane, and there are many other sets that are bases for that plane as well. All of those sets will contain two vectors, illustrating the fact that *every basis of a subspace has the same number of vectors*. This allows us to make the following definition:

Definition 9.2.2: Dimension of a Subspace

The **dimension** of a subspace is the number of elements in a basis for that subspace.

Looking back at Examples 9.2(a), (b) and (d), we then see that $\mathbb{R}^3$ has dimension three, and $\text{span}(S_1)$ has dimension two, and the yz-plane in $\mathbb{R}^3$ has dimension two.

Although its importance may not be obvious to you at this point, here's why we care about a basis rather than any set that spans a subspace:

Theorem 9.2.3: Any vector in a subspace S with basis $\mathcal{B}$ is represented by one, and only one, linear combination of vectors in $\mathcal{B}$.

◇ **Example 9.2(e):** In Example 8.3(d) we determined that the set of all vectors of the form $\mathbf{x} = \begin{bmatrix} a \\ a+b \\ b \\ a-b \end{bmatrix}$, for some real numbers a and b, is a subspace of $\mathbb{R}^4$. Give a basis for that subspace.

The key computation in Example 8.3(d) was

$$\mathbf{x} = \begin{bmatrix} a \\ a+b \\ b \\ a-b \end{bmatrix} = \begin{bmatrix} a \\ a \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ b \\ -b \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

The set of vectors under consideration is spanned by $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$, and we can see that those two vectors are linearly independent (because they aren't scalar multiples of each other, which is sufficient for independence when considering just two vectors). Therefore they form a basis for the subspace of vectors of the given form.

Section 9.2 Exercises

1. For each of the following sub*sets* of $\mathbb{R}^3$, think of each point as a position vector; each set then becomes a set of vectors rather than points. For each,

   - determine whether the set is a sub*space* and
   - if it is *NOT* a subspace, give a reason why it isn't by doing one of the following:
     - ◇ stating that the set does not contain the zero vector
     - ◇ giving a vector that is in the set and a scalar multiple that isn't (show that it isn't)
     - ◇ giving two vectors that in the set and showing that their sum is not in the set
   - if it *IS* a subspace, give a basis for the subspace.

   (a) All points on the horizontal plane at $z = 3$.

   (b) All points on the xz-plane.

   (c) All points on the line containing $\mathbf{u} = [-3, 1, 4]$.

   (d) All points on the lines containing $\mathbf{u} = [-3, 1, 4]$ and $\mathbf{v} = [5, 0, 2]$.

   (e) All points for which $x \geq 0, y \geq 0$ and $z \geq 0$.

(f) All points $\mathbf{x}$ given by $\mathbf{x} = \mathbf{w} + s\mathbf{u} + t\mathbf{v}$, where $\mathbf{w} = [1, 1, 1]$ and $\mathbf{u}$ and $\mathbf{v}$ are as in (d).

(g) All points $\mathbf{x}$ given by $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are as in (d).

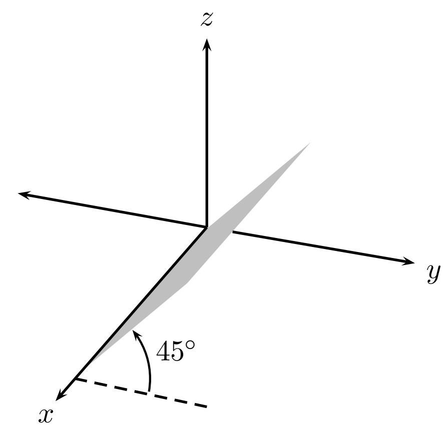
(h) The vector $\mathbf{0}$.

(i) All of $\mathbb{R}^3$.

2. Determine whether each of the following is a subspace. If not, give an appropriate counterexample; if so, give a basis for the subspace.

   (a) The subset of $\mathbb{R}^2$ consisting of all vectors on or to the right of the y-axis.

   (b) The subset of $\mathbb{R}^3$ consisting of all vectors in a plane containing the x-axis and at a 45 degree angle to the xy-plane. See diagram to the right.



3. The xy-plane is a subspace of $\mathbb{R}^3$.

   (a) Give a set of at least two vectors in the xy-plane that is not a basis for that subspace, and tell why it isn't a basis.

   (b) Give a different set of at least two vectors in the xy-plane that is not a basis for that subspace *for a different reason*, and tell why it isn't a basis.

9.3 Bases for the Column Space and Null Space of a Matrix

Performance Criteria:

9. (c) Find the dimension and bases for the column space and null space of a given matrix.

   (d) Given the dimension of the column space and/or null space of the coefficient matrix for a system of equations, say as much as you can about how many solutions the system has.

In a previous section you learned about two special subspaces related to a matrix A, the column space of A and the null space of A. Remember the importance of those two spaces:

A system $A\mathbf{x} = \mathbf{b}$ has a solution if, and only if, $\mathbf{b}$ is in the column space of A.

If the null space of a square matrix A is just the zero vector, A is invertible and $A\mathbf{x} = \mathbf{b}$ has a unique solution for any vector $\mathbf{b}$.

We would now like to be able to find bases for the column space and null space of a given vector A. The following describes how to do this:

Theorem 9.3.1: **Bases for Null Space and Column Space**

- A basis for the column space of a matrix A is the columns of A corresponding to columns of $rref(A)$ that contain leading ones.
- The solution to $A\mathbf{x} = \mathbf{0}$ (which can be easily obtained from $rref(A)$ by augmenting it with a column of zeros) will be an arbitrary linear combination of vectors. Those vectors form a basis for null(A).

◇ **Example 9.3(a):** Find bases for the null space and column space of $A = \begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & 7 & 1 & 4 \\ -2 & 1 & 7 & 7 \end{bmatrix}$.

The reduced row-echelon form of A is shown below and to the left. We can see that the first through third columns contain leading ones, so a basis for the column space of A is the set shown below and to the right.

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} \right\}$$

If we were to augment A with a column of zeros to represent the system $A\mathbf{x} = \mathbf{0}$ and row reduce we'd get the matrix shown above and to the left but with an additional column of zeros on the right. We'd then have x_4 as a free variable t, with $x_1 = -3t$, $x_2 = 7t$ and $x_3 = -4t$. The solution to $A\mathbf{x} = \mathbf{0}$ is any scalar multiple of $\begin{bmatrix} -3 \\ 7 \\ -4 \\ 1 \end{bmatrix}$, so that vector is a basis for the null space of A.

◇ **Example 9.3(b):** Find a basis for the null space and column space of $A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 7 & 1 \\ -2 & 1 & 7 \end{bmatrix}$.

The reduced row-echelon form of this matrix is the identity, so a basis for the column space consists of all the columns of A. If we augment A with the zero vector and row reduce we get a solution of the zero vector, so the null space is just the zero vector (which is of course a basis for itself).

We should note in the last example that the column space is all of $\mathbb{R}^3$, so for any vector $\mathbf{b}$ in $\mathbb{R}^3$ there is a vector $\mathbf{x}$ for which $A\mathbf{x} = \mathbf{b}$. Thus $A\mathbf{x} = \mathbf{b}$ has a solution for every choice of $\mathbf{b}$.

There is an important distinction to be made between a subspace and a basis for a subspace:

- Other than the trivial subspace consisting of the zero vector, *a subspace is an infinite set of vectors.*
- A basis for a subspace *is a finite set of vectors.* In fact a basis consists of relatively few vectors; the basis for any subspace of $\mathbb{R}^n$ contains at most n vectors (and it only contains n vectors if the subspace is all of $\mathbb{R}^n$).

To illustrate, consider the matrix $A = \begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & 7 & 1 & 4 \\ -2 & 1 & 7 & 7 \end{bmatrix}$ from Example 9.3(a). The set $\left\{ \begin{bmatrix} -3 \\ 7 \\ -4 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space of A, whereas the set $\left\{ t \begin{bmatrix} -3 \\ 7 \\ -4 \\ 1 \end{bmatrix} \right\}$ *IS* the null space of A.

We finish this section with a couple definitions and a major theorem of linear algebra. The importance of these will be seen in the next section.

<u>Definition 9.3.2</u>: **Rank and Nullity of a Matrix**

- The **rank** of a matrix A, denoted $\text{rank}(A)$, is the dimension of its column space.
- The **nullity** of a matrix A, denoted $\text{nullity}(A)$, is the dimension of its null space.

> Theorem 9.3.3: **The Rank Theorem**
>
> For an $m \times n$ matrix A, $\text{rank}(A) + \text{nullity}(A) = n$.

Section 9.3 Exercises

1. Consider the matrix $A = \begin{bmatrix} 1 & 1 & -2 \\ -3 & -3 & 6 \\ 2 & 2 & -4 \end{bmatrix}$, which has row-reduced form $\begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. In this exercise you will see how to find a basis for the null space of A. All this means is that *you are looking for a "minimal" set of vectors whose span (all possible linear combinations of them) give all the vectors* $\mathbf{x}$ *for which* $A\mathbf{x} = \mathbf{0}$.

   (a) Give the augmented matrix for the system of equations $A\mathbf{x} = \mathbf{0}$, then give its row reduced form.

   (b) There are two free variables, x_3 and x_2. Let $x_3 = t$ and $x_2 = s$, then find x_1 (in terms of s and t). Give the vector $\mathbf{x}$, in terms of s and t.

   (c) Write $\mathbf{x}$ as the sum of two vectors, one containing only the parameter s and the other containing only the parameter t. Then factor s out of the first vector and t out of the second vector. You now have $\mathbf{x}$ as all linear combinations of two vectors.

   (d) Those two vectors are linearly independent, since neither of them is a scalar multiple of the other, so both are essential in the linear combination you found in (c). They then form a basis for the null space of A. Write this out as a full sentence, "A basis for ...". *A basis is technically a set of vectors, so use the set brackets* $\{\ \ \}$ *appropriately.*

2. Consider the matrix $A = \begin{bmatrix} 1 & -1 & 5 \\ 3 & 1 & 11 \\ 2 & 5 & 3 \end{bmatrix}$

   (a) Solve the system $A\mathbf{x} = \mathbf{0}$. You should get infinitely many solutions containing one or more parameters. Give the general solution, in terms of the parameters. **Give all values in exact form**.

   (b) If you didn't already, you should be able to give the general solution as a linear combination of vectors, with the scalars multiplying them being the parameter(s). Do this.

   (c) The vector or vectors you see in (c) is (are) a basis for the null space of A. Give the basis.

3. When doing part (a) of the previous exercise you should have obtained the row reduced form of the matrix A (of course you augmented it). A basis for the column space of A is the columns of A (*NOT* the columns of the row reduced form of A!) corresponding to the leading variables in the row reduced form of A. Give the basis for the column space of A.

4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 2 \\ 9 \\ -17 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 15 \\ 2 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 8 \\ -8 \\ -4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}.$$

(a) Determine whether each of $\mathbf{u}_1$ and $\mathbf{u}_2$ is in the column space of A.

(b) Find a basis for col(A). **Give your answer with a brief sentence, and indicate that the basis is a set of vectors.**

(c) One of the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ *IS* in the column space of A. Give a linear combination of the *basis vectors* that equals that vector.

5. Again let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 2 \\ 9 \\ -17 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 15 \\ 2 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 8 \\ -8 \\ -4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}.$$

(a) Determine whether each of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is in null(A). Give your answer as a brief sentence.

(b) Determine a basis for null(A), giving your answer in a brief sentence.

(c) Give the linear combinations of the basis vectors of the null space for either of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ that are in the null space.

6. Give a sentence telling the dimensions of the column space and null space of the matrix A from the previous two exercises.

9.4 Solutions to Systems of Equations

Performance Criterion:

9. (e) Determine, from given information about the coefficient matrix A and vector $\mathbf{b}$ of a system $A\mathbf{x} = \mathbf{b}$, whether the system has any solutions and, if it does, whether there is more than one solution.

You may have found the last section to be a bit overwhelming, and you are probably wondering why we bother with all of the definitions in that section. The reason is that those ideas form tools and language for discussing whether a system of equations

(a) has a solution (meaning at least one) and

(b) if it does have a solution, is there only one.

Item (a) above is what mathematicians often refer to as the *existence* question, and item (b) is the *uniqueness* question. Concerns with "existence and uniqueness" of solutions is not restricted to linear algebra; it is a big deal in the study of differential equations as well.

Consider a system of equations $A\mathbf{x} = \mathbf{b}$. We saw previously that the product $A\mathbf{x}$ is the linear combination of the columns of A with the components of $\mathbf{x}$ as the scalars of the linear combination. This means that the system will only have a solution if $\mathbf{b}$ is a linear combination of the columns of A. But all of the linear combinations of the columns of A is just the span of those columns - the column space! the conclusion of this is as follows:

A system of equations $A\mathbf{x} = \mathbf{b}$ has a solution (meaning *at least* one solution) if, and only if, $\mathbf{b}$ is in the column space of A.

Let's look at some consequences of this.

◇ **Example 9.4(a):** Let $A\mathbf{x} = \mathbf{b}$ represent a system of five equations in five unknowns, and suppose that $\text{rank}(A) = 3$. Does the system have (for certain) a solution?

Since the system has five equations in five unknowns, $\mathbf{b}$ is in $\mathbb{R}^5$. Because $\text{rank}(A) = 3$, the column space of A only has dimension three, so it is not all of $\mathbb{R}^5$ (which of course has dimension five). Therefore $\mathbf{b}$ may or may not be in the column space of A, and we can't say for certain that the system has a solution.

◇ **Example 9.4(b):** Let $A\mathbf{x} = \mathbf{b}$ represent a system of three equations in five unknowns, and suppose that $\text{rank}(A) = 3$. Does the system have (for certain) a solution?

Because there are three equations and five unknowns, A is 5×3 and the columns of A are in $\mathbb{R}^3$. Because $\text{rank}(A) = 3$, the column space must then be all of $\mathbb{R}^3$. Therefore $\mathbf{b}$ will be in the column space of A and the system has at least one solution.

