Intermediate Algebra

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Contents

0 To The Student

This textbook is designed to provide you with a basic reference for the topics within. That said, it cannot learn for you, nor can your instructor; ultimately, the responsibility for learning the material lies with you. Before beginning the mathematics, I would like to tell you a little about what research tells us are the best strategies for learning. Here are some of the principles you should adhere to for the greatest success:

- **It's better to recall than to review.** It has been found that re-reading information and examples does little to promote learning. Probably the single most effective activity for learning is attempting to recall information and procedures yourself, rather than reading them or watching someone else do them. The process of trying to recall things you have seen is called *retrieval*.

- **Spaced practice is better than massed practice.** Practicing the same thing over and over (called *massed practice*) is effective for learning very quickly, but it also leads to rapid forgetting as well. It is best to space out, over a period of days and even weeks, your practice of one kind of problem. Doing so will lead to a bit of forgetting that promotes retrieval practice, resulting in more lasting learning. And it has been determined that your brain makes many of its new connections while you sleep!

- **Interleave while spacing.** *Interleaving* refers to mixing up your practice so that you're attempting to recall a variety of information or procedures. Interleaving naturally supports spaced practice.

- **Attempt problems that you have not been shown how to solve.** It is beneficial to attempt things you don't know how to do *if you attempt long enough to struggle a bit*. You will then be more receptive to the correct method of solution when it is presented, or you may discover it yourself!

- **Difficult is better.** You will not strengthen the connections in your brain by going over things that are easy for you. Although your brain is not a muscle, it benefits from being "worked" in a challenging way, just like your body.

- **Connect with what you already know, and try to see the "big picture."** It is rare that you will encounter an idea or a method that is completely unrelated to anything you have already learned. New things are learned better when you see similarities and differences between them and what you already know. Attempting to "see how the pieces fit together" can help strengthen what you learn.

- **Quiz yourself to find out what you *really* do (and don't) know.** Understanding examples done in the book, in class, or on videos can lead to the illusion of knowing a concept or procedure when you really don't. Testing yourself frequently by attempting a variety of exercises without referring to examples is a more accurate indication of the state of your knowledge and understanding. This also provides the added benefit of interleaved retrieval practice.

- **Seek and utilize immediate feedback.** The answers to all of the exercises in the book are in the back. Upon completing any exercise, check your answer right away and correct any misunderstandings you might have. Many of our in-class activities will have answers provided, in one way or another, shortly after doing them.

Through the internet, we now have immediate access to huge amounts of information, some of it good, some of it not! I have attempted to either find or make *quality* videos over all of the concepts and procedures in this book. Anywhere that you see a blue-green box (usually around an example number) in the electronic version of the book you can click the enclosed text to lead you to a similar or relevant example. The majority of these videos were produced by James Sousa and Patrick Jones, and made available through their websites mathispower4u and PatrickJMT. A few other videos produced by Marty Brandl, Regina Parsons and Marc Whitaker are used as well.

I would encourage you to view at least a few of the videos to see whether or not they could be useful for your learning. In keeping with the above suggestions, one good way to use the videos is to view long enough to see the problem to be solved, then pause the video and try solving it yourself. After solving or getting stuck (while trying hard, see the fourth bullet above), forward ahead to see the solution and/or watch how the problem is solved.

It is somewhat inevitable that there will be some errors in this text that I have not caught. As soon as errors are brought to my attention, I will update the online version of the text to reflect those changes. If you are using a hard copy (paper) version of the text, you can look online if you suspect an error. If it appears that there is an uncorrected error, please send me an e-mail at *gregg.waterman@oit.edu* indicating where to find the error.

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1 Expressions and Exponents

1.1 Exponents

1. (a) Evaluate expressions involving positive integer exponents, without and with a calculator.

NOTE: In the past you have taken 5×7 to mean 'five times seven.' From now on we will denote multiplication of two numbers using either a dot, $5 \cdot 7$, or parentheses, $5(7)$. This is done because the symbol $\times$ is easily confused with the letter x, which we will be using to signify an unknown number.

One thing multiplication does for us is to simplify repeated addition of the same number:

$$7 + 7 + 7 + 7 + 7 = 5(7) = 35$$

As you probably know, exponents are used to simplify repeated multiplication:

$$7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 = 7^5$$

So for any natural number (the numbers 1, 2, 3, 4, ...), a^n means n a's multiplied together. This is an exponential expression, the value a is called the **base** and n is the **exponent**.

◇ **Example 1.1(a):** Compute 2^3, 3^2 and 7^1.

Solution: $2^3 = 2 \cdot 2 \cdot 2 = 8\,,\qquad 3^2 = 3 \cdot 3 = 9\,,\qquad 7^1 = 7$

◇ **Example 1.1(b):** Find $\left(\dfrac{2}{3}\right)^4$.

Solution: $\left(\dfrac{2}{3}\right)^4 = \dfrac{2}{3} \cdot \dfrac{2}{3} \cdot \dfrac{2}{3} \cdot \dfrac{2}{3} = \dfrac{16}{81}$. Note for future reference that $\left(\dfrac{2}{3}\right)^4 = \dfrac{2^4}{3^4}$.

◇ **Example 1.1(c):** Compute $(-3)^2$ and $(-2)^3$.

Solution: $(-3)^2 = (-3)(-3) = 9\,,\qquad (-2)^3 = (-2)(-2)(-2) = -8$

In general, a negative number to an even power is positive, and a negative number to an odd power is negative.

◇ **Example 1.1(d):** Compute -3^2.

Solution: In the expression $(-3)^2$ the exponent applies to the number -3, including the sign, so the result is as in the previous example. In the expression -3^2 the exponent applies only to the number 3, and the negative sign is recorded after computing:

$$-3^2 = -(3 \cdot 3) = -9$$

One thing we want to get clear right now is the difference between $(-3)^2$ *and* -3^2. Think about the previous two examples carefully.

◇ **Example 1.1(e):** Compute $2^3 \cdot 2^4$.

Solution: $$2^3 \cdot 2^4 = (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2) = 2^7$$

Summary of Exponents

- Exponents indicate repeated multiplication: $a^n = \overbrace{a \cdot a \cdot a \cdots a}^{n \text{ times}}$
- $a^1 = a$
- $(-a)^n = (-a)(-a)(-a)\cdots(-a)$ and $-a^n = -(a \cdot a \cdot a \cdots a)$

We often refer to 5^2 as 'five squared,' because its value is the area of a square that is five units on a side. Similarly, 'five cubed' refers to the quantity 5^3, the volume of a cube with edges of length five units.

Suppose that you needed to find something like 12^7. You certainly wouldn't want to do it by hand, and it is even a bit annoying to multiply seven twelves on a calculator. Fortunately, your calculator has a key which will do exponents for you.

◇ **Example 1.1(f):** Use your calculator to compute 12^7.

Solution: To compute 12^7, enter the number 12, then hit the $\wedge$ key or the y^x key followed by the number 7, and finish by hitting $=$. You should get 35831808.

