Differential Equations and Their Solutions

Differential equations are simply equations that have derivatives in them. Examples are

$$\frac{dy}{dx} + 5y = 0 \qquad y'' + 2y' + 10y = 0 \qquad y'' + 3y' + 2y = 3\cos t. \tag{1}$$

The **order** of a differential equation is the order of the highest derivative in the equation. The first equation above is a **first order** differential equation, the other two are **second order** differential equations. Most equations commonly encountered are first or second order. In all cases above, y is what we call the dependent variable. The independent variable for the first equation is x , and for the third equation the independent variable is t. We can't determine the independent variable for the second equation - unless told otherwise we can denote it by whatever letter we want.

A solution to a differential equation is an equation (usually a function) whose derivative(s) satisfy the differential equation. For example, a solution to the first equation above is $y = e^{-5x}$, because

$$y = e^{-5x} \quad \Rightarrow \quad y' = -5e^{-5x} \quad \Longrightarrow \quad \frac{dy}{dx} + 5y = -5e^{-5x} + 5(e^{-5x}) = 0.$$

Note that all I am doing here is finding the derivative y' and substituting both it and y into the left side of $\frac{dy}{dx} + 5y = 0$ to see it comes out zero, which it does. Therefore $y = e^{-5x}$ is a solution to $\frac{dy}{dx} + 5y = 0.$

General Solutions to Differential Equations

It is easy to verify that any function of the form $y = Ce^{-5x}$ is a solution to $\frac{dy}{dx} + 5y = 0$. In fact, every solution to this differential equation has the form $y = Ce^{-5x}$, so it is what we call the **general solution** to the differential equation. The general solutions to the three differential equations (1) are

$$y = Ce^{-5x} \qquad y = e^{-x}(C_1 \sin 3x + C_2 \cos 3x) \qquad y = C_1e^{-2t} + C_2e^{-t} + \tfrac{9}{10}\sin t + \tfrac{3}{10}\cos t \tag{2}$$

respectively. Note that the second function could be written as $y = C_1e^{-x}\sin 3x + C_2e^{-x}\cos 3x$. (1) and (2) above illustrate the following:

> **Useful Fact 1:** The order of a differential equation dictates the number of constants in the general solution. The general solution of a first order differential equation contains one unknown constant, the general solution to a second order differential equation contains two unknown constants.

This could be used to rule out an answer on the FE Exam. For example, if the differential equation is $y'' + 5y' + 4y = 0$ and one of the choices for the solution is $y = C_1e^{-4t}$, we know that is not correct because there need to be two constants in the answer.

Solutions to Certain Second Order Differential Equations

The most commonly encountered second order differential equations are ones of the form $ay'' + by' + cy = f(x)$, where a, b and c are constants. We focus now on such differential equations in which it is also the case that $f(x) = 0$ and $a = 1$. An example would be $y'' + 3y' + 2y = 0$. Associated with any such differential equation is another equation called the **auxiliary equation** or **characteristic equation**. It is obtained by replacing the derivatives in the differential equation with an unknown like r to powers corresponding to the derivatives of y, thinking of y itself as the "zeroth derivative." So the auxiliary equation for $y'' + 3y' + 2y = 0$ is $r^2 + 3r + 2 = 0$. We then solve the auxiliary equation by factoring, the quadratic formula or a calculator. (Here is where a calculator like the TI-36 that can quickly solve a quadratic equation could be handy!) The solutions to the auxiliary equation determine the form of the solution to the differential equation as follows:

Useful Facts 2:

- When the auxiliary equation has two distinct real solutions r_1 and r_2, the solution to the ODE is $$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$
- When the auxiliary equation has only one real solution r, the solution to the ODE is $$y = C_1 e^{rx} + C_2 x e^{rx}.$$
- When the auxiliary equation has the form $y'' + cy = 0$ where $c > 0$, the solutions to the auxiliary equation will be $r = \pm\sqrt{c}\, i$ and the solution to the differential equation is $$y = C_1 \sin\sqrt{c}\, x + C_2 \cos\sqrt{c}\, x.$$
- When the solutions to the auxiliary equation are of the form $a \pm bi$, the solution to the differential equation is $$y = e^{ax}(C_1 \sin b\, x + C_2 \cos b\, x).$$

Given (on the FE exam) an equation of the form $y'' + by' + cy = 0$, you may be able to select the correct answer by simply finding the solutions to $r^2 + br + c = 0$ and determining which of the above forms the solution must have.

Something "They" Don't Teach You in a Differential Equations Class!

The auxiliary equation for a first order differential equation like $\dfrac{dy}{dx} + 5y = 0$ is $r + 5 = 0$. The solution to that equation is $r = -5$, and the solution to the differential equation is simply $y = Ce^{-5x}$. This can't be done for all first order differential equations, but it likely can for any that you would see on the FE exam.

Initial Value Problems

Usually we want to know the values of the constants in the general solution to a differential equation. They can be determined by additional information called **initial conditions** or **boundary conditions**. A differential equation, along with such information, is called an **initial value problem** or **boundary value problem**. An example would be something like

$$\frac{dy}{dx} + 5y = 0, \qquad y(0) = 3.$$

The statement $y(0) = 3$ means that $y = 0$ when $x = 3$. A solution to the initial value problem is a function y that satisfies both the differential equation *AND* the initial condition. For the above example we already know that every solution looks like $y = Ce^{-5x}$, so $y(0) = Ce^0 = C$. thus, for the condition $y(0) = 3$ to be true it must be the case that $C = 3$.

The solution to a second order differential equation contains two unknown constants, and two initial conditions are needed to find the values of the constant. Typically there is a condition on both the function and its first derivative, like

$$y'' + 16y = 0, \qquad y(0) = 2,\ y'(0) = -1.$$

Here is how you should go about answering a question on the FE exam about an initial value problem:

Useful Methodology 3: When asked for the solution to a first order initial value problem, check all the given solutions to see which satisfy the initial condition. If more than one does, substitute the given solutions that do satisfy the initial condition into the differential equation to see which one satisfies it.

Useful Methodology 4: When asked for the solution to a second order initial value problem, first attempt to determine the form of the solution, as previously described. If there is more than one answer with the correct form, check those which satisfy the initial condition $y(0) = a.$. If more than one does, you will then have to check the second initial condition $y'(0) = b$ by taking the derivative of the solutio and substituting $x = 0$.

Example: Which of the following is the solution to $y'' + 16y = 0, \quad y(0) = 2,\ y'(0) = -1$?

(a) $y = 2\sin 4x + \cos 4x$

(b) $y = -\frac{1}{4}\sin 4x + 2\cos 4x$

(c) $y = e^{4x} + e^{-4x}$

(d) $y = e^{4x} - e^{-4x}$

Solution: We could actually begin by checking to see which solutions meet the condition $y(0) = 2$; both (b) and (c) do, so our answer is one of those two. If we instead found the correct form of the solution using **Useful Fact 2**, we'd get that the solution must have the form $y = C_1 \sin 4x + C_2 \cos 4x$, so the correct answer is either (a) or (b). Putting both of these ideas together, we arrive at (b) as the correct answer.

Suppose that choice (a) would have been $y = \sin 4x + 2\cos 4x$. Then both (a) and (b) would be the right form, and both would satisfy $y(0) = 2$. You would then need to check the condition $y'(0) = -1$. Let's do that for the new choice (a). Taking the derivative of $y = \sin 4x + 2\cos 4x$ gives $y' = 4\cos 4x - 8\sin 4x$, so $y'(0) = 4 \neq -1$, so $y = \sin 4x + 2\cos 4x$ is not a solution to the initial value problem and the correct choice would again be (b).