Now suppose that we have a system $A\mathbf{x} = \mathbf{b}$ and a vector $\mathbf{x}$ that *IS* a solution to the system. Suppose also that nullity$(A) \neq 0$. Then there is some $\mathbf{y} \neq \mathbf{0}$ such that $A\mathbf{y} = \mathbf{0}$ and

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

This shows that both $\mathbf{x}$ and $\mathbf{x} + \mathbf{y}$ are solutions to the system, so the system does not have a unique solution. The thing that allows this to happen is the fact that the null space of A contains more than just the zero vector. This illustrates the following:

> A system of equations $A\mathbf{x} = \mathbf{b}$ can have a unique solution only if the nullity of A is zero (that is, the null space contains only the zero vector).

Note that this says nothing about whether a system has a solution to begin with; it simply says that if there is a solution and the nullity is zero, then that solution is unique.

◇ **Example 9.4(c):** Consider again a system $A\mathbf{x} = \mathbf{b}$ of three equations in five unknowns, with $\text{rank}(A) = 3$, as in Example 9.4(b). We saw in that example that the system has at least one solution - is there a unique solution?

We note first of all that A is 3×3, so the n of the Rank Theorem is three. We know that $\text{rank}(A)$ is three so, by the Rank Theorem, the nullity is then two. Thus the null space contains more than just the zero vector, so the system does not have a unique solution.

◇ **Example 9.4(d):** Suppose we have a system $A\mathbf{x} = \mathbf{0}$, with $\text{nullity}(A) = 2$. Does the system have a solution and, if it does, is it unique?

Because the system is homogeneous, it has at least one solution, the *zero* vector. But the null space contains more than just the zero vector, so the system has more than one solution, so there is not a unique solution.

Section 9.4 Exercises

1. Suppose that the column space of a 3×3 matrix A has dimension two. What does this tell us about the nature of the solutions to a system $A\mathbf{x} = \mathbf{b}$?

9.5 Chapter 9 Exercises

1. Suppose that the column space of a 3×3 matrix A has dimension two. What does this tell us about the nature of the solutions to a system $A\mathbf{x} = \mathbf{b}$? Show that the vectors $\mathbf{u}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are linearly dependent. Then give $\mathbf{u}_2$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_3$.

2. (a) Give three non-zero linearly dependent vectors in $\mathbb{R}^3$ for which removing any one of the three leaves a linearly independent set.

   (b) Give three non-zero linearly dependent vectors in $\mathbb{R}^3$ for which removing one vector leaves a linearly independent set but removing a different one (of the original three) leaves a linearly dependent set.

3. Let $A\mathbf{x} = \mathbf{b}$ be a system of equations, with A an $m \times n$ matrix where $m = n$ unless specified otherwise. For each situation below, determine whether the system *COULD* have

   (*i*) no solution (*ii*) exactly one solution (*iii*) infinitely many solutions

   Give all possibilities for each.

   (a) $\det(A) = 0$ (b) $\det(A) \neq 0$ (c) $\mathbf{b} = \mathbf{0}$

   (d) $\mathbf{b} = \mathbf{0}$, $\mathbf{A}$ invertible (e) $m < n$ (f) $m > n$

   (g) columns of A linearly independent (h) columns of A linearly dependent

   (i) $\mathbf{b} = \mathbf{0}$, columns of A linearly independent

   (j) $\mathbf{b} = \mathbf{0}$, columns of A linearly dependent

4. Consider the vectors $\mathbf{u} = \begin{bmatrix} 8 \\ -2 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -16 \\ 4 \\ -8 \end{bmatrix}$.

   (a) Is the set of all vectors $\mathbf{x} = \mathbf{u} + t\mathbf{v}$, where t ranges over all real numbers, a subspace of $\mathbb{R}^3$? If so, give a basis for the subspace; if not, tell why not.

   (b) Is the set of all vectors $\mathbf{x} = \mathbf{u} + t\mathbf{w}$, where t ranges over all real numbers, a subspace of $\mathbb{R}^3$? If so, give a basis for the subspace; if not, tell why not.

   (c) Is the set of all vectors $\mathbf{x} = \mathbf{u} + s\mathbf{v} + t\mathbf{w}$, where t ranges over all real numbers, a subspace of $\mathbb{R}^3$? If so, give a basis for the subspace; if not, tell why not.

5. Consider the matrix $A = \begin{bmatrix} 2 & 2 & 2 \\ -2 & 5 & 2 \\ 8 & 1 & 4 \end{bmatrix}$.

(a) Find a basis for row(A), the row space of A. What is the dimension of row(A)? (You can finddirections for finding row(A) in your book or in the definitions and theorems handouts.)

(b) Find a basis for col(A), the column space of A. What is the dimension of col(A)?

6. Give bases for the null and column spaces of the matrix $A = \begin{bmatrix} -6 & 3 & 30 \\ 2 & -1 & -10 \\ -4 & 2 & 20 \end{bmatrix}$.

7. Consider the system of equations $\begin{aligned} x_1 - 2x_2 + 3x_3 &= 4 \\ 2x_1 + x_2 - 4x_3 &= 3 \\ -3x_1 + 4x_2 - x_3 &= -2 \end{aligned}$.

(a) Determine how many solutions the system has.

(b) Give the form $A\mathbf{x} = \mathbf{b}$ for the system.

(c) Find a basis for the column space of A.

(d) Find a basis for the null space of A.

8. Now consider the system of equations $\begin{aligned} x_1 \quad\quad + 2x_3 &= -5 \\ -2x_1 + 5x_2 \quad\quad &= 11 \\ 2x_1 + 5x_2 + 8x_3 &= -7 \end{aligned}$.

(a) Determine how many solutions the system has.

(b) Give the form $A\mathbf{x} = \mathbf{b}$ for the system.

(c) Find a basis for the column space of A.

(d) Find a basis for the null space of A.

9. The system $\begin{aligned} x_1 \quad\quad + 2x_3 &= -1 \\ -2x_1 + 5x_2 \quad\quad &= -1 \\ 2x_1 + 5x_2 + 8x_3 &= -5 \end{aligned}$ *DOES* have a solution. (Infinitely many, actually.) How does the $A\mathbf{x} = \mathbf{b}$ form of this system compare to that of the previous one? How are they similar and how are they different?

10. (a) Give a 3×3 matrix B for which the column space has dimension one. (**Hint:** What kind of subspace of $\mathbb{R}^3$ has dimension one?)

(b) Find a basis for the column space of B.

(c) What should the dimension of the null space of B be?

(d) Find a basis for the null space of B.

10 Linear Transformations

Outcome:

10. Understand linear transformations, their compositions, and their application to homogeneous coordinates. Understand representations of vectors with respect to different bases.

    Performance Criteria:

    (a) Evaluate a transformation.

    (b) Determine the formula for a transformation in $\mathbb{R}^2$ or $\mathbb{R}^3$ that has been described geometrically.

    (c) Determine whether a given transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ is linear. If it isn't, give a counterexample; if it is, demonstrate this algebraically and/or give the standard matrix representation of the transformation.

    (d) Draw an arrow diagram illustrating a transformation that is linear, or that is not linear.

    (e) For a transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, show the graphical differences between (a) $T(c\mathbf{u})$ and $cT\mathbf{u}$, and (b) $T(\mathbf{u}+\mathbf{v})$ and $T\mathbf{u}+T\mathbf{v}$.

    (f) Find the composition of two transformations.

    (g) Find matrices that perform combinations of dilations, reflections, rotations and translations in homogenous coordinates.

10.1 Transformations of Vectors

Performance Criteria:

10. (a) Evaluate a transformation.
    (b) Determine the formula for a transformation in $\mathbb{R}^2$ or $\mathbb{R}^3$ that has been described geometrically.

Back in a "regular" algebra class you might have considered a function like $f(x) = \sqrt{x+5}$, and you may have discussed the fact that this function is only valid for certain values of x. When considering functions more carefully, we usually "declare" the function before defining it:

$$\text{Let } f : [-5, \infty) \to \mathbb{R} \text{ be defined by } f(x) = \sqrt{x+5}$$

Here the set $[-5, \infty)$ of allowable "inputs" is called the **domain** of the function, and the set $\mathbb{R}$ is sometimes called the **codomain** or **target set**. Those of you with programming experience will recognize the process of first declaring the function, then defining it. Later you might "call" the function, which in math we refer to as "evaluating" it.

In a similar manner we can define functions from one vector space to another, like

$$\text{Define } T : \mathbb{R}^2 \to \mathbb{R}^3 \text{ by } T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 \\ x_1^2 \end{bmatrix}$$

We will call such a function a **transformation**, hence the use of the letter T. (When we have a second transformation, we'll usually call it S.) The word "transformation" implies that one vector is transformed into another vector. It should be clear how a transformation works:

◇ **Example 10.1(a):** Find $T\left(\begin{bmatrix} -3 \\ 5 \end{bmatrix}\right)$ for the transformation defined above.

$$T\left(\begin{bmatrix} -3 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} -3+5 \\ 5 \\ (-3)^2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$$

It gets a bit tiresome to write both parentheses and brackets, so from now on we will dispense with the parentheses and just write

$$T\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$$

At this point we should note that you have encountered other kinds of transformations. For example, taking the derivative of a function results in another function,

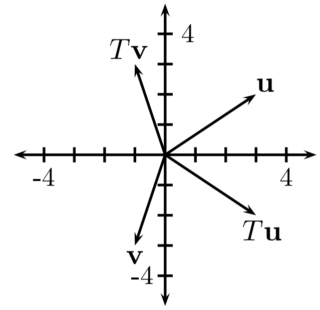
$$\frac{d}{dx}(x^3 - 5x^2 + 2x - 1) = 3x^2 - 10x + 2,$$

so the action of taking a derivative can be thought of as a transformation. Such transformations are often called **operators**.

Sometimes we will wish to determine the formula for a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ or $\mathbb{R}^3$ to $\mathbb{R}^3$ that has been described geometrically.

◇ **Example 10.1(b):** Determine the formula for the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects vectors across the x-axis.

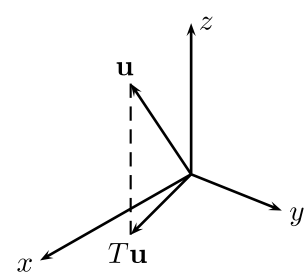
First we might wish to draw a picture to see what such a transformation does to a vector. To the right we see the vectors $\mathbf{u} = [3, 2]$ and $\mathbf{v} = [-1, -3]$, and their transformations $T\mathbf{u} = [3, -2]$ and $T\mathbf{v} = [-1, 3]$. From these we see that what the transformation does is change the sign of the second component of a vector. Therefore



$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

◇ **Example 10.1(c):** Determine the formula for the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that projects vectors onto the xy-plane.

It is a little more difficult to draw a picture for this one, but to the right you can see an attempt to illustrate the action of this transformation on a vector $\mathbf{u}$. Note that in projecting a vector onto the xy-plane, the x- and y-coordinates stay the same, but the z-coordinate becomes zero. The formula for the transformation is then



$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

We conclude this section with a very important observation. Consider the matrix

$$A = \begin{bmatrix} 5 & 1 \\ 0 & -3 \\ -1 & 2 \end{bmatrix}$$

and define $T_A\mathbf{x} = A\mathbf{x}$ for every vector for which $A\mathbf{x}$ is defined. This transformation acts on vectors in $\mathbb{R}^2$ and "returns" vectors in $\mathbb{R}^3$. That is, $T_A : \mathbb{R}^2 \to \mathbb{R}^3$. In general, we can use any $m \times n$ matrix A to define a transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ in this manner. In the next section we will see that such transformations have a desirable characteristic, and that every transformation with that characteristic can be represented by multiplication by a matrix.

10.2 Linear Transformations

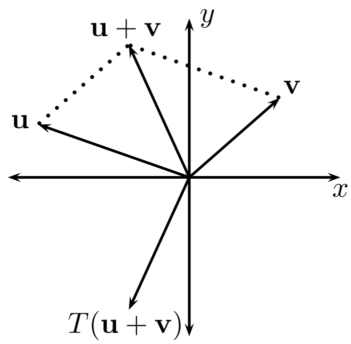
Performance Criteria:

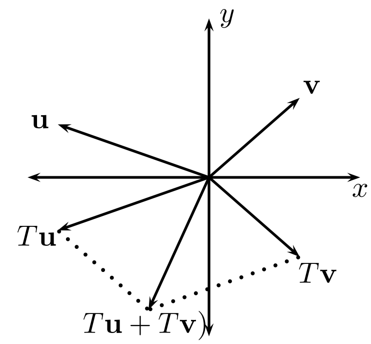
10. (c) Determine whether a given transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ is linear. If it isn't, give a counterexample; if it is, demonstrate this algebraically and/or give the standard matrix representation of the transformation.

    (d) Draw an arrow diagram illustrating a transformation that is linear, or that is not linear.

    (e) For a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, show the graphical differences between (a) $T(c\mathbf{u})$ and $cT\mathbf{u}$, and (b) $T(\mathbf{u}+\mathbf{v})$ and $T\mathbf{u}+T\mathbf{v}$.

To begin this section, recall the transformation from Example 10.1(b) that reflects vectors in $\mathbb{R}^2$ across the x-axis. In the drawing below and to the left we see two vectors $\mathbf{u}$ and $\mathbf{v}$ that are added, and then the vector $\mathbf{u}+\mathbf{v}$ is reflected across the x-axis. In the drawing below and to the right the same vectors $\mathbf{u}$ and $\mathbf{v}$ are reflected across the x-axis *first*, then the resulting vectors $T\mathbf{u}$ and $T\mathbf{v}$ are added.