◇ **Example 1.1(g):** Compute $(-5)^6$ using your calculator.

Solution: To compute $(-5)^6$ you don't need to use the parentheses that your calculator has. Just enter -5, then use the $\wedge$ key or the y^x key with the exponent 6. To compute -5^6 simply compute 5^6 *then* change the sign.

Section 1.1 Exercises To Solutions

1. Simplify each of the following *without using a calculator*.

   (a) 3^3 (b) $\left(\frac{1}{2}\right)^6$ (c) 10^2 (d) -5^3

   (e) $(-5)^3$ (f) $(-5)^2$ (g) -5^2 (h) $\left(-\frac{4}{3}\right)^2$

2. Use your calculator to find the value of each of the following exponential expressions:

   (a) 4^9 (b) -3^{10} (c) $(-3)^{10}$ (d) 5^{12}

3. Try using your calculator to find the value of 23^8. You will get something that you may not recognize; we will learn about this soon.

1.2 Evaluating Numerical Expressions

1. (b) Apply order of operations to evaluate polynomial and rational expressions without a calculator.

   (c) Apply order of operations to evaluate polynomial and rational expressions with a calculator.

An **expression** is a collection of numbers that are to be combined using mathematical operations like addition, multiplication, and so on. Some examples are

$$12 - 4(3+2) \qquad \frac{3(5)-7}{4} \qquad x + 3y \qquad \frac{4r-s}{t}$$

The letters x, y, r, s and t represent numbers whose values we don't know - hence they are called **unknowns**. We will call the first two expressions **arithmetic expressions** or **numerical expressions**, meaning that all of the numbers involved are known. More often in this course we will be dealing with **algebraic expressions**, which are expressions containing unknown numbers that are represented by letters. The second two expressions are examples of these. As you may already know, $3(5)$ means three times five, and $3y$ means 3 times y. The long horizontal bar indicates that a division is to be performed.

The act of determining the single numerical value for a numerical expression is called **evaluating** the expression. *We want to be certain that everyone gets the same value when evaluating an expression*, so when evaluating most expressions we will need to refer to what is called the **order of operations**. Consider the expression $3 + 5(4)$, which contains an addition and a multiplication. If we do the addition first we get 32 and if we do the multiplication first we get 23. So, depending on which operation we do first, we can get two different values for this expression. The order of operations tell us the order in which the operations in an expression are performed; they were developed to make sure that we all interpret an expression in the same way.

Order of Operations

(1) **Parentheses:** Operations in parentheses must be performed first. If there is more than one operation within a set of parentheses, then the operations within the parentheses must follow the remaining rules.

(2) **Exponents:** Exponents are applied next - remember that they apply only to the number that they are directly "attached" to.

(3) **Multiplication and Division:** These both have equal priority. When it is not clear from other things which to do first, do them from left to right.

(4) **Addition and Subtraction:** These also have equal priority, and are done from left to right.

Evaluating Numerical Expressions

◇ **Example 1.2(a):** Evaluate the expression $3 + 5(4)$.

Solution: There are two operations, an addition and a multiplication. Since multiplication is higher on the list than addition, the multiplication is done first, giving $3 + 20$. We then perform the addition to get 23.

NOTE: You will usually be asked to show how an expression is evaluated, step-by-step. One way to do this is to write the original expression followed by an equal sign, followed by the expression that results when the first operation is performed, followed by another equal sign and the result after the second operation is performed. This is repeated until the final result is obtained. The process from Example 1 would then be illustrated as shown below and to the left.

$$3 + 5(4) = 3 + 20 = 23 \qquad\qquad \begin{array}{c} 3 + 5(4) \\ 3 + 20 \\ 23 \end{array}$$

Another option is to write the original expression, then write each step below the previous one, as shown above and to the right. *Note that, either way, the original expression is always given first!*

◇ **Example 1.2(b):** Evaluate the expression $(3 + 5)4$.

Solution: Compare this expression with the expression from Example 1.2(a). The same numbers and operations are involved, but parentheses have been inserted to tell us to do the addition first in this case:

$$(3 + 5)4 = 8(4) = 32$$

◇ **Example 1.2(c):** Evaluate $9 - 4 + 6$.

Solution: The expression contains both an addition and a subtraction, neither of which is necessarily to be done before the other. In this case, we simply work from left to right:

$$\begin{array}{c} 9 - 4 + 7 \\ 5 + 7 \\ 12 \end{array}$$

If we had intended for the addition to be done first, we would have to use parentheses to indicate that:

$$9 - (4 + 7) = 9 - 11 = -2$$

Note that the two results are different!

Sometimes we want parentheses inside parentheses; in those cases we usually use the 'square brackets' [ ] instead of ( ) for the 'outer' set. An example is the expression $5[3 - 2(7 + 1)]$. Here we do what is in the parentheses first, then the square brackets.

◇ **Example 1.2(d):** Evaluate $5[3-2(7+1)]$.

Solution:

$$\begin{gathered} 5[3-2(7+1)] \\ 5[3-2(8)] \\ 5[3-16] \\ 5[-13] \\ -65 \end{gathered}$$

NOTE: Soon we will discuss the distributive property, which you are probably already familiar with. It could be used in the previous example, but *it is simpler to just follow the order of operations when working with known numbers*, as in this last example. For other situations we will have to use the distributive property.

A fraction bar in an expression indicates division of the expression above the bar (the **numerator**) by the expression below the bar (the **denominator**). In this case the bar also acts as two sets of parentheses, one enclosing the numerator and one enclosing the denominator. In other words, something like

$$\frac{3(7)-5}{2+8} \qquad \text{means} \qquad \frac{(3(7)-5)}{(2+8)}$$

We don't usually put in the parentheses shown in the second form of the expression above, but we do need to understand that they are implied.

◇ **Example 1.2(e):** Evaluate $\dfrac{3(7)-5}{2+8}$.

$$\frac{3(7)-5}{2+8} = \frac{21-5}{10} = \frac{16}{10} = \frac{8}{5}$$

NOTE: Fractions must always be reduced when the numerator and denominator contain common factors. They *can* be left in what is called **improper form**, which means that the numerator is greater than the denominator.

There will be times that you will want to evaluate expressions using your calculator. Your calculator 'knows' the order of operations, which can be an advantage or a disadvantage. When using your calculator, you need to 'think the way it does!' Here are two examples:

◇ **Example 1.2(f):** Evaluate the expression $5[3-2(7+1)]$ using your calculator.

Solution: Our calculators do not have brackets, but we just use parentheses instead, and we use $\times$ for multiplication. So this is entered into our calculators as

$$5 \times (3 - 2 \times (7+1)) =$$

Try it, making sure you get -65 as your result! (Be sure to include the $=$ sign at the end to complete the calculation.)

◇ **Example 1.2(g):** Use your calculator to evaluate $\dfrac{3(7)-5}{2+8}$.

