FE Exam Preparation - Mathematics

General Advice(?): Get the calculator that you are going to use and the Reference Handbook well ahead of time and get used to working with both. Maybe use the calculator to do some of your classwork, just to become familiar with where the various operations and functions are. **Be aware of this**: Due to the fact that the exam is multiple choice, with answers provided, *I would suggest that you not "completely" solve any exercise unless it is absolutely necessary.* Here are some ways to solve problems without actually working them out completely:

- When appropriate, draw a fairly accurate picture and estimate; the scales on pictures provided with the exercises seem to be very accurate, so that estimates can be made from the pictures.
- When you are asked to solve either an algebraic equation, a differential equation or a system of equations, you can check the solutions that are provided, rather than actually solving the equation(s).
- You can also test solutions to initial value problems by checking the solutions satisfy the initial conditions.
- To determine whether two trig expressions are equivalent, just substitute an angle into both and see if they come out the same. DO NOT substitute special angles like $30°$, $45°$, $60°$, etc. - use something like $20°$ instead.
- When working a problem that contains an angle whose measure is in degrees and minutes, just round to the nearest whole degree - the choices of answers will vary enough that you can determine the correct answer from your slightly incorrect answer.
- Evaluate limits numerically, using your calculator. For example, to evaluate $\lim_{x\to 0} \dfrac{\sin(3x)}{e^{2x}-1}$, "plug in" something like 0.001 for x and see what you get.

Here are some mathematical facts that I found useful when working the exam problems:

- If $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.
- When the equation of a line is put in the form $y = mx + b$, m is the slope of the line. Parallel lines have the same slope and perpendicular lines have slopes that are negative reciprocals.
- All properties of arithmetic that hold for numbers also hold for matrices, except that *matrix multiplication is NOT commutative.* That is, for two matrices $\mathbf{A}$ and $\mathbf{B}$ it is not necessarily true that $\mathbf{AB} = \mathbf{BA}$. (It can be true, but usually isn't.)
- The dot product of two vectors is a scalar, and it is zero if the vectors are perpendicular. The cross product of two vectors is a vector, and *it is perpendicular to both of the original vectors.*
- A geometric progression is a sequence of numbers for which each number is a constant multiple of the previous number.
- The solution of a first-order differential equation with no initial condition will contain one arbitrary constant. The solution of a second-order differential equation with no initial conditions given will contain two arbitrary constants.

- Conic sections:
  - The equation of a parabola contains x and y, with one squared and one not.
  - The equation of a circle contains x^2 and y^2 terms. If they are on the same side of the equation as each other, both have *the same positive* coefficients.
  - The equation of an ellipse contains x^2 and y^2 terms. If they are on the same side of the equation as each other, both have *positive* coefficients.
  - The equation of a hyperbola contains x^2 and y^2 terms. If they are on the same side of the equation as each other, their coefficients have *opposite signs*.
  - If $y = \log_a x$, then $a^y = x$.
  - $log_a x^r = r \log_a x$, $\log_a(xy) = \log_a x + \log_a y$, $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

Quickies: These are questions in the *FE Review Manual* that can be answered fairly quickly using a picture, your calculator, the reference manual or very little paper-and-pencil calculation. Give yourself about 30 seconds to a minute on each.

Geo/Trig Sample Problems: 1,5

Geo/Trig Exam Problems: 1,2,3,6, 8-12,14,16,17

Algebra Sample Problems: 1,2,5

Algebra Exam Problems: 1,2,6-9,12-16,20,29,30

Prob/Stat Exam Problems: 10-12

Calculus Sample Problems: 1,4,5

Calculus Exam Problems: 1,3,4,5,6,12,19-21

Differential Equations Sample Problems: 1,2

Differential Equations Exam Problems: 1,2,4,5,7,15,17,18

A Brief Review of (Introduction to?) Differential Equations

A differential equation (DE) is simply an equation containing derivatives. Here are two examples:

$$\frac{dy}{dx} + 3y = 0 \;, \qquad\qquad y'' + y' - 6y = \sin x$$

The solution to an algebraic equation like $2x - 7 = 3$ is a *number* that makes the equation true. **The solution to a differential equation is a FUNCTION that makes the equation true.** The **order** of a differential equation is the highest order derivative in the equation - the first equation above is first order, and the second equation is second order. I doubt you would see anything other than a first or second order equation on the exam.