Note that $T(\mathbf{u}+\mathbf{v}) = T\mathbf{u}+T\mathbf{v}$. Not all transformations have this property, but those that do have it, along with an additional property, are very important:

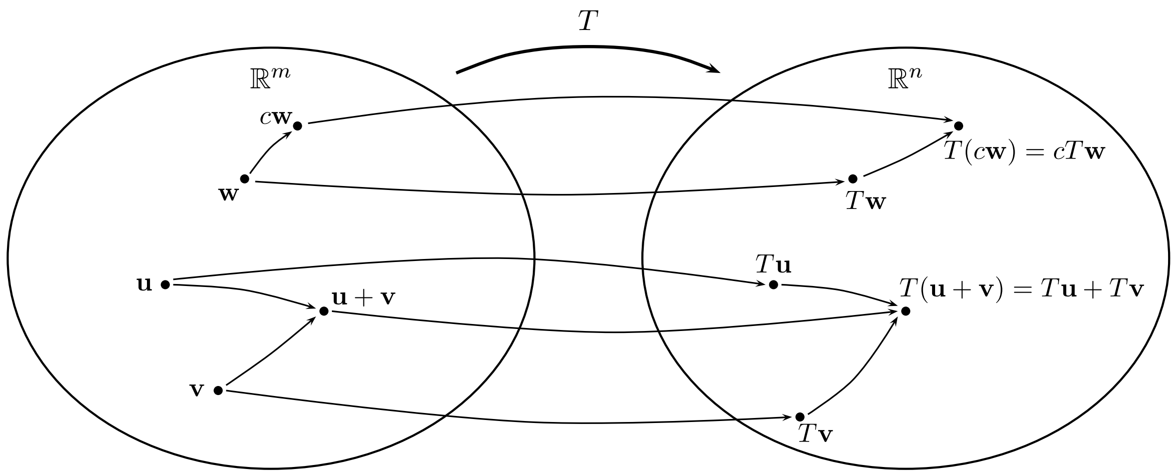
<u>Definition 10.2.1</u>: **Linear Transformation**

A transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is called a **linear transformation** if, for every scalar c and every pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^m$

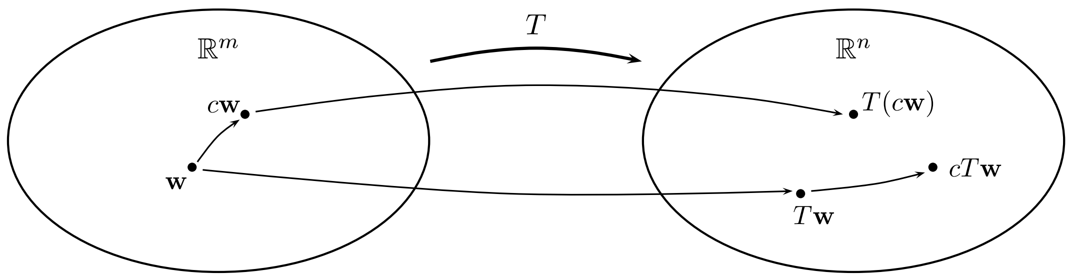
1) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and

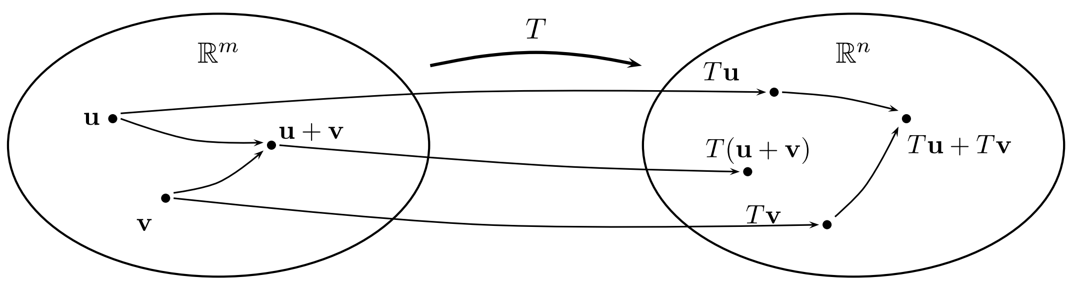
2) $T(c\mathbf{u}) = cT(\mathbf{u})$.

Note that the above statement describes how a transformation T interacts with the two operations of vectors, addition and scalar multiplication. It tells us that if we take two vectors in the domain and add them in the domain, then transform the result, we will get the same thing as if we transform the vectors individually first, then add the results in the codomain. We will also get the same thing if we multiply a vector by a scalar and then transform as we will if we transform first, *then* multiply by the scalar.This is illustrated in the *mapping diagram* at the top of the next page.



The following two mapping diagrams are for transformations R and S that *ARE NOT* linear:





◇ **Example 10.2(a):** Let A be an $m \times n$ matrix. Is $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T_A\mathbf{x} = A\mathbf{x}$ a linear transformation?

We know from properties of multiplying a vector by a matrix that

$$T_A(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T_A\mathbf{u} + T_A\mathbf{v}, \qquad T_A(c\mathbf{u}) = A(c\mathbf{u}) = cA\mathbf{u} = cT_A\mathbf{u}.$$

Therefore T_A is a linear transformation.

◇ **Example 10.2(b):** Is $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \\ x_1^2 \end{bmatrix}$ a linear transformation? If so, show that it is; if not, give a counterexample demonstrating that.

A good way to begin such an exercise is to try the two properties of a linear transformation for some specific vectors and scalars. If either condition is not met, then we have our counterexample, and if both hold we need to show they hold in general. Usually it is a bit simpler to check the condition $T(c\mathbf{u}) = cT\mathbf{u}$. In our case, if $c = 2$ and $vecu = [2, 3, 4]$,

$$T\left(2\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = T\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 14 \\ 8 \\ 36 \end{bmatrix} \quad \text{and} \quad 2T\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 14 \\ 8 \\ 18 \end{bmatrix}$$

Because $T(c\mathbf{u}) \neq cT\mathbf{u}$ for our choices of c and u, T is not a linear transformation.

The next example shows the process required to show in general that a transformation is linear.

◇ **Example 10.2(c):** Determine whether $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 - x_3 \end{bmatrix}$ is linear. If it is, prove it in general; if it isn't, give a specific counterexample.

First let's try a specific scalar $c = 2$ and two specific vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix}$. (I threw the negative in there just in case something funny happens when everything is positive.) Then

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix}\right) = T\begin{bmatrix} 5 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ -12 \end{bmatrix}$$

and

$$T\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + T\begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -11 \end{bmatrix} = \begin{bmatrix} 2 \\ -12 \end{bmatrix}$$

so the first condition of linearity *appears* to hold. Let's prove it in general. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be arbitrary (that is, randomly selected) vectors in $\mathbb{R}^3$. Then

$$T(\mathbf{u}+\mathbf{v}) = T\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = T\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix} = \begin{bmatrix} u_1+v_1+u_2+v_2 \\ (u_2+v_2)-(u_3+v_3) \end{bmatrix} =$$

$$\begin{bmatrix} u_1+u_2+v_1+v_2 \\ (u_2-u_3)+(v_2-v_3) \end{bmatrix} = \begin{bmatrix} u_1+u_2 \\ u_2-u_3 \end{bmatrix} + \begin{bmatrix} v_1+v_2 \\ v_2-v_3 \end{bmatrix} = T\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + T\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = T(\mathbf{u})+T(\mathbf{v})$$

This shows that the first condition of linearity holds in general. Let $\mathbf{u}$ again be arbitrary, along with the scalar c. Then

$$T(c\mathbf{u}) = T\left(c\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = T\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} = \begin{bmatrix} cu_1 + cu_2 \\ cu_2 - cu_3 \end{bmatrix} =$$

$$\begin{bmatrix} c(u_1 + u_2) \\ c(u_2 - u_3) \end{bmatrix} = c\begin{bmatrix} u_1 + u_2 \\ u_2 - u_3 \end{bmatrix} = cT\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = cT(\mathbf{u})$$

so the second condition holds as well, and T is a linear transformation.

There is a handy fact associated with linear transformations:

> THEOREM 10.2.2: If T is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$.

Note that this does not say that if $T(\mathbf{0}) = \mathbf{0}$, then T is a linear transformation, as you will see below. However, the contrapositive of the above statement tells us that *if* $T(\mathbf{0}) \neq \mathbf{0}$, *then* T *is not a linear transformation*.

When working with coordinate systems, one operation we often need to carry out is a **translation**, which means a shift of all points the same distance and direction. The transformation in the following example is a translation in $\mathbb{R}^2$.

◇ **Example 10.2(d):** Let a and b be any real numbers, with not both of them zero. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + a \\ x_2 + b \end{bmatrix}$. Is T a linear transformation?

Since $T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (since not both a and b are zero), T is not a linear transformation.

◇ **Example 10.2(e):** Determine whether $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 x_2 \end{bmatrix}$ is linear. If it is, prove it in general; if it isn't, give a specific counterexample.

It is easy to see that $T(\mathbf{0}) = \mathbf{0}$, so we can't immediately rule out T being linear, as we did in the last example. Let's do a quick check of the first condition of the definition of a linear transformation with an example. Let $\mathbf{u} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Then

$$T(\mathbf{u} + \mathbf{v}) = T\left(\begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = T\begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ -12 \end{bmatrix}$$

and

$$T\mathbf{u} + T\mathbf{v} = T\begin{bmatrix} -3 \\ 2 \end{bmatrix} + T\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

Clearly $T(\mathbf{u} + \mathbf{v}) \neq T\mathbf{u} + T\mathbf{v}$, so T is not a linear transformation.

Recall from Example 10.2(a) that if A be an $m \times n$ matrix, then $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation. It turns out that the converse of this is true as well:

> THEOREM 10.2.3: **Matrix of a Linear Transformation**
>
> If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, then there is a matrix A such that $T(\mathbf{x}) = A(\mathbf{x})$ for every $\mathbf{x}$ in $\mathbb{R}^m$. We will call A the matrix that represents the transformation.

As it is cumbersome and confusing the represent a linear transformation by the letter T and the matrix representing the transformation by the letter A, we will instead adopt the following convention: We'll denote the transformation itself by T, and the matrix of the transformation by $[T]$.

◇ **Example 10.2(f):** Find the matrix $[T]$ of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ of Example 10.2(c), defined by $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 - x_3 \end{bmatrix}$.

We can see that $[T]$ needs to have three columns and two rows in order for the multiplication to be defined, and that we need to have

$$\begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 - x_3 \end{bmatrix}$$

From this we can see that the first row of the matrix needs to be $1 \;\; 1 \;\; 0$ and the second row needs to be $0 \;\; 1 \;\; -1$. The matrix representing T is then $[T] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$.

The sort of "visual inspection" method used above can at times be inefficient, especially when trying to find the matrix of a linear transformation based on a geometric description of the action of the transformation. To see a more effective method, let's look at any linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. Suppose that the matrix of of the transformation is $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then for the two standard basis vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$$T\mathbf{e}_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \quad \text{and} \quad T\mathbf{e}_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

This indicates that the columns of $[T]$ are the vectors $T\mathbf{e}_1$ and $T\mathbf{e}_2$. In general we have the following:

Theorem 10.2.4

Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ be the standard basis vectors of $\mathbb{R}^m$, and suppose that $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation. Then the columns of $[T]$ are the vectors obtained when T acts on each of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$. We indicate this by

$$[T] = [\, T\mathbf{e}_1 \; T\mathbf{e}_2 \; \cdots \; T\mathbf{e}_m \,]$$

◇ **Example 10.2(g):** Let T be the transformation in $\mathbb{R}^2$ that rotates all vectors counterclockwise by ninety degrees. This is a linear transformation; use the previous theorem to determine its matrix $[T]$.

It should be clear that $T\mathbf{e}_1 = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $T\mathbf{e}_2 = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Then

$$[T] = [\, T\mathbf{e}_1 \; T\mathbf{e}_2 \,] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Section 10.2 Exercises

1. Two transformations from $\mathbb{R}^3$ to $\mathbb{R}^2$ are given below. One is linear and one is not. For the one that is, give the matrix of the transformation. For the one that is not, give a *specific* counterexample showing that the transformation violates the definition of a linear transformation. $(T(c\mathbf{u}) = cT\mathbf{u},\ T(\mathbf{u}+\mathbf{v}) = T\mathbf{u} + T\mathbf{v})$

   (a) $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_3 \\ x_1 + x_2 + x_3 \end{bmatrix}$ (b) $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1x_2 + x_3 \\ x_1 \end{bmatrix}$

2. For each of the following transformations, give

   - the matrix of the transformation if it is linear,
   - a **specific** counterexample showing that it is not linear, if it is not.

   (a) The transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_3 \\ x_1 + x_2 \end{bmatrix}$.

   (b) The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 1 \\ x_2 - 1 \end{bmatrix}$

3. Two transformations from $\mathbb{R}^3$ to $\mathbb{R}^2$ are given below. One is linear and one is not. For the one that is, give the matrix of the transformation. For the one that is not, give a *specific* counterexample showing that the transformation violates the definition of a linear transformation. $(T(c\mathbf{u}) = cT\mathbf{u},\ T(\mathbf{u}+\mathbf{v}) = T\mathbf{u} + T\mathbf{v})$

(a) $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_3 \\ x_1 + x_2 + x_3 \end{bmatrix}$ (b) $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1x_2 + x_3 \\ x_1 \end{bmatrix}$

4. For each of the following transformations, give

- the matrix of the transformation if it is linear,
- a **specific** counterexample showing that it is not linear, if it is not.

(a) The transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_3 \\ x_1 + x_2 \end{bmatrix}$.

(b) The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 1 \\ x_2 - 1 \end{bmatrix}$

5. (a) The transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 3x_2 - 5x_1 \\ x_1 \end{bmatrix}$ is linear.

Show that, for vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $T(\mathbf{u} + \mathbf{v}) = T\mathbf{u} + T\mathbf{v}$. Do this via a string of equal expressions, beginning with $T(\mathbf{u} + \mathbf{v})$ and ending with $T\mathbf{u} + T\mathbf{v}$.

(b) Give the matrix for the transformation from part (a).

6. For each of the following, a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by describing its action on a vector $\mathbf{x} = [x_1, x_2]$. For each transformation, determine whether it is linear by

- finding $T(c\mathbf{u})$ and $c(T\mathbf{u})$ and seeing if they are equal,
- finding $T(\mathbf{u} + \mathbf{v})$ and $T(\mathbf{u}) + T(\mathbf{v})$ and seeing if they are equal.

For any that you find to be linear, say so. For any that are not, say so and produce a **specific** counterexample to one of the two conditions for linearity.

(a) $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 + x_2 \end{bmatrix}$ (b) $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 x_2 \end{bmatrix}$

(c) $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}$ (d) $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ x_1 - x_2 \end{bmatrix}$

7. For each of the transformations from the previous exercise that are linear, give the standard matrix of the transformation.

10.3 Compositions of Transformations

Performance Criterion:

10. (f) Find the composition of two transformations.