Solution: The key to evaluating something like this is to recall that it should be interpreted as $\dfrac{(3(7)-5)}{(2+8)}$. Thus it is entered in the calculator as

$$(3 \times 7 - 5) \div (2+8) =$$

Try it; the result should be 1.6, the decimal equivalent of $\frac{8}{5}$.

Evaluating Algebraic Expressions

As stated previously, an algebraic expression is one containing unknown numbers (often only one) that are represented by letters. On many occasions we will evaluate algebraic expressions for given values of the unknown or unknowns in the expression. Suppose that we are asked to evaluate

$$3x - 5y \qquad \text{for} \qquad x = 2,\ y = -7$$

This means to replace x with 2 and y with -7 in the algebraic expression, and evaluate the resulting numerical expression.

◇ **Example 1.2(h):** Evaluate $3x - 5y$ for $x = 2,\ y = -7$.

Solution: A good strategy for evaluating expressions like these is to replace each unknown with a set of parentheses with space between them, then fill the parentheses with the numbers and evaluate:

$$3x - 5y \quad \Rightarrow \quad 3(\ \ ) - 5(\ \ ) \quad \Rightarrow \quad 3(2) - 5(-7) = 6 + 35 = 41$$

All we would show when doing this is the sequence $3(2) - 5(-7) = 6 + 35 = 41$; the rest of what I've shown is just what we would be thinking.

◇ **Example 1.2(i):** Evaluate $5x - 2x^2$ for $x = 3$.

Solution:

$$5(3) - 2(3)^2$$
$$15 - 2(9)$$
$$15 - 18$$
$$-3$$

Notice how the order of operations were followed when evaluating this expression, in computing $(3)^2$ *before* multiplying by two.

◇ **Example 1.2(j):** Evaluate $5x - 2x^2$ for $x = -3$.

Solution: Here one must be a little bit careful because we are evaluating the expression for a *negative* value of x:

$$5(-3) - 2(-3)^2 = -15 - 2(9) = -15 - 18 = -33$$

Section 1.2 Exercises To Solutions

1. Evaluate each of the following expressions, without using a calculator. Give any answers that are not whole numbers as fractions in reduced form.

(a) $5-3(7)$ (b) $\dfrac{8+4}{2}-1+5(4)$ (c) $[2(3+7)-5](4-6)$

(d) $\dfrac{10-7}{4}+\dfrac{4+1}{2(3)}$ (e) $3(5-1)-5(4+1)$ (f) $(2-3)[5+2(7-1)]$

2. Evaluate each of the expressions from Exercise 1 using your calculator, *without writing down any intermediate values*. Use the parentheses ( ) on your calculator, and remember that the calculator "knows" the order of operations. If rounding is necessary, round to the hundredth's place, which is two places past the decimal.

3. Evaluate each expression for the given value or values of the unknowns.

(a) $2l+2w, \quad l=13,\ w=5$ (b) $\frac{1}{2}bh, \quad b=7,\ h=3$

(c) $\dfrac{6t}{t-1}, \quad t=5$ (d) $6x-3(x+2), \quad x=4$

4. Evaluate each of the following arithmetic expressions without using a calculator. Remember that exponents take precedence over all other operations except those taking place in parentheses. Note also that *an exponent applies only to the numbers (or unknown) that it is directly "attached" to.*

(a) $5^2-2(7)^2$ (b) $(5+4)(-2)^3$ (c) $8+4(-3)^2$

(d) $3(5)^2-7(5)+14$ (e) $4(-3)-2(-3)^2$ (f) $\dfrac{100-5(3)^2}{33}$

(g) $-(-2)^2+5(-2)$ (h) $4(3)-3^2$ (i) $4(-3)-(-3)^2$

5. Evaluate each of the above expressions with a calculator, without recording any intermediate values.

6. Evaluate each of the following algebraic expressions for the given value of the unknown *without using a calculator*.

(a) $x^2-5x+3, \quad x=4$ (b) $x^2-5x+3, \quad x=-4$

(c) $(3x-5)^2 \quad x=-1$ (d) $x^2+3x+4, \quad x=5$

7. Use your calculator to evaluate each of the following:

(a) $P(1+r)^t$ for $P=800$, $r=0.05$ and $t=7$ (b) $-16t^2+48t+5, \quad t=2$

8. (a) Evaluate $5x-3x$ and $2x$ for $x=4$. What do you notice?

(b) Evaluate the same two expressions for $x=10$.

(c) Evaluate $5(x+7)$ and $5x+35$ for $x=3$ and $x=-7$.

1.3 Simplifying Linear Algebraic Expressions

> 1. (d) Apply order of operations and the distributive property to simplify linear algebraic expressions.

In the previous section we learned how to evaluate numerical expressions. In this section we'll be working with algebraic expressions, and our objective with such expressions is usually to 'simplify' them. When doing Exercise 8 from the previous section you should have noticed that the expressions $5x - 3x$ and $2x$ give the same results when evaluating for $x = 4$. We might think this is just an accident, but the same thing happens when evaluating for $x = 10$. When two algebraic expressions give the same result when evaluated for any value of the unknown (the same value in *BOTH* expressions) we say the expressions are equal. In this case we write $5x - 3x = 2x$. This should make sense intuitively; if we have five of some number and we remove (subtract) three of the same number, then two of the unknown number should remain.

Similarly, $5(x + 7) = 5x + 35$. This can be understood as saying that we can add two numbers and multiply the result by five, but we will get the same result if we multiply each number individually by five first, *then* add those two results. You probably know that this is a result of the distributive property, which we'll discuss soon.

A large part of this course will be devoted to simplifying expressions, which is the process of taking an expression and finding an equivalent expression that is somehow 'simpler.' We will have two main tools for doing this: 'combining like terms' and the distributive property. (Combining like terms is really the distributive property, as we'll see.) An example of combining like terms is that $5x - 3x$, as you saw before, is equivalent to $2x$. We can think that x is some unknown amount, then see that we start with five of that amount and remove three *of the same amount*, resulting in two of the unknown amount left. We write $5x - 3x = 2x$. Similarly, $13t + 5t = 18t$. Note that $2l + 2w$ cannot be simplified because l and w are likely (but not necessarily) different amounts. Or, $2l$ and $2w$ are not "like terms," so they can't be combined.

◇ **Example 1.3(a):** Combine like terms when possible: $8x + 3x$, $8x + 3y$, $8x^2 + 3x$

Solution: Like the other examples just given, $8x + 3x = 11x$. We cannot simplify $8x + 3y$ because x and y are likely different values, and we can also not simplify $8x^2 + 3x$ because $x^2 \neq x$ (unless $x = 1$ or $x = 0$).

In a later section we'll see examples similar to the last one above, in which we can combine some like terms.

We mentioned previously the distributive property:

> **The Distributive Property**
>
> For three numbers a, b and c, any of which might be unknowns,
>
> $$a(b + c) = ab + ac \qquad \text{and} \qquad a(b - c) = ab - ac$$

◇ **Example 1.3(b):** Simplify $8(x-2)$.