Solving a differential equation essentially involves "undoing" the derivatives in the equation, which (sort of) means integrating. When we integrate a function, we obtain an arbitrary constant, and if we were to integrate twice we would get two constants. **The solution to a first order DE contains one arbitrary constant, and the solution to a second order equation contains two arbitrary constants.**

A differential equation combined with the value of the function (and maybe the first derivative) is what is called an **initial value problem** (IVP). Here is an example:

$$\frac{dy}{dx} + 3y = 0 \;, \qquad\qquad y(0) = 2 \tag{1}$$

The solution to an initial value problem *DOES NOT* contain arbitrary constants, because the initial value(s) is (are) used to determine the arbitrary constant(s). **The solution to an IVP is a function that satisfies BOTH the differential equation and the initial value(s)**. The function $y = 5e^{-3x}$ *IS NOT* a solution to the given initial value problem; you should verify that it in fact makes the equation true, but does not satisfy the initial condition. Can you guess how to modify this function to make it the solution to the IVP?

Second Order Differential Equations

The only kind of second order DE that you could expect to see on the exam is what is called a linear, constant coefficient equation. (You need not bother yourself with being familiar with that language!) Here are some examples:

$$y'' + y' - 6y = 0 \;, \quad \frac{d^2x}{dt^2} + 25x = 0 \;, \quad \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 11y = 0 \;, \quad y'' + 6y' + 9y = 0 \tag{2}$$

Note that in the first, third and fourth equations, the function is denoted by y. In the third equation we see that y is a function of x; in the first and fourth equations y could be a function of anything - it is most likely that it is a function of either x or t. The function for the second one is denoted by x, which is a function of t.

The solutions to such equations take one of four forms:

$$y = C_1e^{ax} + C_2e^{bx} \;, \qquad y = C_1e^{ax} + C_2xe^{ax} \;,$$

$$y = C_1\sin(ax) + C_2\cos(ax) \;, \qquad y = e^{ax}[C_1\sin(bx) + C_2\cos(bx)]$$

So how do we tell which kind of solution we have? Well, it goes like this:

1. From each DE we obtain something that is called the **characteristic equation** or **auxiliary equation**. This equation is simply obtained by replacing each derivative with r to the power of the order of the derivative - we think of the function itself as being the "zeroth" derivative. So for the four DEs above, the characteristic equations are

$$r^2 + r - 6 = 0 \;, \qquad r^2 + 25 = 0 \;, \qquad r^2 - 6r + 11 = 0 \;, \qquad r^2 + 6r + 9 = 0$$

2. We then solve the characteristic equation, either by factoring or using the quadratic formula (or completing the square, if you are efficient at that). Solving these characteristic equations we get
$$r = -3, 2, \qquad r = -5i, 5i, \qquad r = 3 \pm i\sqrt{2}, \qquad r = -3$$

3. The solution to the DE is then determined from the solutions to the characteristic equation:

   - When the characteristic equation has two real roots, the solution has the first form. The solution to the first differential equation is then $y = C_1 e^{-3x} + C_2 e^{2x}$.
   - When the characteristic equation has one real root, the solution has the second form. The solution to the fourth differential equation is then $y = C_1 e^{-3x} + C_2 x e^{-3x}$.
   - When the characteristic equation has purely imaginary roots, the solution has the first form. The solution to the second DE is $x = C_1 \sin(5t) + C_2 \cos(5t)$.
   - When the characteristic equation has complex roots, the solution has the fourth form. The solution to the third DE is $y = e^{3x}[C_1 \sin(\sqrt{2}\,t) + C_2 \cos(\sqrt{2}\,t)]$.

NOTE: There is no way to tell whether the solutions to the first and fourth equations from (2) are functions of x or t (or of some other variable) from the information given. I just gave the solution functions in terms of those two to show what y *could* be a function of.

Non-Homogeneous Second-Order Differential Equations

Technically speaking, the equations in (2) above are all what are called **homogeneous** equations, which is essentially because their right hand sides are zero. The equation

$$y'' + y' - 6y = \sin(3x) \tag{3}$$

is a non-homogeneous second order equation, since the right hand side is not zero. (The fact that the right hand side is a function of x tells us that the solution function y is a function of x.) **The solution to a non-homogeneous DE is the solution to the homogenous equation formed by making the right hand side zero, plus an additional term or terms that look something like the right hand side**. For example, the solution to (3) is

$$y = C_1 e^{-3x} + C_2 e^{2x} - \frac{5}{78}\sin(3x) + \frac{1}{78}\cos(3x).$$

The method for determining the coefficients of $\sin(3x)$ and $\cos(3x)$ is a bit too complicated for a brief treatment. *It is generally not necessary to be able to find the coefficients when answering FE exam questions.*

Here are a couple final notes:

- The solution to the IVP (1) is $y = 2e^{-3x}$.
- Those of you who have had a class in differential equations might remember that the reason for the trigonometric functions in the solutions to second order equations is due to the facts that
$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{and} \qquad e^{a+bi} = e^a e^{ib} = e^a(\cos b + i\sin b)$$