It is likely that at some point in your past you have seen the concept of the composition of two functions; if the functions were denoted by f and g, one composition of them is the new function $f \circ g$. We call this new function "f of g", and we must describe how it works. This is simple - for any x, $(f \circ g)(x) = f[g(x)]$. That is, g acts on x, and f then acts on the result. There is another composition, $g \circ f$, which is defined the same way (but, of course, in the opposite order). For specific functions, you were probably asked to find the new rule for these two compositions. Here's a reminder of how that is done:

◇ **Example 10.3(a):** For the functions $f(x) = 2x - 1$ and $g(x) = 4x - x^2$, find the formulas for the composition functions $f \circ g$ and $g \circ f$.

Basic algebra gives us

$$(f \circ g)(x) = f[g(x)] = f[4x - x^2] = 2(4x - x^2) - 1 = 8x - 2x^2 - 1 = -2x^2 + 8x - 1$$

and

$$(g \circ f)(x) = g[f(x)] = g[2x - 1] = 4(2x - 1) - (2x - 1)^2 =$$

$$(8x - 4) - (4x^2 - 4x + 1) = 8x - 4 - 4x^2 + 4x - 1 = -4x^2 + 12x - 5$$

The formulas are then $(f \circ g)(x) = -2x^2 + 8x - 1$ and $(g \circ f)(x) = -4x^2 + 12x - 5$.

Worthy of note here is that the two compositions $f \circ g$ and $g \circ f$ are not the same!

One thing that was probably glossed over when you first saw this concept was the fact that the range (all possible outputs) of the first function to act must fall within the domain (allowable inputs) of the second function to act. Suppose, for example, that $f(x) = \sqrt{x - 4}$ and $g(x) = x^2$. The function f will be undefined unless x is at least four; we indicate this by writing $f : [4, \infty) \to \mathbb{R}$. This means that we need to restrict g in such a way as to make sure that $g(x) \geq 4$ if we wish to form the composition $f \circ g$. One simple way to do this is to restrict the domain of g to $[2, \infty)$. (We could include the interval $(-\infty, -2]$ also, but for the sake of simplicity we will just use the positive interval.) The range of g is then $[4, \infty)$, which coincides with the domain of f. We now see how these ideas apply to transformations, and we see how to carry out a process like that of Example 10.3(a) for transformations.

◇ **Example 10.3(b):** Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$S \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1^2 \\ x_2 x_3 \end{bmatrix}, \qquad T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ 2x_2 - x_1 \end{bmatrix}$$

Determine whether each of the compositions $S \circ T$ and $T \circ S$ exists, and find a formula for either of them that do.

Since the domain of S is $\mathbb{R}^3$ and the range of T is a subset of $\mathbb{R}^2$, the composition $S \circ T$ does not exist. The range of S falls within the domain of T, so the composition $T \circ S$ does exist. Its equation is found by

$$(T \circ S)(\mathbf{x}) = T\left(S\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = T\begin{bmatrix} x_1^2 \\ x_2x_3 \end{bmatrix} = \begin{bmatrix} x_1^2 + 3x_2x_3 \\ 2x_2x_3 - x_1^2 \end{bmatrix}$$

Let's formally define what we mean by a composition of two transformations.

Definition 10.3.1 Composition of Transformations

Let $S : \mathbb{R}^p \to \mathbb{R}^n$ and $T : \mathbb{R}^m \to \mathbb{R}^p$ be transformations. The **composition** of S and T, denoted by $S \circ T$, is the transformation $S \circ T : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$(S \circ T)\,\mathbf{x} = S(T\mathbf{x})$$

for all vectors $\mathbf{x}$ in $\mathbb{R}^m$.

Although the above definition is valid for compositions of any transformations between vector spaces, we are primarily interested in linear transformations. Recall that any linear transformation between vector spaces can be represented by matrix multiplication for some matrix. Suppose that $S : \mathbb{R}^3 \to \mathbb{R}^3$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ are linear transformations that can be represented by the matrices

$$[S] = \begin{bmatrix} 3 & -1 & 5 \\ 0 & 2 & 1 \\ 4 & 0 & -3 \end{bmatrix} \qquad \text{and} \qquad [T] = \begin{bmatrix} 2 & 7 \\ -6 & 1 \\ 1 & -4 \end{bmatrix}$$

respectively.

◇ **Example 10.3(c):** For the transformations S and T just defined, find $(S \circ T)\,\mathbf{x} = (S \circ T)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then find the matrix of the transformation $S \circ T$.

We see that

$$\begin{aligned} (S \circ T)\,\mathbf{x} = S(T\mathbf{x}) &= S\left(\begin{bmatrix} 2 & 7 \\ -6 & 1 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \\ &= S\begin{bmatrix} 2x_1 + 7x_2 \\ -6x_1 + x_2 \\ x_1 - 4x_2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -1 & 5 \\ 0 & 2 & 1 \\ 4 & 0 & -3 \end{bmatrix}\begin{bmatrix} 2x_1 + 7x_2 \\ -6x_1 + x_2 \\ x_1 - 4x_2 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 3(2x_1+7x_2)-(-6x_1+x_2)+5(x_1-4x_2) \\ 0(2x_1+7x_2)+2(-6x_1+x_2)+(x_1-4x_2) \\ 4(2x_1+7x_2)+0(-6x_1+x_2)-3(x_1-4x_2) \end{bmatrix}$$

$$= \begin{bmatrix} 17x_1+0x_2 \\ -11x_1-2x_2 \\ 5x_1+40x_2 \end{bmatrix}$$

From this we can see that the matrix of $S \circ T$ is $[S \circ T] = \begin{bmatrix} 17 & 0 \\ -11 & -2 \\ 5 & 40 \end{bmatrix}$.

Recall that the linear transformations of this example have matrices $[S]$ and $[T]$, and we find that

$$[S][T] = \begin{bmatrix} 3 & -1 & 5 \\ 0 & 2 & 1 \\ 4 & 0 & -3 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -6 & 1 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 17 & 0 \\ -11 & -2 \\ 5 & 40 \end{bmatrix}.$$

This illustrates the following:

> Theorem 10.3.2 **Matrix of a Composition**
>
> Let $S : \mathbb{R}^p \to \mathbb{R}^n$ and $T : \mathbb{R}^m \to \mathbb{R}^p$ be linear transformations with matrices $[S]$ and $[T]$. Then
> $$[S \circ T] = [S][T]$$

Section 10.3 Exercises

1. Consider the linear transformations $S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ 2x \\ -3y \end{bmatrix}$, $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5x-y \\ x+4y \end{bmatrix}$.

   (a) Since both of these are linear transformations, there are matrices A and B representing them. Give those two matrices (A for S, B for T).

   (b) Give equations for either (or both) of the compositions $S \circ T$ and $T \circ S$ that exist.

   (c) Give the matrix for either (or both) of the compositions that exist.

   (d) Find either (or both) of AB and BA that exist.

   (e) What did you notice in parts (c) and (d)? **Answer this with a complete sentence.**

10.4 Chapter 10 Exercises

1. Consider the matrix $A = \begin{bmatrix} 1 & 1 & -3 \\ -1 & 1 & 1 \end{bmatrix}$, which row reduces to $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}$.

   (a) Give a basis for the null space of A.

   (b) Give a basis for the column space of A.

   (c) The transformation T defined by $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation. Fill in each blank with the correct $\mathbb{R}^n$:

   $$T : \underline{\hspace{1.5cm}} \rightarrow \underline{\hspace{1.5cm}}$$

   (d) A mapping $f : C \rightarrow D$ is **one-to-one** if whenever $x \neq y$, it is also true that $f(x) \neq f(y)$. Give a specific example showing that the mapping T defined in (c) is *NOT* one-to-one. (**Hint:** I did not include parts (a) and (b) just to keep you off the streets a bit longer!)

   (e) A mapping $f : C \rightarrow D$ is onto if for every element y in D there is some element x in C such that $y = f(x)$. Tell why we know that the transformation T from (c) is onto, **using one sentence and language from our class**.

5. Now consider the matrix $B = \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 2 & 1 \end{bmatrix}$, which row reduces to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

   (a) Give a basis for the null space of B.

   (b) Give a basis for the column space of B.

   (c) The transformation T defined by $T(\mathbf{x}) = B\mathbf{x}$ is a linear transformation. Fill in each blank with the correct $\mathbb{R}^n$:

   $$T : \underline{\hspace{1.5cm}} \rightarrow \underline{\hspace{1.5cm}}$$

   (d) Explain, in a single sentence and using language from our class, why the transformation T defined in (c) is not onto.

11 Eigenvalues, Eigenspaces and Diagonalization

Outcome:

11. Understand eigenvalues and eigenspaces, diagonalization.

    Performance Criteria:

    (a) Determine whether a given vector is an eigenvector for a matrix; if it is, give the corresponding eigenvalue.

    (b) Determine eigenvectors and corresponding eigenvalues for linear transformations in $\mathbb{R}^2$ or $\mathbb{R}^3$ that are described geometrically.

    (c) Find the characteristic polynomial for a 2×2 or 3×3 matrix. Use it to find the eigenvalues of the matrix.

    (d) Give the eigenspace E_j corresponding to an eigenvalue λ_j of a matrix.

    (e) Diagonalize a matrix; know the forms of the matrices P and D from $P^{-1}AP = D$.

    (f) Write a system of linear differential equations in matrix-vector form. Write the initial conditions in vector form.

    (g) Solve a system of two linear differential equations; solve an initial value problem for a system of two linear differential equations.

11.1 An Introduction to Eigenvalues and Eigenvectors

Performance Criteria:

11. (a) Determine whether a given vector is an eigenvector for a matrix; if it is, give the corresponding eigenvalue.

    (b) Determine eigenvectors and corresponding eigenvalues for linear transformations in $\mathbb{R}^2$ or $\mathbb{R}^3$ that are described geometrically.

Recall that the two main features of a vector in $\mathbb{R}^n$ are direction and magnitude. In general, when we multiply a vector $\mathbf{x}$ in $\mathbb{R}^n$ by an $n \times n$ matrix A, the result $A\mathbf{x}$ is a new vector in $\mathbb{R}^n$ whose direction and magnitude are different than those of $\mathbf{x}$. For every square matrix A there are some vectors whose directions are not changed (other than perhaps having their directions reversed) when multiplied by the matrix. That is, multiplying $\mathbf{x}$ by A gives the same result as multiplying $\mathbf{x}$ by a scalar. It is very useful for certain applications to identify which vectors those are, and what the corresponding scalar is. Let's use the following example to get started:

◇ **Example 11.1(a):** Multiply the matrix $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$ times the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and determine whether multiplication by A is the same as multiplying by a scalar in either case.

$$A\mathbf{u} = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -22 \\ 18 \end{bmatrix}, \qquad A\mathbf{v} = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

For the first multiplication there appears to be nothing special going on. For the second multiplication, the effect of multiplying $\mathbf{v}$ by A is the same as simply multiplying $\mathbf{v}$ by -1. Note also that

$$\begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -6 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}, \qquad \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 8 \\ -4 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

It appears that if we multiply any scalar multiple of $\mathbf{v}$ by A the same thing happens; the result is simply the negative of the vector. That is, $A\mathbf{x} = (-1)\mathbf{x}$ for every scalar multiple of $\mathbf{x}$.

We say that $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and all of its scalar multiples are **eigenvectors** of A, with *corresponding* **eigenvalue** -1. Here is the formal definition of an eigenvalue and eigenvector:

> <u>Definition 11.1.1</u>: **Eigenvalues and Eigenvectors**
>
> A scalar λ is called an **eigenvalue** of a matrix A if there is a *nonzero* vector $\mathbf{x}$ such that
> $$A\mathbf{x} = \lambda\mathbf{x}.$$
> The vector $\mathbf{x}$ is an **eigenvector** corresponding to the eigenvalue λ.

One comment is in order at this point. Suppose that $\mathbf{x}$ has n components. Then $\lambda\mathbf{x}$ does as well, so A must have n rows. However, for the multiplication $A\mathbf{x}$ to be possible, A must also have n columns. For this reason, *only square matrices have eigenvalues and eigenvectors.* We now see how to determine whether a vector is an eigenvector of a matrix.

◇ **Example 11.1(b):** Determine whether either of the vectors $\mathbf{w}_1 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$ are eigenvectors for the matrix $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$ of Example 11.1(a). If either is, give the corresponding eigenvalue.

We see that

$$A\mathbf{w}_1 = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -10 \\ 7 \end{bmatrix} \quad \text{and} \quad A\mathbf{w}_2 = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \end{bmatrix}$$

$\mathbf{w}_1$ is not an eigenvector of A because there is no scalar λ such that $A\mathbf{w}_1$ is equal to $\lambda\mathbf{w}_1$. $\mathbf{w}_2$ *IS* an eigenvector, with corresponding eigenvalue 2, because $A\mathbf{w}_2 = 2\mathbf{w}_2$.

Note that for the 2×2 matrix A of Examples 11.1(a) and (b) we have seen two eigenvalues now. It turns out that those are the only two eigenvalues, which illustrates the following:

> <u>Theorem 11.1.2</u>: The number of eigenvalues of an $n \times n$ matrix is at most n.

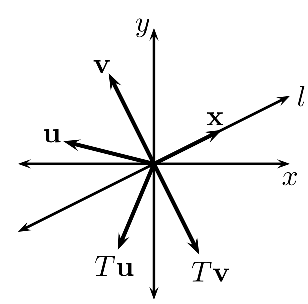
Do not let the use of the Greek letter lambda intimidate you - it is simply some scalar! It is tradition to use λ to represent eigenvalues. Now suppose that $\mathbf{x}$ is an eigenvector of an $n \times n$ matrix A, with corresponding eigenvalue λ, and let c be any scalar. Then for the vector $c\mathbf{x}$ we have
$$A(c\mathbf{x}) = c(A\mathbf{x}) = c(\lambda\mathbf{x}) = (c\lambda)\mathbf{x} = \lambda(c\mathbf{x})$$
This shows that any scalar multiple of $\mathbf{x}$ is also an eigenvector of A with the same eigenvalue λ. The set of all scalar multiples of $\mathbf{x}$ is of course a subspace of $\mathbb{R}^n$, and we call it the **eigenspace** corresponding to λ. $\mathbf{x}$, or any scalar multiple of it, is a basis for the eigenspace. The two eigenspaces you have seen so far have dimension one, but an eigenspace can have a higher dimension.

> Definition 11.1.3: **Eigenspace Corresponding to an Eigenvalue**
>
> For a given eigenvalue λ_j of an $n \times n$ matrix A, the **eigenspace** E_j corresponding to λ is the set of all eigenvectors corresponding to λ_j. It is a subspace of $\mathbb{R}^n$.