Solution: Applying the subtraction version of the distributive property, we have

$$8(x-2) = 8(x) - 8(2) = 8x - 16$$

We usually leave out the middle step above - I just put it in to make it completely clear what is happening.

◇ **Example 1.3(c):** Simplify $3(x+4)+7x$.

Solution: In this case we must apply the **distributive property** before combining like terms:

$$3(x+4)+7x = 3x+12+7x = 10x+12$$

Whenever you are uncertain about whether you correctly simplified an expression, it is a good idea to check your work by evaluating both the original expression and the final result for some number value(s). *It is best not to use zero or one when doing this!* Let's test the original expression and the result for $x=2$:

$$3(2+4)+7(2) = 3(6)+14 = 18+14 = 32 \qquad \text{and} \qquad 10(2)+12 = 20+12 = 32$$

Note that we do not distribute to evaluate $3(2+4)$, but apply order of operations instead. Since we got the same result from evaluating both the original expression and the simplified expression for $x=2$, it is likely that that we simplified the expression correctly.

◇ **Example 1.3(d):** Simplify $2x-5(x+1)$.

Solution: It is *VERY* important to note that when we see something like this we must distribute not only the number five, but the minus sign as well:

$$2x-5(x+1) = 2x-5(x)-5(1) = 2x-5x-5 = -3x-5$$

We usually don't show the second step; I put it in to try to emphasize what is happening.

◇ **Example 1.3(e):** Simplify $3-4(3x-2)$.

Solution: Don't forget that subtracting a negative amounts to adding a positive:

$$3-4(3x-2)$$
$$3-4(3x)-4(-2)$$
$$3-12x-(-8)$$
$$3-12x+8$$
$$11-12x$$

Again, we won't usually show the second step.

◇ **Example 1.3(f):** Simplify $17x + 4[15 - 3(x + 7)]$.

Solution: When there are parentheses inside brackets, it is usually easiest to distribute to eliminate the parentheses first, combine like terms within the brackets, then distribute to eliminate the brackets:

$$17x + 4[15 - 3(x + 7)] = 17x + 4[15 - 3x - 21] =$$

$$17x + 4[-3x - 6] = 17x - 12x - 24 = 5x - 24$$

Simplifying Linear Expressions

To simplify a linear expression, take the following steps.

- Apply the distributive property to eliminate *the innermost parentheses*, taking care to distribute negatives.
- Combine like terms.
- Repeat the above two steps for any parentheses or brackets that remain.

Section 1.3 Exercises

To Solutions

1. Consider the expression $3x - 5(x - 2)$.

   (a) Evaluate the expression for $x = 3$. *Use order of operations, not the distributive property.*

   (b) Simplify the expression; for this you will need the distributive property.

   (c) Evaluate your answer to (b) for $x = 3$. Your answer should be the same as you got for (a).

2. Simplify each of the following.

   (a) $3(2x - 4) - 5(x - 1)$ (b) $2x + 3(7 - x) + 5$

   (c) $5t + 4(t - 1)$ (d) $6 - (2x + 4)$

3. Simplify each expression. Refer to Example 1.3(f) if you are not sure what to do.

   (a) $x - 7[3x - (2 - x)]$ (b) $-3[2x - 4(3x + 1)]$

   (c) $4[7 - 2(x + 1)]$ (d) $(x - 7) + [3x - (x + 2)]$

   (e) $x - 7[x - (2x + 3)]$ (f) $4x - 5[3(2x - 1) - 7x]$

4. Simplify the expressions with unknowns in them, evaluate the numerical expressions. When evaluating the numerical expressions, *DO NOT* use the distributive property; instead, just apply the order of operations.

(a) $(2+5)[3-(7+2)]$

(b) $2x+5[x-(7x+2)]$

(c) $3(2x-5)-4(x+1)$

(d) $2+5(3-7+2)$

(e) $3(2-5)-4(3+1)$

(f) $-2(2x+1)+5x(3-1)$

1.4 Algebraic Expressions With Exponents

1. (e) Simplify algebraic expressions with positive integer exponents.

In this section we will simplify algebraic expressions containing exponents. We will see that this can be done by one of two ways. A person can either work each such problem "from scratch" by simply applying the definition of an exponent, or one can use instead some "rules" that we will now derive. Note the following:

$$x^3 \cdot x^2 = (x \cdot x \cdot x) \cdot (x \cdot x) = x \cdot x \cdot x \cdot x \cdot x = x^5 = x^{3+2}$$

and

$$(x^3)^2 = (x \cdot x \cdot x)^2 = (x \cdot x \cdot x) \cdot (x \cdot x \cdot x) = x^6 = x^{3 \cdot 2}$$

These indicate that the following rules hold for exponents: $x^m x^n = x^{m+n}$ and $(x^m)^n = x^{mn}$. *It is easy to confuse these two rules, but if you think back to the two examples given you can always sort out which is which.* We can also see that

$$(xy)^3 = (xy)(xy)(xy) = (x \cdot x \cdot x)(y \cdot y \cdot y) = x^3 y^3$$

This demonstrates the rule that $(xy)^m = x^m y^m$.

◇ **Example 1.4(a):** Simplify $(5x)^2$.

Solution: By the third rule above, $(5x)^2 = 5^2 x^2 = 25x^2$. We can also "barehand" it:

$$(5x)^2 = (5x)(5x) = 5 \cdot 5 \cdot x \cdot x = 25x^2$$

◇ **Example 1.4(b):** Simplify $(5x^7)^2$.

Solution: Combining the third rule with the second, $(5x^7)^2 = 5^2(x^7)^2 = 25x^{7 \cdot 2} = 25x^{14}$. We can also use the definition of an exponent and the first rule above:

$$(5x^7)^2 = (5x^7)(5x^7) = 5 \cdot 5 \cdot x^7 \cdot x^7 = 25x^{7+7} = 25x^{14}$$

◇ **Example 1.4(c):** Simplify $(4x^5)(2x^7)$.

Solution: $$(4x^5)(2x^7) = 4 \cdot 2 \cdot x^5 \cdot x^7 = 8x^{12}$$

◇ **Example 1.4(d):** Simplify $(4x^5)(2x^7)^3$.

Solution: $$(4x^5)(2x^7)^3 = 4x^5 \cdot 2^3(x^7)^3 = 4x^5 \cdot 8x^{21} = 32x^{26}$$

In Example 1.1(b) we saw that

$$\left(\frac{2}{3}\right)^4 = \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} = \frac{16}{81}$$

This indicates that $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$. We can also see that

$$\frac{x^8}{x^3} = \frac{x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot}{x\cdot x\cdot x} = \frac{x}{x}\cdot\frac{x}{x}\cdot\frac{x}{x}\cdot\frac{x\cdot x\cdot x\cdot x\cdot x}{1} = 1\cdot 1\cdot 1\cdot x^5 = x^5 = x^{8-3}$$

so $\frac{x^m}{x^n} = x^{m-n}$, at least when $m > n$.

Rules of Exponents

- $x^m x^n = x^{m+n}$
- $(x^m)^n = x^{mn}$
- $(xy)^m = x^m y^m$
- $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$
- $\frac{x^m}{x^n} = x^{m-n}$ if $m > n$

At this point we should consider the last rule to only be valid when $m > n$. When this is not the case we get a computation like this:

$$\frac{x^3}{x^8} = \frac{x\cdot x\cdot x}{x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot} = \frac{x}{x}\cdot\frac{x}{x}\cdot\frac{x}{x}\cdot\frac{1}{x\cdot x\cdot x\cdot x\cdot x} = 1\cdot 1\cdot 1\cdot\frac{1}{x^5} = \frac{1}{x^5}$$

We can now use the rules for exponents and the principle just shown to simplify fractions containing exponents.

◇ **Example 1.4(e):** Simplify $\frac{15x^2}{10x^7}$.

Solution: Here we can separate the given fraction into the product of two fractions, reduce each, and multiply the result back together:

$$\frac{15x^2}{10x^7} = \frac{15}{10}\cdot\frac{x^2}{x^7} = \frac{3}{2}\cdot\frac{1}{x^5} = \frac{3}{2x^5}$$

◇ **Example 1.4(f):** Simplify $\frac{(4x^2)^3}{8x^{10}}$.

Solution: In this case we must first apply the third power to the numerator (as prescribed by the order of operations), then we can reduce the fraction in the same way as in the previous example.

$$\frac{(4x^2)^3}{8x^{10}} = \frac{4^3(x^2)^3}{8x^{10}} = \frac{64x^6}{8x^{10}} = \frac{8}{x^4}$$

◇ **Example 1.4(g):** Simplify $\left(\frac{21x^9}{35x^2}\right)^3$.

Solution: For this type of exercise it is generally simplest to reduce the fraction inside the parentheses first, as done below, then apply the fourth rule.

$$\left(\frac{21x^9}{35x^2}\right)^3 = \left(\frac{3x^7}{5}\right)^3 = \frac{(3x^7)^3}{5^3} = \frac{27x^{21}}{125}$$

Section 1.4 Exercises To Solutions

1. Simplify each of the following.

   (a) $(4x)(3x^7)$ (b) $(2xy^5)^3$ (c) $(3y^5)^2$ (d) $(2x^3y)^4$

2. Simplify each of the following exponential expressions. Give all answers without negative exponents (which in theory you are not supposed to know about at this point).

   (a) $x^2 \cdot x^5$ (b) $(y^7)^2$ (c) $(s^2t^3)^5$ (d) $\frac{r^7s}{r^2s^4}$

   (e) $\frac{-24x^3y^5}{16x^4y^2}$ (f) $3(z^2)^5$ (g) $\frac{x^3y^6}{x^3y^3}$ (h) $\left(\frac{8x^2y}{4x^4}\right)^3$

3. Evaluate each *without using a calculator*.

   (a) $(-3)^2$ (b) -3^2 (c) $\left(\frac{5}{2}\right)^3$ (d) $(5-2)(4+1)^2$

   (e) $(7-3)[5-2(3+1)]$ (f) $7-3[5-2(3-1)]$ (g) $\frac{8+3(3-1)}{-2(1+3)}$

4. Evaluate each expression for the given value of the unknown.

   (a) a^2-5a+2 for $a=-4$ (b) $\frac{3x-x^2}{x-3}$ for $x=-1$

5. Simplify each:

   (a) $8x-3(2x+5)$ (b) $8-3[2x-(4+4)]$ (c) $2(5x-1)-3(x+2)$

1.5 Negative Exponents and Scientific Notation

1. (f) Evaluate and simplify numerical expressions involving integer (including zero) exponents without use of a calculator.

   (g) Change numbers from decimal form to scientific notation and vice versa without using a calculator.

Negative Exponents

In the last section we saw that $\frac{x^8}{x^3} = x^5$ and $\frac{x^3}{x^8} = \frac{1}{x^5}$. In the first case the result can be obtained using the rule $\frac{x^m}{x^n} = x^{m-n}$. If we were to apply the same rule to the second situation we would get $\frac{x^3}{x^8} = x^{3-8} = x^{-5}$, which would make sense if x^{-5} was equal to $\frac{1}{x^5}$. So to make things work out we define negative exponents by $x^{-n} = \frac{1}{x^n}$, and we can then remove the condition that $m > n$ for the fifth rule of exponents.

◇ **Example 1.5(a):** Simplify 5^{-3}.

Solution:

$$5^{-3} = \frac{1}{5^3} = \frac{1}{125}$$

◇ **Example 1.5(b):** Simplify $(-3)^{-4}$

Solution:

$$(-3)^{-4} = \frac{1}{(-3)^4} = \frac{1}{81}$$

One must be careful not to confuse the sign of the exponent with the sign of the number that it is applied to. *A negative exponent does not cause a number to be negative!*

Now we note two things. First, we know that $\frac{x^n}{x^n} = 1$, but if we again apply the rule $\frac{x^m}{x^n} = x^{m-n}$ we get $\frac{x^n}{x^n} = x^{n-n} = x^0$. Therefore it must be the case that $x^0 = 1$ regardless of what x is (well, unless x is zero). Second, remembering that to divide by a fraction we really multiply by its reciprocal,

$$\left(\frac{3}{5}\right)^{-2} = \frac{1}{(\frac{3}{5})^2} = \frac{1}{\frac{9}{25}} = 1 \div \frac{9}{25} = 1 \cdot \frac{25}{9} = \left(\frac{5}{3}\right)^2$$

Here we see that $\left(\frac{3}{5}\right)^{-2}$ is equivalent to $\left(\frac{5}{3}\right)^2$, giving us our final rule for negative and zero exponents.

Rules of Negative and Zero Exponents

- $x^{-n} = \dfrac{1}{x^n}$
- $\left(\dfrac{x}{y}\right)^{-n} = \left(\dfrac{y}{x}\right)^n$
- $x^0 = 1$ for all values of x except $x = 0$. 0^0 is undefined.

Scientific Notation

Suppose that we are considering the number of molecules in some amount of a substance, and we decide that there are 2,340,000,000 molecules. There might actually be 2,341,257,956 molecules, but we usually can't determine the value this exactly, so perhaps 2,340,000,000 is the best we can do. Now this number is two billion three hundred and forty million, which is the same as 2.34 billion. A billion happens to be 10^9, so

$$2,340,000,000 = 2.34 \times 10^9$$

The second form of this number is what is called the **scientific notation** form of the number. Scientific notation is used to give very large or small numbers. The scientific notation form of a number consists of a decimal number with just one digit to the left of the decimal point, times a power of ten. Notice that to change 2,340,000,000 to 2.34 we must move the decimal point 9 places to the left; this is where we get the power of ten when changing a number from decimal form to scientific notation form.

◇ **Example 1.5(c):** Change 7100 to scientific notation.

Solution: We move the decimal point three places to the left to get 7.1. We must then multiply by 10^3 so that the actual value of the number will be 7100, rather than 7.1. The final result is then 7.1×10^3.