So far we have been looking at eigenvectors and eigenvalues from a purely algebraic viewpoint, but looking to see if the equation $A\mathbf{x} = \lambda\mathbf{x}$ held for some vector $\mathbf{x}$ and some scalar λ. It is useful to have some geometric understanding of eigenvectors and eigenvalues as well. In the next two examples we consider eigenvectors and eigenvalues of two linear transformations in $\mathbb{R}^2$ from a geometric standpoint. Note that the equation $A\mathbf{x} = \lambda x$ tells us that $\mathbf{x}$ is an eigenvector if the action of A on it leaves its direction unchanged or opposite of what it was.

◇ **Example 11.1(c):** Consider the transformation T that reflects very vector in $\mathbb{R}^2$ over the line l with equation $y = \frac{1}{2}x$; this is a linear transformation. Determine the eigenvectors and corresponding eigenvalues for this transformation.

We begin by observing that any vector that lies on l will be unchanged by the reflection, so it will be an eigenvector, with eigenvalue $\lambda = 1$. These vectors are all the scalar multiples of $\mathbf{x} = [2, 1]$; see the picture to the right. A vector not on the line, $\mathbf{u}$, is shown along with its reflection $T\mathbf{u}$ as well. We can see that its direction is changed, so it is not an eigenvector. However, for any vector $\mathbf{v}$ that is perpendicular to l we have $T\mathbf{v} = -\mathbf{v}$. Therefore any such vector is an eigenvector with eigenvalue $\lambda = -1$. Those vectors are all the scalar multiples of $\mathbf{x} = [-1, 2]$.



◇ **Example 11.1(d):** Let T be the transformation T that rotates every vector in $\mathbb{R}^2$ by thirty degrees counterclockwise; this is a linear transformation. Determine the eigenvectors and corresponding eigenvalues for this transformation.

Because every vector in $\mathbb{R}^2$ will be rotated by thirty degrees, the direction of every vector will be altered, so there are no eigenvectors for this transformation.

Our conclusion in Example 11.1(d) is correct in one sense, but incorrect in another. Geometrically, in a way that we can see, the conclusion is correct. Algebraically, the transformation has eigenvectors, but their components are complex numbers, and the corresponding eigenvalues are complex numbers as well. In this course we will consider only real eigenvectors and eigenvalues.

Section 11.1 Exercises

1. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$.

   (a) Find $A\mathbf{x}$ for each of the following vectors: $\mathbf{x}_1 = [3,6]$, $\mathbf{x}_2 = [2,-1]$, $\mathbf{x}_3 = [1,5]$, $\mathbf{x}_4 = [-3,-3]$, $\mathbf{x}_5 = [2,2]$

   (b) Give the vectors from part (a) that are eigenvectors and, for each, give the corresponding eigenvalue.

   (c) Give one of the eigenvalues that you have found. Then give the general form for *any* eigenvector corresponding to that eigenvalue.

   (d) Repeat part (c) for the other eigenvalue that you have found.

2. Now consider the matrix $B = \begin{bmatrix} 5 & -2 \\ 6 & -2 \end{bmatrix}$. One eigenvalue of this matrix is $\lambda = 2$. The objective of this exercise is to describe the associated eigenspace.

   (a) Since 2 is an eigenvalue of B, for some vector $\mathbf{x} = [x_1, x_2]$ we must have $B\mathbf{x} = 2\mathbf{x}$. Write the system of equations represented by this equation.

   (b) You should be able to rewrite your system of equations from (a) in a form that is equivalent to the vector equation $C\mathbf{x} = \mathbf{0}$. Do that and solve for the vector $\mathbf{x}$.

   (c) Describe the eigenspace of B associated with the eigenvector $\lambda = 2$.

11.2 Finding Eigenvalues and Eigenvectors

Performance Criteria:

11. (c) Find the characteristic polynomial for a 2×2 or 3×3 matrix. Use it to find the eigenvalues of the matrix.

    (d) Give the eigenspace E_j corresponding to an eigenvalue λ_j of a matrix.

So where are we now? We know what eigenvectors, eigenvalues and eigenspaces are, and we know how to determine whether a vector is an eigenvector of a matrix. There are two big questions at this point:

- Why do we care about eigenvalues and eigenvectors?
- If we are just given a square matrix A, how do we find its eigenvalues and eigenvectors?

We will not see the answer to the first question for a bit, but we'll now tackle answering the second question. We begin by rearranging the eigenvalue/eigenvector equation $A\mathbf{x} = \lambda\mathbf{x}$ a bit. First, we can subtract $\lambda\mathbf{x}$ from both sides to get

$$A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}.$$

Note that the right side of this equation must be the zero *vector*, because both $A\mathbf{x}$ and $\lambda\mathbf{x}$ are vectors. At this point we want to factor $\mathbf{x}$ out of the left side, but if we do so carelessly we will get a factor of $A - \lambda$, which makes no sense because A is a matrix and λ is a scalar! Note, however, that multiplying a vector by the scalar λ is the same as multiplying by a diagonal vector with all diagonal entries being λ, and that matrix is just λI. Therefore we can replace $\lambda\mathbf{x}$ with $(\lambda I)\mathbf{x}$, allowing us to factor $\mathbf{x}$ out:

$$A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0} \qquad \Rightarrow \qquad A\mathbf{x} - (\lambda I)\mathbf{x} = \mathbf{0} \qquad \Rightarrow \qquad (A - \lambda I)\mathbf{x} = \mathbf{0}$$

Now $A - \lambda I$ is just a matrix - let's call it B for now. Any *nonzero* (by definition) vector $\mathbf{x}$ that is a solution to $B\mathbf{x} = \mathbf{0}$ is an eigenvector for A. Clearly the zero vector is a solution to $B\mathbf{x} = \mathbf{0}$, and if B is invertible that will be the only solution. But since eigenvectors are nonzero vectors, A will have eigenvectors only if B is not invertible. Recall that one test for invertibility of a matrix is whether its determinant is nonzero. For B to not be invertible, then, its determinant must be zero. But B is $A - \lambda I$, so we want to find values of λ for which $\det(A - \lambda I) = 0$. (Note that the determinant of a matrix is a scalar, so the zero here is just the scalar zero.) We introduce a bit of special language that we use to discuss what is happening here:

Definition 11.2.1: **Characteristic Polynomial and Equation**

Taking λ to be an unknown, $\det(A - \lambda I)$ is a polynomial called the **characteristic polynomial** of A. The equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** for A, and its solutions are the eigenvalues of A.

Before looking at a specific example, you would probably find it useful to go back and look at Examples 7.4(a),(b) and (c), and to recall the following.

Determinant of a 2×2 Matrix

The determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A) = ad - bc$.

Determinant of a 3×3 Matrix

To find the determinant of a 3×3 matrix,

- Augment the matrix with its first two columns.
- Find the product down each of the three complete "downward diagonals" of the augmented matrix, and the product up each of the three "upward diagonals."
- Add the products from the downward diagonals and subtract each of the products from the upward diagonals. The result is the determinant.

Now we're ready to look at a specific example of how to find the eigenvalues of a matrix.

◇ **Example 11.2(a):** Find the eigenvalues of the matrix $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$.

We need to find the characteristic polynomial $\det(A - \lambda I)$, then set it equal to zero and solve.

$$A - \lambda I = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -4-\lambda & -6 \\ 3 & 5-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (-4-\lambda)(5-\lambda) - (3)(-6)) = (-20 - \lambda + \lambda^2) + 18 = \lambda^2 - \lambda - 2$$

We now factor this and set it equal to zero to find the eigenvalues:

$$\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0 \qquad \Longrightarrow \qquad \lambda = 2, -1$$

We use subscripts to distinguish the different eigenvalues: $\lambda_1 = 2,\ \lambda_2 = -1$.

We now need to find the eigenvectors or, more generally, the eigenspaces, corresponding to each eigenvalue. We defined eigenspaces in the previous section, but here we will give a slightly different (but equivalent) definition.

Definition 11.2.2: **Eigenspace Corresponding to an Eigenvalue**

For a given eigenvalue λ_j of an $n \times n$ matrix A, the **eigenspace** E_j corresponding to λ_j is the set of all solutions to the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}.$$

It is a subspace of $\mathbb{R}^n$.

Note that we indicate the correspondence of an eigenspace with an eigenvalue by subscripting them with the same number.

◇ **Example 11.2(b):** Find the eigenspace E_1 of the matrix $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$ corresponding to the eigenvalue $\lambda_1 = 2$.

For $\lambda_1 = 2$ we have $A - \lambda I = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -6 & -6 \\ 3 & 3 \end{bmatrix}$

The augmented matrix of the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$ is then $\begin{bmatrix} -6 & -6 & 0 \\ 3 & 3 & 0 \end{bmatrix}$, which reduces to $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The solution to this system is all vectors of the form $t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. We can then describe the eigenspace E_1 corresponding to $\lambda_1 = 2$ by either

$$E_1 = \left\{ t\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \qquad \text{or} \qquad E_1 = \text{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

It would be beneficial for the reader to repeat the above process for the second eigenvalue $\lambda_2 = -1$ and check the answer against what was seen in Example 11.1(a).

When first seen, the whole process for finding eigenvalues and eigenvectors can be a bit bewildering! Here is a summary of the process:

Finding Eigenvalues and Bases for Eigenspaces

The following procedure will give the eigenvalues and corresponding eigenspaces for a square matrix A.

1) Find $\det(A - \lambda I)$. This is the characteristic polynomial of A.

2) Set the characteristic polynomial equal to zero and solve for λ to get the eigenvalues.

3) For a given eigenvalue λ_i, solve the system $(A - \lambda_i I)\mathbf{x} = \mathbf{0}$. The set of solutions is the eigenspace corresponding to λ_i. The vector or vectors whose linear combinations make up the eigenspace are a basis for the eigenspace.

Section 11.2 Exercises

1. Use the methods of Examples 11.2(a) and (b) to find the other eigenvalue and its corresponding eigenspace.

2. Consider the matrix $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

   (a) Find characteristic polynomial by computing $\det(A - \lambda I)$. This is pretty easy by either a cofactor expansion along the first row *OR* the diagonal method. Either way, you will initially have two terms that both have a factor of $1 - \lambda$ in them. *Do not expand (multiply out) these terms - instead, factor the common factor of* $1 - \lambda$ *out of both*, then combine and simplify the rest.

   (b) Give the characteristic equation for matrix A, which is obtained by setting the characteristic polynomial equal to zero. (Remember that you are doing this because the equation $A\mathbf{x} = \lambda\mathbf{x}$ will only have solutions $\mathbf{x} \neq \mathbf{0}$ if $\det(A - \lambda I) = 0$.) Find the roots (eigenvalues) of the equation by factoring.

   (c) One of your eigenvalues should be one; let's refer to it as λ_1. Find a basis for the eigenspace E_1 corresponding to $\lambda = 1$ by solving the equation $(A - I)\mathbf{x} = \mathbf{0}$. ($(A - \lambda I)$ becomes $(A - I)$ because $\lambda_1 = 1$.) Conclude by giving the eigenspace E_1 using correct notation; you can write it as an arbitrary scalar times the basis vector or as the span of the basis vector:

   $$E_1 = \left\{ t \begin{bmatrix} \quad \end{bmatrix} \right\} \qquad OR \qquad E_1 = \text{span}\left(\begin{bmatrix} \quad \end{bmatrix} \right)$$

   (d) Describe the eigenspaces corresponding to the other two eigenvalues. *Make it clear which eigenspace is associated with which eigenvalue.*

   (e) (Optional) You can check your answers by multiplying each eigenvector by the original matrix A to see if the result is the same as multiplying the eigenvector by the corresponding eigenvalue. In other words, if the eigenvector is $\mathbf{x}$, check to see that $A\mathbf{x} = \lambda\mathbf{x}$.

11.3 Diagonalization of Matrices

Performance Criterion:

11. (e) Diagonalize a matrix; know the forms of the matrices P and D from $P^{-1}AP = D$.

We begin with an example involving the matrix A from Examples 11.1(a) and (b).

◇ **Example 11.3(a):** For $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$ and $P = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$, find the product $P^{-1}AP$.

First we obtain $P^{-1} = \dfrac{1}{(-1)(1)-(1)(-2)}\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$. Then

$$P^{-1}AP = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

We want to make note of a few things here:

- The columns of the matrix P are eigenvectors for A.
- The matrix $D = P^{-1}AP$ is a diagonal matrix.
- The diagonal entries of D are the eigenvalues of A, in the order of the corresponding eigenvectors in P.

For a square matrix A, the process of creating such a matrix D in this manner is called **diagonalization** of A. This cannot always be done, but often it can. (We will fret about exactly when it can be done later.) The point of the rest of this section is to see a use or two of this idea.

Before getting to the key application of this section we will consider the following. Suppose that we wish to find the kth power of a 2×2 matrix A with eigenvalues λ_1 and λ_2 and having corresponding eigenvectors that are the columns of P. Then solving $P^{-1}AP = D$ for A gives $A = PDP^{-1}$ and

$$\begin{aligned} A^k = (PDP^{-1})^k &= (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1}) \\ &= PD(P^{-1}P)D(P^{-1}P)\cdots(P^{-1}P)DP^{-1} \\ &= PDDD\cdots DP^{-1} \\ &= PD^kP^{-1} \\ &= P\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^k P^{-1} \\ &= P\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} P^{-1} \end{aligned}$$

Therefore, once we have determined the eigenvalues and eigenvectors of A we can simply take each eigenvector to the kth power, then put the results in a diagonal matrix and multiply *once* by P on the left and P^{-1} on the right.

◇ **Example 11.3(b):** Diagonalize the matrix $A = \begin{bmatrix} 3 & 12 & -21 \\ -1 & -6 & 13 \\ 0 & -2 & 6 \end{bmatrix}$.

First we find the eigenvalues by solving $\det(A - \lambda I) = 0$:

$$\begin{aligned}
\det \begin{bmatrix} 3-\lambda & 12 & -21 \\ -1 & -6-\lambda & 13 \\ 0 & -2 & 6-\lambda \end{bmatrix} &= (3-\lambda)(-6-\lambda)(6-\lambda) - 42 + 26(3-\lambda) + 12(6-\lambda) \\
&= (-18 + 3\lambda + \lambda^2)(6-\lambda) - 42 + 78 - 26\lambda + 72 - 12\lambda \\
&= -108 + 18\lambda + 18\lambda - 3\lambda^2 + 6\lambda^2 - \lambda^3 + 108 - 38\lambda \\
&= -\lambda^3 + 3\lambda^2 - 2\lambda \\
&= -\lambda(\lambda^2 - 3\lambda + 2) \\
&= -\lambda(\lambda - 2)(\lambda - 1)
\end{aligned}$$

The eigenvalues of A are then $\lambda = 0, 1, 2$. We now find an eigenvector corresponding to $\lambda = 0$ by solving the system $(A - \lambda I)\mathbf{x} = 0$. The augmented matrix and its row-reduced form are shown below:

$$\begin{bmatrix} 3 & 12 & -21 & 0 \\ -1 & -6 & 13 & 0 \\ 0 & -2 & 6 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{array}{l} \text{Let } x_3 = 1. \\ \text{Then } x_2 = 3 \\ \text{and } x_1 = -5 \end{array}$$

The eigenspace corresponding to the eigenvalue $\lambda = 0$ is then the span of the vector $\mathbf{v}_1 = [-5, 3, 1]$. For $\lambda = 1$ we have

$$\begin{bmatrix} 2 & 12 & -21 & 0 \\ -1 & -7 & 13 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & \frac{9}{2} & 0 \\ 0 & 1 & -\frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{array}{l} \text{Let } x_3 = 2. \\ \text{Then } x_2 = 5 \\ \text{and } x_1 = -9 \end{array}$$

The eigenspace corresponding to the eigenvalue $\lambda = 1$ is then the span of the vector $\mathbf{v}_2 = [-9, 5, 2]$ (obtained by multiplying the solution vector by two in order to get a vector with integer components). Finally, for $\lambda = 2$ we have

$$\begin{bmatrix} 1 & 12 & -21 & 0 \\ -1 & -8 & 13 & 0 \\ 0 & -2 & 4 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{array}{l} \text{Let } x_3 = 1. \\ \text{Then } x_2 = 2 \\ \text{and } x_1 = -3 \end{array}$$

so the eigenspace corresponding to the eigenvalue $\lambda = 2$ is then the span of the vector $\mathbf{v}_3 = [-3, 2, 1]$. The diagonalization of A is then $D = P^{-1}AP$, where

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \text{and} \qquad P = \begin{bmatrix} -5 & -9 & -3 \\ 3 & 5 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Section 11.3 Exercises

1. Consider matrix again the matrix $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ from Exercise 2 of Section 11.2.

   (a) Let P be a matrix whose columns are eigenvectors for the matrix A. (The basis vectors for each of the three eigenspaces will do.) Give P and P^{-1}, using your calculator to find P^{-1}.

   (b) Find $P^{-1}AP$, using your calculator if you wish. The result should be a diagonal matrix with the eigenvalues on its diagonal. If it isn't, check your work from Exercise 4.

2. Now let $B = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

   (a) Find characteristic polynomial by computing $\det(B - \lambda I)$. If you expand along the second column you will obtain a characteristic polynomial that already has a factor of $2 - \lambda$.

   (b) Give the characteristic *equation* (make sure it has an equal sign!) for matrix B. Find the roots (eigenvalues) by factoring. *Note that in this case one of the eigenvalues is repeated.* This is not a problem.

   (c) Find and describe (as in Exercise 1(c)) the eigenspace corresponding to each eigenvalue. The repeated eigenvalue will have *TWO* eigenvectors, so that particular eigenspace has dimension two. State your results as sentences, and use set notation for the bases.

3. Repeat the process from Exercise 1 for the matrix B from Exercise 2.

11.4 Solving Systems of Differential Equations

Performance Criteria:

11. (f) Write a system of linear differential equations in matrix-vector form. Write the initial conditions in vector form.

    (g) Solve a system of two linear differential equations; solve an initial value problem for a system of two linear differential equations.

We now get to the centerpiece of this section. Recall that the solution to the initial value problem $x'(t) = kx(t)$, $x(0) = C$ is $x(t) = Ce^{kt}$. Now let's consider the **system of two differential equations**

$$\begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= 3x_1 + 2x_2, \end{aligned}$$

where x_1 and x_2 are functions of t. Note that the two equations are **coupled**; the equation containing the derivative x_1' contains the function x_1 itself, but also contains x_2. The same sort of situation occurs with x_2'. The key to solving this system is to *uncouple* the two equations, and eigenvalues and eigenvectors will allow us to do that!

We will also add in the initial conditions $x_1(0) = 10$, $x_2(0) = 5$. If we let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ we can rewrite the system of equations and initial conditions as follows:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

which can be condensed to

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \tag{1}$$

This is the matrix initial value problem that is completely analogous to $x'(t) = kx(t)$, $x(0) = C$.

Before proceeding farther we note that the matrix A has eigenvectors $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with corresponding eigenvalues $\lambda = 4$ and $\lambda = -1$. Thus, if $P = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ we then have $P^{-1}AP = D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ and $A = PDP^{-1}$.

We can substitute this last expression for A into the vector differential equation in (1) to get $\mathbf{x}' = PDP^{-1}\mathbf{x}$. If we now multiply both sides on the left by P^{-1} we get $P^{-1}\mathbf{x}' = DP^{-1}\mathbf{x}$. We now let $\mathbf{y} = P^{-1}\mathbf{x}$; Since P^{-1} is simply a matrix of constants, we then have $\mathbf{y}' = (P^{-1}\mathbf{x})' = P^{-1}\mathbf{x}'$ also. Making these two substitutions into $P^{-1}\mathbf{x}' = DP^{-1}\mathbf{x}$ gives us $\mathbf{y}' = D\mathbf{y}$. By the same substitution we also have $\mathbf{y}(0) = P^{-1}\mathbf{x}(0)$. We now have the new initial value problem

$$\mathbf{y}' = D\mathbf{y}, \quad \mathbf{y}(0) = P^{-1}\mathbf{x}(0). \tag{3}$$

Here the vector $\mathbf{y}$ is simply the unknown vector $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and $\mathbf{y}(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}$ which can be determined by $\mathbf{y}(0) = P^{-1}\mathbf{x}(0)$. Because the coefficient matrix of the system in (3) is diagonal, the two differential equations can be uncoupled and solved to find $\mathbf{y}$. Now since $\mathbf{y} = P^{-1}\mathbf{x}$ we also have $\mathbf{x} = P\mathbf{y}$, so after we find $\mathbf{y}$ we can find $\mathbf{x}$ by simply multiplying $\mathbf{y}$ by P.

So now it is time for you to make all of this happen!

1. Write the system $\mathbf{y}' = D\mathbf{y}$ in that form, then as two differential equations. Solve the differential equations. There will be two arbitrary constants; distinguish them by letting one be C_1 and the other C_2. solve the two equations to find $y_1(t)$ and $y_2(t)$.

2. Find P^{-1} and use it to find $\mathbf{y}(0)$. Use $y_1(0)$ and $y_2(0)$ to find the constants in your two differential equations.

3. Use $\mathbf{x} = P\mathbf{y}$ to find x. Finish by giving the functions $x_1(t)$ and $x_2(t)$.

4. Check your final answer by doing the following. If your answer doesn't check, go back and find your error. I had to do that, so you might as well also!

   (a) Make sure that $x_1(0) = 10$ and $x_2(0) = 5$.

   (b) Put x_1 and x_2 into the equations (1) and make sure you get true statements.

A Index of Symbols

$\mathbb{R}$	real number	7
rref	reduced row echelon form	10
$\mathbb{R}^2,\ \mathbb{R}^3$	two and three dimensional Euclidean space	33, 34
a, b, c, x, y, z	scalars	37
$\mathbf{u},\ \mathbf{v}\mathbf{w}, \mathbf{x}$	vectors	37
$\overrightarrow{OP}$	position vector from the origin to point P	37
$\overrightarrow{PQ}$	vector from point P to point Q	37
$\Vert\mathbf{v}\Vert$	magnitude (or length) of vector $\mathbf{v}$	38
$\mathbf{u}\cdot\mathbf{v}$	dot product of $\mathbf{u}$ and $\mathbf{v}$	56
$\mathrm{proj}_{\mathbf{v}}\mathbf{u}$	projection of $\mathbf{u}$ onto $\mathbf{v}$	57, 58
$A,\ B$	matrices	64
A^T	transpose of a matrix	64
$A(i, j)$	ith row, jth column of matrix A	64
I_n	$n \times n$ identity matrix	64
$m \times n$	dimensions of a matrix	64, 65
a_{ij}	i, jth entry of matrix A	65
$\mathbf{a}_{i*},\ \mathbf{a}_{*j}$	ith row and jth column of matrix A	66
$A\mathbf{u}$	matrix A times vector u	66
AB	product of matrices A and B	77
A^{-1}	inverse of the matrix A	81
L	lower triangular matrix	91
U	upper triangular matrix	91
$\det(A)$	determinant of matrix A	95
$\mathcal{S}$	a finite set of vectors	104
$\mathrm{col}(A)$	column space of A	116
$\mathrm{null}(A)$	null space of A	117
$\bar{\mathbf{x}}$	least squares solution	118
$\mathbf{e}$	error vector for least squares approximation	120
$\Vert\mathbf{e}\Vert$	error for least squares approximation	120
$\mathbf{e}_j$	jth standard bassi vector	132
$\mathcal{B}$	basis	132
$T,\ S$	transformations	146
$[T]$	matrix for the linear transformation T	152
$S \circ T$	composition of transformations S and T	154, 155
$\lambda,\ \lambda_j$	eigenvalues	160
E_j	eigenspace corresponding to eigenvalue λ_j	164
$P^{-1}AP$	diagonalization of matrix A	166

B Solutions to Exercises

B.1 Chapter 1 Solutions

Section 1.2 Solutions

1. (a) $(-2, 1)$ (b) $(3, 4)$ (c) $(3, 2)$
   (d) $(1, -1)$ (e) $(-1, 4)$ (f) $(-2, \frac{1}{3})$
2. (a) $(4, 1)$ (b) $(0, -3)$ (c) $(2, 2)$

Section 1.3 Solutions

1. The coefficient matrix is $\begin{bmatrix} 1 & 1 & -3 \\ -3 & 2 & -1 \\ 2 & 1 & -4 \end{bmatrix}$ and the augmented matrix is $\begin{bmatrix} 1 & 1 & -3 & 1 \\ -3 & 2 & -1 & 7 \\ 2 & 1 & -4 & 0 \end{bmatrix}$
2. Matrices A, B and D are in row-echelon form.
3. Matrices A and B are in reduced row-echelon form.
4. $\begin{bmatrix} 1 & 1 & -3 & 1 \\ -3 & 2 & -1 & 7 \\ 2 & 1 & -4 & 0 \end{bmatrix} \begin{array}{c} 3R_1 + R_2 \to R_2 \\ \Longrightarrow \\ -2R_1 + R_3 \to R_3 \end{array} \begin{bmatrix} 1 & 1 & -3 & 1 \\ 0 & 5 & -10 & 10 \\ 0 & -1 & 2 & -2 \end{bmatrix}$
5. (a) $\begin{bmatrix} 1 & 5 & -7 & 3 \\ -5 & 3 & -1 & 0 \\ 4 & 0 & 8 & -1 \end{bmatrix} \begin{array}{c} 5R_1 + R_2 \to R_2 \\ \Longrightarrow \\ -4R_1 + R_3 \to R_3 \end{array} \begin{bmatrix} 1 & 5 & -7 & 3 \\ 0 & 28 & -36 & 15 \\ 0 & -20 & 36 & -13 \end{bmatrix}$

   (b) $\begin{bmatrix} 2 & -8 & -1 & 5 \\ 0 & -2 & 0 & 0 \\ 0 & 6 & -5 & 2 \end{bmatrix} \begin{array}{c} \Longrightarrow \\ 3R_2 + R_3 \to R_3 \end{array} \begin{bmatrix} 2 & -8 & -1 & 5 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -5 & 2 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 3 & 5 & -2 \\ 0 & 2 & -8 & 1 \end{bmatrix} \begin{array}{c} \Longrightarrow \\ -\frac{2}{3}R_2 + R_3 \to R_3 \end{array} \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 3 & 5 & -2 \\ 0 & 0 & -\frac{34}{3} & \frac{7}{3} \end{bmatrix}$
6. (a) $(-4, \frac{1}{2}, -4)$ (b) $(33, -4, 1)$ (c) $(7, 0, -2)$
7. (a) $(2, 3, -1)$ (b) $(-2, 1, 2)$ (c) $(-1, 2, 1)$
8. Same as Exercise 7.

Section 1.4 Solutions

2. (a) $y = -\frac{3}{4}x^2 + \frac{5}{2}x - \frac{3}{4}$
3. (a) fourth degree (b) $y = \frac{2}{5}x + \frac{23}{30}x^2 - \frac{1}{10}x^3 - \frac{1}{15}x^4$ or $y = 0.4x + 0.77x^2 - 0.1x^3 - 0.07x^4$
4. $z = 3.765 + 0.353x - 1.235y$
5. $t_1 = 52.6°$, $t_2 = 57.3°$, $t_3 = 61.6°$, $t_4 = 53.9°$, $t_5 = 57.1°$, $t_6 = 60.2°$
7. $t_1 = 44.6°$, $t_2 = 49.6°$, $t_3 = 38.6°$

B.2 Chapter 2 Solutions

Section 2.1 Solutions

1. (a) $(6,3,4)$, $(0,0,4)$, $(-2,-1,4)$, $(2,1,4)$

   (b) $x-2y=0$, $z=4$, we can determine z (c) $(2t,t,4)$

2. (a) $y=2t+2$, $x=t-1$

3. (a) no solution (b) $(0,-2,t)$ (c) $(7,t,2)$ (d) no solution (e) $(5,3,1)$

   (f) $(-1,2,0)$ (g) $(1-2s+t,s,t)$ (h) no solution (i) $(2.5t-4,t,-5)$

4. $t=-2$: $(5,-8,-2,4)$, $t=-1$: $(2,-3,-1,4)$, $t=0$: $(-1,2,0,4)$, $t=1$: $(-4,7,1,4)$, $t=2$: $(-7,12,2,4)$

5. The leading variables are x and y, the free variable is z, and the rank of the coefficient matrix is two.

6. $(t-1,2t+2,t)$, $t=-1$: $(-2,0,-1)$, $t=0$: $(-1,2,0)$, $t=1$: $(0,4,1)$, $t=2$: $(1,6,2)$

   There is no solution to the system with the next reduced matrix given.