◇ **Example 1.5(d):** In addition to changing a decimal to scientific notation, we will at times want to change the other direction; change 4.09×10^6 to decimal form.

Solution: The decimal point must be moved six places to the right because of the exponent of six. Therefore the decimal equivalent to 4.09×10^6 is $4,090,000$.

A small number like 0.00000056 is equal to 5.6 times 0.0000001, but $0.0000001 = \frac{1}{10000000} = 10^{-7}$. Therefore we can write

$$0.00000056 = 5.6 \times 10^{-7}$$

The number seven indicates how many places the decimal must move, but since the number is small, the exponent must be negative. Similarly, to change a number like 8.45×10^{-3} to decimal form, we know that the number is small because of the negative exponent, and we know that we must move the decimal to the left to get a small number. Thus

$$8.45 \times 10^{-3} = 0.00845$$

Section 1.5 Exercises To Solutions

1. Evaluate each of the following.

(a) 5^{-2} (b) 2^{-4} (c) $(-2)^{-4}$ (d) -3^{-2}

2. Evaluate each of the following.

(a) 3^{-1} (b) $\left(\frac{3}{4}\right)^{-2}$ (c) $(5)^0$ (d) $(-5)^{-2}$

(e) $\left(\frac{4}{3}\right)^{-1}$ (f) $\left(\frac{3}{10}\right)^{-3}$ (g) $(-4)^2$ (h) $\left(-\frac{1}{2}\right)^{-5}$

3. Change each of the following large numbers into scientific notation.

(a) 344,000 (b) 156,700,000,000,000

4. Change each of the following into decimal form.

(a) 9.85×10^4 (b) 7.328×10^1

5. If the number given is in decimal form, change it to scientific notation form. If the number given is in scientific notation form, change it to decimal form.

(a) 4300 (b) 1.67×10^{-3} (c) 0.00043 (d) $150,700$

(e) 0.0000369 (f) 2.65×10^3 (g) 3.142×10^7 (h) 0.0062

6. Simplify each of the following exponential expressions. Give all answers without negative exponents (which in theory you are not supposed to know about at this point).

(a) s^2t^3 (b) $\frac{(6x)^2}{(2x^5)^3}$ (c) $\frac{2a^2b^3}{4a^2}$

(d) $\frac{-48ab^{10}}{-32a^4c^3}$ (e) $(2x^3)(5x^2)$ (f) $(-5z^2)^3z^7$

7. Evaluate each of the following expressions, without using a calculator. Give any answers that are not whole numbers as fractions in reduced form.

(a) $-(-5)^2 - 3(-5) + 1$ (b) $(3-1) - 4(5+1) + 2$ (c) $4(7) - \frac{42}{9-5}$

(d) $-2^2 + 5(2)$ (e) $\frac{4(4+2)}{19-5}$ (f) $-5^2 - 3(5) + 1$

8. Evaluate each of the expressions in Exercise 7 with a calculator, *without computing and writing down any intermediate results.*

9. Evaluate each of the following algebraic expressions for the given value of the unknown *without using a calculator*.

(a) $x^2 - 4x \qquad x = -2$ (b) $\frac{1}{9-x^2}, \qquad x = 2$

2 Solving Linear and Polynomial Equations

2.1 Equations and Their Solutions

2. (a) Determine whether a value is a solution to an equation.

An equation is simply two algebraic expressions connected by an equal sign. The equations that we will deal with for now will contain only one unknown, denoted with some letter, often x. Our objective will be to find all **solutions** to a given equation. A solution is any number that the unknown can be replaced with to give a true statement.

◇ **Example 2.1(a):** Determine whether 1 and 3 are solutions to $(x+2)^2 = 9x$.

Solution: We replace x with 1, then with 3, seeing if each makes the equation true. We will use $\stackrel{?}{=}$ instead of $=$ because we are not sure whether the two sides will in fact be equal in each case.

$$x=1: \quad \begin{aligned} (1+2)^2 &\stackrel{?}{=} 9(1) \\ 3^2 &\stackrel{?}{=} 9 \\ 9 &= 9 \end{aligned} \qquad\qquad x=3: \quad \begin{aligned} (3+2)^2 &\stackrel{?}{=} 9(3) \\ 5^2 &\stackrel{?}{=} 27 \\ 25 &\neq 27 \end{aligned}$$

Therefore 1 is a solution to the equation but 3 is not.

We will be spending a great deal of time trying to find all solutions to a given equation; we call the process of finding those solutions **solving the equation**. When asked to solve an equation, *the objective is to find all solutions by the most efficient way possible!* In some cases this might just involve looking at the equation:

◇ **Example 2.1(b):** Solve $x+5=11$.

Solution: Here we can simply look at the equation and see that $x=6$ is the only solution.

The following example will prove to be extremely important soon.

◇ **Example 2.1(c):** Solve $(x+5)(x-4)=0$.

Solution: In this case, if $x=-5$ the left side becomes $(0)(-4)=0$, so $x=-5$ is a solution to the equation. For the same reason, $x=4$ is also a solution.

◇ **Example 2.1(d):** Solve $x^2-9=0$.

Solution: It should be clear that $x=3$ is a solution to this equation, but note that $x=-3$ is as well!

Section 2.1 Exercises

To Solutions

1. Determine whether each of the given values is a solution to the equation.

   (a) $6x + 2 = 9x - 4$, $x = -2$, $x = 4$

   (b) $(x + 2)^2 = 9x$, $x = 4$, $x = 0$

   (c) $2(x - 6) - 7(x - 3) = 14$, $x = -1$

   (d) $16x^2 = 9$, $x = \frac{3}{4}$, $x = -\frac{3}{4}$

   (e) $\sqrt{x + 5} = x^2 - 3x - 1$, $x = 4$, $x = -4$

   (f) $x^5 + 4x^2 = 3$, $x = 1$, $x = -1$

   (g) $\dfrac{\sqrt{x + 10}}{2} = x + 8$, $x = 4$, $x = -6$

   (h) $x^4 + 16 = 8x^2$, $x = -2$, $x = 2$

2. Consider the equation $(x + 2)(x - 7) = 0$.

   (a) Is $x = 2$ a solution to the equation?

   (b) Is $x = 7$ a solution to the equation?

   (c) Give another solution to the equation.

3. Consider the equation $x^2 = 5x$.

   (a) Is $x = 5$ a solution?

   (b) Find another solution by just looking at the equation. (**Hint:** The two easiest numbers to do arithmetic with are zero and one!)

4. Consider the equation $(x - 4)(x^2 + 2x - 15) = 0$.

   (a) Is $x = -5$ a solution to the equation?

   (b) Give a solution to the equation (other than -5 if it turned out to be a solution).

5. Evaluate each of the following.

   (a) $\left(\frac{3}{4}\right)^0$ (b) -5^3 (c) $(-2)^{-1}$ (d) $\left(-\frac{3}{4}\right)^2$

6. If the number given is in decimal form, change it to scientific notation form. If the number given is in scientific notation form, change it to decimal form.