   $(-2s+t+5,s,t,-4)$, $s=t=0$: $(5,0,0,-4)$, $s=1,\ t=0$: $(3,1,0,-4)$, $s=0,\ t=1$: $(6,0,1,-4)$

Section 2.5 Solutions

1. (a) $x_1=-1$, $x_2=0$

   (b) $x_1=t-4$, $x_2=-2t+5$, $x_3=t$, $x_4=2$, $(-4,5,0,2)$, $(-3,3,1,2)$, $(-2,1,2,2)$, $(-5,7,-1,2)$, etc.

   (c) No solution

2. (a) $(t-13,t,3,4)$

   (b) $(-14,-1,3,4)$, $(-13,0,3,4)$, $(-12,1,3,4)$, $(-11,2,3,4)$, $(0,13,3,4)$, $(1,14,3,4)$

   (c) Change the 2 in the third row to a zero.

B.3 Chapter 3 Solutions

Section 3.1 Solutions

1. (a) Plane, intersects the x, y and z-axes at $(3,0,0)$, $(0,6,0)$ and $(0,0,-2)$, respectively.

   (b) Plane, intersects the x and z-axes at $(6,0,0)$ and $(0,0,2)$, respectively. Does not intersect the y-axis.

   (c) Plane, intersects only the y-axis, at $(0,-6,0)$.

   (d) Not a plane.

   (e) Plane, intersects the x, y and z-axes at $(-6,0,0)$, $(0,3,0)$ and $(0,0,-2)$, respectively.

2. Any equation of the form $ax+by=c$, where c and at leat one of a or b is *NOT* zero. The plane intersects the x-axis at $(\frac{c}{a},0,0)$ if $a\neq 0$ and the z-axis at $(0,0,\frac{c}{b})$ if $b\neq 0$.

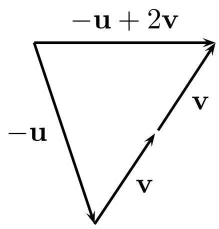
3. $y=4$

Section 3.2 Solutions

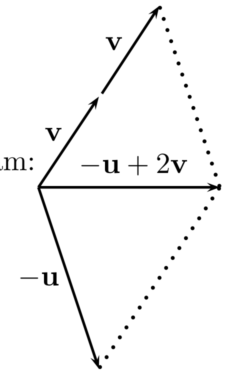
1. (a) $\overrightarrow{PQ} = [17, -6, -15]$, $\|\overrightarrow{PQ}\| = \sqrt{550} = 23.4$
   (b) $\overrightarrow{PQ} = [12, -3]$, $\|\overrightarrow{PQ}\| = \sqrt{153} = 12.4$
   (c) $\overrightarrow{PQ} = [10, -1, -7, 9]$, $\|\overrightarrow{PQ}\| = \sqrt{231} = 15.2$

Section 3.3 Solutions

1. Tip-to-tail:



Parallelogram:



2. $\begin{bmatrix} -45 \\ 30 \end{bmatrix}$

3. $\begin{bmatrix} -c_1 - 8c_2 \\ 3c_1 + c_2 \\ -6c_1 + 4c_2 \end{bmatrix}$

5. (a) $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

   (b) $\mathbf{v} = \begin{bmatrix} 8 \\ -6 \end{bmatrix}$ is any vector of the form $(-\frac{1}{2}t + \frac{7}{2}) \begin{bmatrix} 3 \\ -1 \end{bmatrix} + (\frac{1}{2}t - \frac{5}{2}) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

6. (a) $a_1 = \frac{3}{2}$, $a_2 = 4$, $a_3 = -\frac{1}{2}$
   (b) There are no such b_1, b_2 and b_3
   (c) $\det(A) = -42$, $\det(B) = 0$

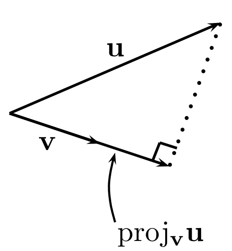
Section 3.4 Solutions

2. (a) $\text{proj}_{\mathbf{b}}\mathbf{v} = \dfrac{15 - 2}{25 + 4} \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \dfrac{13}{29} \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{65}{29} \\ -\frac{26}{29} \end{bmatrix}$

   (b) $\text{proj}_{\mathbf{b}}\mathbf{v} = \dfrac{10 - 0}{4 + 1} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

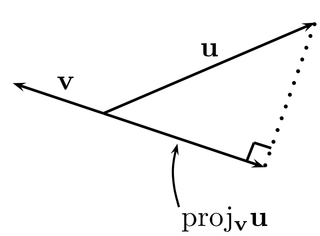
   (c) $\text{proj}_{\mathbf{b}}\mathbf{v} = \dfrac{-6 - 4}{16 + 4} \begin{bmatrix} -2 \\ -4 \end{bmatrix} = -\dfrac{1}{2} \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

3.
(a)

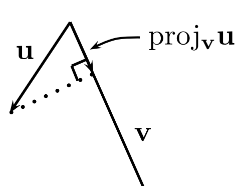


(tail is where $\mathbf{u}$ and $\mathbf{v}$ meet)

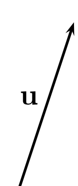
(b)



(c)



(d)



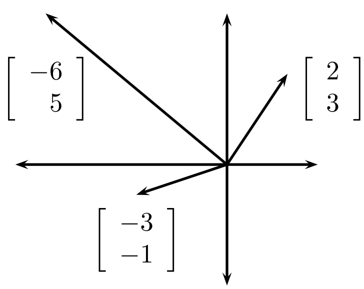
B.4 Chapter 4 Solutions

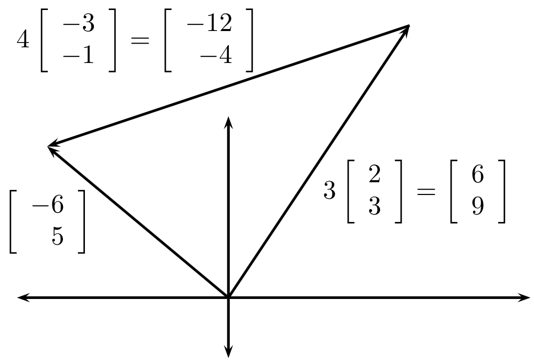
Section 4.1 Solutions

1. (a) $x\begin{bmatrix}1\\-3\\2\end{bmatrix}+y\begin{bmatrix}1\\2\\1\end{bmatrix}+z\begin{bmatrix}-3\\-1\\-4\end{bmatrix}=\begin{bmatrix}1\\7\\0\end{bmatrix}$

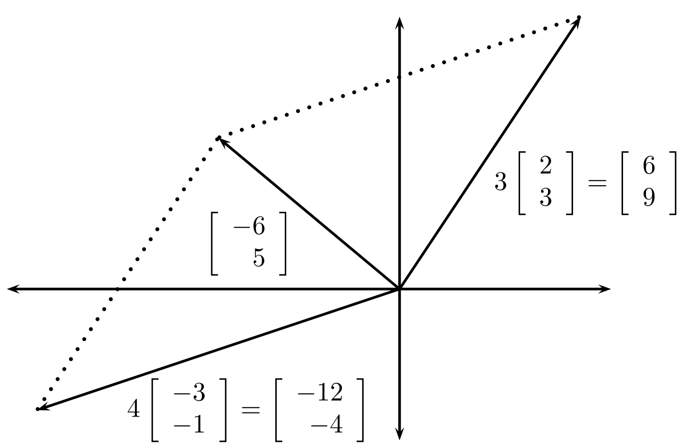
   (b) $x_1\begin{bmatrix}5\\0\\2\end{bmatrix}+x_2\begin{bmatrix}0\\2\\1\end{bmatrix}+x_3\begin{bmatrix}1\\3\\-4\end{bmatrix}=\begin{bmatrix}-1\\0\\2\end{bmatrix}$

2. $x\begin{bmatrix}2\\3\end{bmatrix}+y\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-6\\5\end{bmatrix} \implies 3\begin{bmatrix}2\\3\end{bmatrix}+4\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-6\\5\end{bmatrix}$





tip-to-tail method



parallelogram method

Section 4.3 Solutions

1. (a) $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + t\begin{bmatrix} -1 \\ 7 \\ -4 \end{bmatrix}$ (b) $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + s\begin{bmatrix} -1 \\ 7 \\ -4 \end{bmatrix} + t\begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix}$

   (c) $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

2. The vector equation of the plane is $\mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} + s\begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix} + t\begin{bmatrix} 6 \\ -3 \\ -8 \end{bmatrix}$. Letting $s = 1$ and $t = 1$ gives the point $(9, -1, -7)$

Section 4.4 Solutions

1. (a) $\mathbf{x} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (b) $\mathbf{x} = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

   (c) $\mathbf{x} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

Section 4.5 Solutions

1. We need to solve the vector equation $s\begin{bmatrix} 1 \\ 7 \\ -4 \end{bmatrix} + t\begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 13.9 \\ 51.4 \\ -30.1 \end{bmatrix}$. (Make sure you see where this comes from!) The result is $s = 3.7$, $t = -5.1$

B.5 Chapter 5 Solutions

Section 5.1 Solutions

1. (a) A is 3×3, B is 3×2, C is 3×4

   (b) $b_{13} = 4$, $c_{32} = 2$

2. (a) $\begin{bmatrix} 2 & 0 & 0 \\ -5 & 1 & 0 \\ 7 & 3 & -8 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -8 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -5 & 7 \\ -5 & 1 & 3 \\ 7 & 3 & -8 \end{bmatrix}$

   (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & -1 & 5 \\ 0 & 4 & 3 \\ 0 & 0 & -8 \end{bmatrix}$ (f) see (c)

   (g) see (c) (h) see (b)

3. $A^T = \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & 7 \\ 5 & -2 & 0 \end{bmatrix}$ $\quad B^T = \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & 7 \end{bmatrix}$ $\quad C^T = \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & 7 \\ -1 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}$

4. $B + D = \begin{bmatrix} 2 & -3 \\ -2 & 3 \\ 5 & 11 \end{bmatrix}$, $\quad B - D = \begin{bmatrix} 0 & 3 \\ -4 & -1 \\ 3 & 3 \end{bmatrix}$, $\quad D - B = \begin{bmatrix} 0 & -3 \\ 4 & 1 \\ -3 & -3 \end{bmatrix}$

Section 5.2 Solutions

1. $\begin{bmatrix} 3 \\ -12 \\ 9 \end{bmatrix}$, $\begin{bmatrix} -23 \\ 28 \end{bmatrix}$ $\quad$ 2. $A = \begin{bmatrix} 3 & -5 \\ 1 & 1 \end{bmatrix}$ $\quad$ 3. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\quad I\mathbf{x} = \mathbf{x}$

B.6 Chapter 6 Solutions

Section 6.1 Solutions

1. $\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 8 & -7 \end{bmatrix}$

2. $A^2 = \begin{bmatrix} -15 & 5 \\ -3 & -14 \end{bmatrix}$ $\quad AF = \begin{bmatrix} 30 & 45 \\ 0 & 12 \end{bmatrix}$ $\quad BC = \begin{bmatrix} -69 \\ -25 \end{bmatrix}$

$BD = \begin{bmatrix} 62 & -25 & -2 \\ -121 & 80 & 36 \end{bmatrix}$ $\quad CE = \begin{bmatrix} -25 & 5 & -10 \\ 20 & -4 & 8 \\ -35 & 7 & -14 \end{bmatrix}$ $\quad DC = \begin{bmatrix} -51 \\ 27 \\ -1 \end{bmatrix}$

$D^2 = \begin{bmatrix} 39 & 3 & 18 \\ -48 & 18 & -7 \\ 1 & 4 & 5 \end{bmatrix}$ $\quad EC = \begin{bmatrix} -43 \end{bmatrix}$ $\quad ED = \begin{bmatrix} 37 & -2 & 13 \end{bmatrix}$

$FA = \begin{bmatrix} 3 & 9 \\ -27 & 39 \end{bmatrix}$ $\quad FB = \begin{bmatrix} 10 & -35 & 10 \\ 6 & 147 & -18 \end{bmatrix}$ $\quad F^2 = \begin{bmatrix} -2 & -11 \\ 66 & 75 \end{bmatrix}$

3. The $(2, 1)$ entry is -8 and the $(3, 2)$ entry is -13.

4. $a_{31}a_{12} + a_{32}a_{22} + a_{33}a_{32} + a_{34}a_{42} + a_{35}a_{52}$

Section 6.3 Solutions

1. Since $\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 8 & -4 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \neq I_2$, the matrices are not inverses.

2. (a) $[I_2 \mid B] = \begin{bmatrix} 1 & 0 & -\frac{5}{2} & \frac{3}{2} \\ 0 & 1 & 2 & -1 \end{bmatrix}$

(b) $AB = BA = I_2$ $\quad$ (c) B is the inverse of A

Section 6.5 Solutions

1. (a) $A^T = \begin{bmatrix} -5 & 0 & 2 \\ 1 & 4 & -3 \end{bmatrix}$ (b) $A^TA = \begin{bmatrix} 29 & -11 \\ -11 & 26 \end{bmatrix}$, $AA^T = \begin{bmatrix} 26 & 4 & -13 \\ 4 & 16 & -12 \\ -13 & -12 & 13 \end{bmatrix}$

   (c) A^TA and AA^T are both square, symmetric matrices

2. For AB to not exist, B has to be a matrix with number of rows not equalling three. For AC to exist, C must have three rows.

3. If they are not square they can't be inverses. If they are square, compute AB and see if it is the identity matrix. If it is they are inverses; if it is not, they aren't inverses.

4. (a) $AB = I_2$

   (b) The product BA doesn't even exist, so it can't be an identity matrix.