   (a) 0.0514 (b) 8.94×10^5 (c) 6.5×10^{-4} (d) 7300

2.2 Solving Linear Equations

2. (b) Solve linear equations.

A linear equation is an equation in which the unknown is to the first power and is not under a root or in the bottom of a fraction. Some examples of linear equations are

$$5x + 7 = 13 \qquad \frac{w}{10} - \frac{4}{15} = \frac{w}{5} \qquad 3(x+5) - 7(x-1) = 2x - 4$$

Some equations that are *NOT* linear are

$$\sqrt{t-4} = t + 1 \qquad x^3 = 5x^2 - 6x \qquad \frac{5}{x^2-4} = \frac{2}{x-2} - \frac{7}{x+2}$$

The procedure for solving linear equations is fairly simple. The cornerstone idea is that we can add or subtract the same amount from both sides of an equation, or we can multiply or divide both sides by the same amount (other than zero). Formally we state these ideas as follows:

Properties of Equality

- For any number c, if $a = b$ then $a + c = b + c$ and $a - c = b - c$.
- For any number $c \neq 0$, if $a = b$ then $a \cdot c = b \cdot c$ and $\dfrac{a}{c} = \dfrac{b}{c}$.

Here are two simple examples of solving a linear equation - you should be familiar with these types of equations and the methods for solving them.

◇ **Example 2.2(a):** Solve $5x + 3 = 23$. Another Example

Solution: Here we can see that the solution to this equation is $x = 4$. If instead we had an equation like $5x + 3 = 20$, we would solve the equation as follows:

$$\begin{aligned} 5x + 3 &= 20 && \text{the given equation} \\ 5x &= 17 && \text{subtract } 3 \text{ from both sides} \\ x &= \tfrac{17}{5} && \text{divide both sides by } 5 \end{aligned}$$

◇ **Example 2.2(b):** Solve $3x - 4 = 7x + 2$.

Solution: Here we need to get the two x terms together on one side, and the numbers 4 and 2 on the other. In this case, we subtract $3x$ from both sides to avoid having a negative coefficient for x, and subtract 2 from both sides as well:

$3x - 4 = 7x + 2$	the original equation
$-4 = 4x + 2$	subtract $3x$ from both sides
$-6 = 4x$	subtract 2 from both sides
$\frac{-6}{4} = x$	divide both sides by 4
$x = -\frac{3}{2}$	reduce the resulting fraction, a negative divided by a positive is a negative

Of course we can instead begin by subtracting $7x$ from both sides and adding four to both sides:

$3x - 4 = 7x + 2$	the original equation
$-4x - 4 = 2$	subtract $7x$ from both sides
$-4x = 6$	add 4 to both sides
$x = \frac{6}{-4}$	divide both sides by -4
$x = -\frac{3}{2}$	reduce the resulting fraction

Some equations will involve a few more steps, but shouldn't be any more difficult to solve if you learn a few basic techniques and work carefully. Here is the general procedure that we use when solving linear equations:

Solving Linear Equations

(1) If the equation contains parentheses or other grouping, eliminate them by applying the distributive property.

(2) If the equation contains fractions, eliminate them by multiplying both sides of the equation by the least common denominator (or any common denominator.)

(3) Add or subtract terms from both sides to get all terms with the unknown on one side, all without the unknown on the other. Combine like terms.

(4) Divide both sides by the coefficient of the unknown.

NOTE: The **coefficient** of the unknown is the number that it is multiplied by.

We ALWAYS show our work when solving an equation by first writing the original equation, then showing each successive step below the previous one, as shown in Examples 2.2(a) and (b). **The last line should be the solutions(s), labelled as shown in the previous examples.**

◇ **Example 2.2(c):** Solve $2(x-6)-7(x-3)=14$. Another Example

Solution: In this case we begin by distributing to eliminate parentheses, then we combine like terms. At that point we will have something like the equation in Example 2.2(a).

$2(x-6)-7(x-3) = 14$	the original equation
$2x-6-7x+21 = 14$	distribute the 2 and -7 on the left side
$-5x+15 = 14$	combine like terms on the left side
$-5x = -1$	subtract 15 from both sides
$x = \frac{-1}{-15}$	divide both sides by -15
$x = \frac{1}{15}$	a negative divided by a negative is a positive

◇ **Example 2.2(d):** Solve $\frac{2}{3}x-\frac{1}{5}=\frac{3}{10}$. Another Example Another Example

Solution: It is possible to solve this equation in the manner used in Example 2.2(a) if you are good with either fractions or your calculator. Here we will see a different method, *that has applications to more difficult equations for which your calculator won't be able to help you as much*. We begin by multiplying both sides of the equation by a number that 3, 5, and 10 all go into, preferably the smallest such number, thirty.

$\frac{2}{3}x-\frac{1}{5} = \frac{3}{10}$	the original equation
$30\left(\frac{2}{3}x-\frac{1}{5}\right) = 30\left(\frac{3}{10}\right)$	multiply both sides by 30
$30\left(\frac{2}{3}x\right)-30\left(\frac{1}{5}\right) = 30\left(\frac{3}{10}\right)$	distribute the 30 to both terms on the left side
$\overset{10}{\cancel{30}}\left(\frac{2}{\cancel{3}}x\right)-\overset{6}{\cancel{30}}\left(\frac{1}{\cancel{5}}\right) = \overset{3}{\cancel{30}}\left(\frac{3}{\cancel{10}}\right)$	multiply each fraction by 30, cancelling first
$20x-6 = 9$	result from multiplying each fraction by 30
$20x = 15$	add 6 to both sides
$x = \frac{3}{4}$	divide both sides by 20 and reduce

Section 2.2 Exercises

To Solutions

1. Solve each of the following equations.

(a) $4x+7=5$

(b) $2x-21=-4x+39$

(c) $\frac{9}{5}x-1=2x$

(d) $7(x-2)=x+2(x+3)$

(e) $8x=10-3x$

(f) $\frac{x}{4}+\frac{1}{2}=1-\frac{x}{8}$

(g) $2(x+4)-5(x+10)=6$

(h) $5x+1=2x+8$

2. For the following,

- evaluate numerical expressions (those not containing unknowns)
- simplify algebraic expressions (those containing unknowns)

(a) $10x - 3(2x + 4)$ (b) 3^{-2} (c) $10 - 3(2 - 4)$

(d) $(-3)^2$ (e) $12x + 5[2x - 3(x - 1)]$ (f) -3^2

3. Evaluate $-2x^2 + 5x - 3$ for $x = -3$

4. Simplify each of the following exponential expressions. Give all answers without negative exponents.

(a) $(2x^7)^4$ (b) $\dfrac{6z^3}{3z^4}$ (c) $3s^5(-7s)$ (d) $-\dfrac{36x^6y^8}{24x^3y^9}$

2.3 Adding, Subtracting and Multiplying Polynomials

> 2. (c) Add and subtract polynomial expressions.
>   (d) Multiply polynomial expressions.