5. $\begin{bmatrix} 2 & -1 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{bmatrix} \overset{rref}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0.36 & 0.09 \\ 0 & 1 & 0.27 & 0.18 \end{bmatrix}$ so the inverse is $\begin{bmatrix} 0.36 & 0.09 \\ 0.27 & 0.18 \end{bmatrix}$

B.7 Chapter 7 Solutions

Section 7.1 Solutions

1. $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix}$

2. (a) $\begin{bmatrix} 1 & 1 & -3 \\ -3 & 2 & -1 \\ 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & -3 & 1 \\ 1 & 1 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

Section 7.3 Solutions

2. (a) $\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ (b) $A^{-1} = \dfrac{1}{10-(-12)} \begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix} = \dfrac{1}{22} \begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix}$

   (c) $\dfrac{1}{22} \begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix} = \dfrac{1}{22} \begin{bmatrix} 22 & 0 \\ 0 & 22 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   (d)
$$\begin{aligned}
\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\
\frac{1}{22} \begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix} \left(\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) &= \frac{1}{22} \begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\
\left(\frac{1}{22} \begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \frac{1}{22} \begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \frac{1}{22} \begin{bmatrix} 29 \\ -10 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \frac{29}{22} \\ -\frac{10}{22} \end{bmatrix}
\end{aligned}$$

   (e)
$$\begin{bmatrix} 2 & -3 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -3 & 1 & 0 \\ 0 & 11 & -2 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -3 & 1 & 0 \\ 0 & 1 & -\frac{2}{11} & \frac{1}{11} \end{bmatrix}$$
$$\begin{bmatrix} 2 & 0 & \frac{5}{11} & \frac{3}{11} \\ 0 & 1 & -\frac{2}{11} & \frac{1}{11} \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & \frac{5}{22} & \frac{3}{22} \\ 0 & 1 & -\frac{2}{11} & \frac{1}{11} \end{bmatrix}$$

3. (a) $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ (b) $A^{-1} = \dfrac{1}{(5)(3)-(2)(7)}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 5 & 7 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} \overset{rref}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 3 & -7 \\ 0 & 1 & -2 & 5 \end{bmatrix}$, so $A^{-1} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

(d)
$$\begin{aligned}
\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -1 \\ 4 \end{bmatrix} \\
\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}\left(\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} \\
\left(\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -31 \\ 22 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -31 \\ 22 \end{bmatrix} \\
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -31 \\ 22 \end{bmatrix}
\end{aligned}$$

Section 7.4 Solutions

1. The determinant of the coefficient matrix is zero, so the system *DOES NOT* have a unique solution.

2. (a) If the determinant of A is zero, then the system has no solution or infinitely many solutions.
   (b) If the determinant of A is not zero, then the system has a unique solution.

3. (a) If the determinant of A is zero, then the system has infinitely many solutions. (It can't have no solutions, because $\mathbf{x} = \mathbf{0}$ is a solution.
   (b) If the determinant of A is not zero, then the system has the unique solution $\mathbf{x} = \mathbf{0}$.

Section 7.6 Solutions

1.
$$\begin{aligned}
\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
\frac{1}{7}\begin{bmatrix} 5 & -2 \\ -4 & 3 \end{bmatrix}\left(\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \frac{1}{7}\begin{bmatrix} 5 & -2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
\left(\frac{1}{7}\begin{bmatrix} 5 & -2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{7}\begin{bmatrix} -7 \\ 7 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\end{aligned}$$

2. (a) $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $\begin{bmatrix} -4 \\ 5 \end{bmatrix}$ (b) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 \\ 5 & -2 \end{bmatrix}$, $\begin{bmatrix} -4 \\ 5 \end{bmatrix}$

B.8 Chapter 8 Solutions

Section 8.1 Solutions

1. (a) The span of the set is the x-axis.

   (b) The span of the set is the xz-plane, or the plane $y = 0$.

   (c) The span of the set is all of $\mathbb{R}^2$.

   (d) The span of the set is the origin.

   (e) The span of the set is a line through the origin and the point $(1, 2, 3)$.

2. (a) $\mathbf{w}$ is not in the span of S. (b) $\mathbf{w}$ is not in the span of S.

   (c) $\begin{bmatrix} 8 \\ 38 \\ -14 \\ 11 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -4 \\ -3 \\ 7 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ 6 \\ -4 \\ 5 \end{bmatrix}$ (d) $\begin{bmatrix} 3 \\ 7 \\ -4 \end{bmatrix} = -4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 11 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Section 8.3 Solutions

1. (a) Not a subspace, doesn't contain the zero vector.

   (b) Subspace. (c) Subspace.

   (d) Not a subspace, the vector $\begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$ is not on either line because it is not a scalar multiple of either vector.

   (e) Not a subspace, the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in the set, but $-2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$ is not.

   (f) The set is a plane not containing the zero vector, so it is not a subspace.

   (g) This is a plane containing the origin, so it is a subspace.

   (h) The vector $\mathbf{0}$ is a subspace. (i) Subspace.

2. (a) The set is not a subspace because it does not contain the zero vector. We can tell this because $\mathbf{u}$ and $\mathbf{v}$ are not scalar multiples of each other.

   (b) The set is a subspace, and either of the vectors $\mathbf{u}$ or $\mathbf{w}$ by itself is a basis, as is any scalar multiple of either of them.

Section 8.4 Solutions

1. (a) $c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ -9 \\ 17 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & 3 & -2 & -9 \\ -1 & -4 & 6 & 17 \end{bmatrix}$

   $\mathbf{u}_1$ is in the column space.

   (b) $-3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ -9 \\ 17 \end{bmatrix}$.

(c) $c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \\ 2 \end{bmatrix}$, $\qquad \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 3 & -2 & 15 \\ -1 & -4 & 6 & 2 \end{bmatrix}$

$\mathbf{u}_2$ is not in the column space.

2. $\mathbf{v}_1$ is in $\text{null}(A)$ and $\mathbf{v}_2$ is not.

Section 8.6 Solutions

1. (a) Here the three vectors need to all be scalar multiples of each other.

   (b) The span of two nonzero vectors that are not scalar multiples of each other is always a plane, no matter what space we are in. If we create a linear combination of them in which neither is multiplied by zero we obtain another vector that is in the same plane, so the span does not increase when this new vector is included, yet the new vector is not a scalar multiple of either of the original two vectors.

   (c) In order to have a chance of spanning $\mathbb{R}^3$, a set of vectors must contain at least three vectors.

2. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 10 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ would be one example. $\mathbf{v}_1$ and $\mathbf{v}_2$ must be scalar multiples of each other, and $\mathbf{v}_3$ must *NOT* be a scalar multiple of either of them.

3. Any three vectors such that none is a scalar multiple of either of the other two, and that have third components equal to zero, will do it.

B.9 Chapter 9 Solutions

Section 9.1 Solutions

1. (a) $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$.

   (b) The zero vector (in other words, $c_1 = c_2 = c_3 = 0$) is a solution. If it is the *ONLY* solution, then the vectors are linearly independent.

   (c) $\begin{aligned} -5c_1 + 5c_2 + 5c_3 &= 0 \\ 9c_1 + 0c_2 + 9c_3 &= 0 \\ 4c_1 + 6c_2 + 16c_3 &= 0 \end{aligned} \implies \begin{bmatrix} -5 & 5 & 5 & 0 \\ 9 & 0 & 9 & 0 \\ 4 & 6 & 16 & 0 \end{bmatrix}$

   (d) $c_1 = -1,\ c_2 = -2,\ c_3 = 1$ OR $c_1 = 1,\ c_2 = 2,\ c_3 = -1$ OR any scalar multiple of these.

   (e) $\mathbf{u}_1 = -2\mathbf{u}_2 + \mathbf{u}_3,\ \mathbf{u}_2 = -\frac{1}{2}\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_3,\ \mathbf{u}_3 = \mathbf{u}_1 + 2\mathbf{u}_2$

2. Solving $c_1 \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 1 \\ -6 \end{bmatrix} + c_3 \begin{bmatrix} -4 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ gives the general solution $c_1 = -\frac{3}{2}t$, $c_2 = \frac{1}{2}t,\ c_3 = t$. Therefore $\mathbf{w} = \frac{3}{2}\mathbf{u} - \frac{1}{2}\mathbf{v}$. (Also $\mathbf{u} = \frac{1}{3}\mathbf{v} + \frac{2}{3}\mathbf{w}$ and $\mathbf{v} = 3\mathbf{u} - 2\mathbf{w}$.)

3. For the following, use the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$.

(a) $\mathbf{u}$ is not in span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v_3})$, but $\mathbf{w}$ is.

(b) $-\frac{6}{5}\mathbf{v}_1 + \frac{1}{5}\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$.

(c) $\mathbf{v}_1 = \frac{1}{6}\mathbf{v}_2 + \frac{5}{6}\mathbf{v}_3$.

Section 9.2 Solutions

1. (a) Not a subspace, doesn't contain the zero vector.

   (b) Subspace, a basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

   (c) Subspace, $\mathbf{u}$ is a basis.

   (d) Not a subspace, the vector $\begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$ is not on either line because it is not a scalar multiple of either vector.

   (e) Not a subspace, the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in the set, but $-2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$ is not.

   (f) It can be shown that $\mathbf{w}$, $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, so the set is a plane not containing the zero vector, so it is not a subspace.

   (g) This is a plane containing the origin, so it is a subspace. The set $\{\mathbf{u}, \mathbf{v}\}$ is a basis.

   (h) The vector $\mathbf{0}$ is a subspace.

   (i) All of $\mathbb{R}^3$ is a subspace.

2. (a) Not a subspace. $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is in the set, but $-1\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$.

   (b) Subspace. A basis would be $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Section 9.3 Solutions

2. (a) $x_1 = -4t$, $x_2 = t$ and $x_3 = t$.

   (b) $\mathbf{x} = t\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$

   (c) $\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$

3. A basis for the column space of A is $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} \right\}$.

4. (a) $\mathbf{u}_1$ is in the column space and $\mathbf{u}_2$ is not.

   (b) A basis for col(A) is $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} \right\}$.

   (c) $\mathbf{u}_1 = -3\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$.

5. (a) $\mathbf{v}_1$ is in null(A) and $\mathbf{v}_2$ is not.

(b) A basis for null(A) is $\left\{\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}\right\}$.   (c) $\mathbf{v}_1 = -2\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

6. The column space has dimension two and the null space has dimension one.

Section 9.5 Solutions

1. Solving $c_1\begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives $c_1 = -\frac{1}{2}$, $c_2 = \frac{1}{2}$ and $c_3 = 1$. We can then write $\mathbf{u}_2 = \mathbf{u}_1 - 2\mathbf{u}_3$.

2. (a) Almost any three vectors in $\mathbb{R}^3$ will do, as long as no two are scalar multiples of each other.

   (b) Here we want three vectors where two of them are scalar multiples of each other.

3. (a) (*i*), (*iii*)   (b) (*ii*)   (c) (*ii*), (*iii*)   (d) (*ii*)

   (e) (*i*), (*iii*)   (f) (*i*), (*ii*), (*iii*)   (g) (*ii*)   (h) (*i*), (*iii*)

   (i) (*ii*)   (j) (*iii*)

4. (a) The set is not a subspace because it does not contain the zero vector. We can tell this because $\mathbf{u}$ and $\mathbf{v}$ are not scalar multiples of each other.

   (b) The set is a subspace, and either of the vectors $\mathbf{u}$ or $\mathbf{w}$ by itself is a basis, as is any scalar multiple of either of them.

   (c) The set is a plane. When $s = 0$ and $t = \frac{1}{2}$, $\mathbf{x}$ is the zero vector, so the plane is a subspace.

B.10 Chapter 10 Solutions

Section 10.2 Solutions

1. (a) $T(\mathbf{u}+\mathbf{v}) = T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = T\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix} = \begin{bmatrix} u_1+v_1+2(u_2+v_2) \\ 3(u_2+v_2)-5(u_1+v_1) \\ u_1+v_1 \end{bmatrix} =$

   $$= \begin{bmatrix} u_1+2u_2 \\ 3u_2-5u_1 \\ u_1 \end{bmatrix} + \begin{bmatrix} v_1+2v_2 \\ 3v_2-5v_1 \\ v_1 \end{bmatrix} = T\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + T\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = T(\mathbf{u}) + T(\mathbf{v})$$

   (b) $A = \begin{bmatrix} 1 & 2 \\ -5 & 3 \\ 1 & 0 \end{bmatrix}$

2. (a) The transformation is linear.

   (b) Not linear: $T\left(2\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right) = T\begin{bmatrix} 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 16 \\ 60 \end{bmatrix}$ and $2T\begin{bmatrix} 3 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 8 \\ 15 \end{bmatrix} = \begin{bmatrix} 16 \\ 30 \end{bmatrix}$,

   so $T\left(2\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right) \neq 2T\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

(c) Not linear: $T\left(-2\begin{bmatrix} 3 \\ -5 \end{bmatrix}\right) = T\begin{bmatrix} -6 \\ 10 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$ and $-2T\begin{bmatrix} 3 \\ -5 \end{bmatrix} = -2\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -6 \\ -10 \end{bmatrix}$, so $T\left(-2\begin{bmatrix} 3 \\ -5 \end{bmatrix}\right) \neq -2T\begin{bmatrix} 3 \\ -5 \end{bmatrix}$

(d) The transformation is linear.

3. (a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 3 & 0 \\ 1 & -1 \end{bmatrix}$

Section 10.4 Solutions

1. (a) Linear, $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

(b) Not linear: $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right) = T\begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 44 \\ 5 \end{bmatrix}$ and $T\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + T\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 26 \\ 4 \end{bmatrix} = \begin{bmatrix} 31 \\ 5 \end{bmatrix}$, so $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right) \neq T\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + T\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

2. (a) Linear, $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

(b) Not linear. Note that $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, which violates Theorem 10.2.2.

3. (a) $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -1 \\ 1 & 4 \end{bmatrix}$

(b) $(S \circ T)\begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} 5x - y \\ x + 4y \end{bmatrix} = \begin{bmatrix} 6x + 3y \\ 10x - 2y \\ -3x - 12y \end{bmatrix}$

(c) The matrix of $S \circ T$ is $\begin{bmatrix} 6 & 3 \\ 10 & -2 \\ -3 & -12 \end{bmatrix}$ (d) $AB = \begin{bmatrix} 6 & 3 \\ 10 & -2 \\ -3 & -12 \end{bmatrix}$

(e) The matrix of $S \circ T$ is AB.

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