A **polynomial** is an *expression* consisting of the powers $x, x^2, x^3, \dots$ of some unknown (in this case denoted by x), each multiplied by some number, then added or subtracted along with possibly a number. Some examples are

$$x^3 - 5x^2 + 7x - 9, \qquad 7a^4 - 3a^2 + 5, \qquad 3x + 2, \qquad 48t - 16t^2$$

There is some specific language used in discussing polynomials:

- The highest power of the unknown is called the **degree** of the polynomial. The first polynomial above is third degree, the second is fourth degree, the third polynomial is first degree, and the last one is second degree.
- Second degree polynomials are often called **quadratic polynomials**, or just "quadratics."
- Each power of the unknown, along the number that it is multiplied by and the sign before it is called a **term** of the polynomial. The terms of the first polynomial above are x^3, $-5x^2$, $7x$ and -9. The terms of the last one are $48t$ and $-16t^2$.
- The number that is added or subtracted, along with its sign, is also a term of the polynomial, called the **constant term**. The terms of the first polynomial are then x^3, $-5x^2$, $7x$, and -9, with -9 being the constant term. The constant terms of the second and third polynomials are 5 and 2, respectively.
- Each term has a degree, which is the power of the unknown in that term. For example, the term $-5x^2$ of the first polynomial is the second-degree term.
- The number (with its sign) that a power of the unknown is multiplied by is called the **coefficient** of that power of the unknown. In the first polynomial, -5 is the coefficient of x^2 and 7 is the coefficient of x.

From the examples we note several additional things:

- The unknown can be represented by any letter, and the letter doesn't really matter. The polynomial $s^3 - 5s^2 + 7s - 9$ is the same as $x^3 - 5x^2 + 7x - 9$; the letter used to represent the unknown makes no difference when we go to evaluate the polynomial for assorted values of the unknown.
- It is not necessary that all powers of the unknown less than the degree be represented.
- There need not be a constant term.
- The terms of a polynomial can be arranged in any order, although we usually arrange them so that the degree of each term is lower than the one before it.

One final item of importance is that a number alone can be thought of as a polynomial of degree zero, since something like $3x^0$ is really just 3 times 1, or 3.

If there is more than one term of a particular degree in a polynomial, we usually combine them using the distributive property. For example, $-4x^7 + 2x^7 = (-4+2)x^7 = -2x^7$.

◇ **Example 2.3(a):** Combine the like terms: $3x^2 - 5x + 2 - x^2 - 8x - 5$

Solution: The like terms are $3x^2$ and $-x^2$, $-5x$ and $-8x$, and 2 and -5. We combine them like this:

$$3x^2 - 5x + 2 - x^2 - 8x - 5 = 3x^2 - x^2 - 5x - 8x + 2 - 5 = 2x^2 - 13x - 3$$

We don't usually show the middle step above.

Adding or Subtracting Polynomials

To add or subtract polynomials we follow these steps:

> **Adding or Subtracting Polynomials**
>
> - Get rid of the parentheses on the first polynomial. Distribute the negative sign to all terms of the second polynomial when subtracting, simply get rid of the parentheses on the second when adding.
> - Combine like terms.

Here are some examples:

◇ **Example 2.3(b):** Add $(x^2 - 7x + 3) + (6x^2 + x - 9)$. Another Example

Solution: Because the polynomials are being added and there is no number in front of either set of parentheses to be distributed, we can just get rid of the parentheses and combine like terms:

$$(x^2 - 7x + 3) + (6x^2 + x - 9) = x^2 - 7x + 3 + 6x^2 + x - 9 = 7x^2 - 6x - 6$$

◇ **Example 2.3(c):** Subtract $(5x^2 - 3x + 1) - (3x^2 - 8x + 3)$. Another Example

Solution: In this case the subtraction has to distribute to all terms of the second polynomial:

$$\begin{aligned}(5x^2 - 3x + 1) - (3x^2 - 8x + 3) &= 5x^2 - 3x + 1 - 3x^2 - (-8x) - 3\\ &= 5x^2 - 3x + 1 - 3x^2 + 8x - 3\\ &= 2x^2 + 5x - 2\end{aligned}$$

We will usually not show the second step above, but instead jump to the third step or not even show any intermediate steps at all.

Note the difference in how the work is shown in the above two examples. In Example 2.3(b) we are really just simplifying the expression $(x^2 - 7x + 3) + (6x^2 + x - 9)$, and in 2.3(c) the expression $(5x^2 - 3x + 1) - (3x^2 - 8x + 3)$ is being simplified. If we have room we sometimes show our work horizontally as in Example 2.3(b). When there is not enough room to do so we usually show one step to the right, then work downward after that, like in Example 2.3(c).

Multiplying Polynomials

Recall that to simplify something like $5(2x - 7)$ we multiply the entire quantity $2x - 7$ by five, by distributing:

$$5(2x - 7) = 5(2x) + 5(-7) = 10x - 35$$

We do the very same thing with something more complicated:

$$7x^2(x^2 - 5x + 2) = (7x^2)(x^2) + (7x^2)(-5x) + (7x^2)(2) = 7x^4 - 35x^3 + 14x^2$$

The first part of this, $7x^2$ is called a **monomial**, meaning a polynomial with just one term. Notice that we simply distribute the monomial to each term of the polynomial.

Now suppose we want to simplify $(x - 3)(x^2 - 5x + 2)$. To do this we have to "double-distribute." First we distribute the $(x-3)$ factor to each term of x^2-5x+2, then we distribute those terms into $(x - 3)$:

$$\begin{aligned}(x - 3)(x^2 - 5x + 2) &= (x - 3)x^2 + (x - 3)(-5x) + (x - 3)2 \\ &= (x)(x^2) + (-3)(x^2) + (x)(-5x) + (-3)(-5x) + (x)(2) + (-3)(2)\end{aligned}$$

At this point we are not done, but let's make an observation. Notice that we have each term of the first polynomial $x-3 = x+(-3)$ times each term of the second, $x^2-5x+2 = x^2+(-5x)+2$. This is precisely how we multiply two polynomials, we *multiply each term of the first times each term of the second and add the results*. Finishing the above gives us a final result of

$$x^3 - 3x^2 - 5x^2 + 15x + 2x - 6 = x^3 - 8x^2 + 17x - 6$$

Multiplying Polynomials

- To multiply a monomial times a polynomial, distribute the monomial to each term of the polynomial and multiply.
- To multiply two polynomials, first distribute each term of the first polynomial to each term of the second and multiply. Be careful with signs! Then combine like terms.

◇ **Example 2.3(d):** Multiply $3x^2(x + 1)$. Another Example

$$3x^2(x + 1) = 3x^2(x) + 3x^2(1) = 3x^3 + 3x^2$$

In the above calculation we would not generally show the second step, and from here on I will no longer show such intermediate steps.

◇ **Example 2.3(e):** Multiply $(x+5)(x-3)$. Another Example

$$(x+5)(x-3) = x^2 - 3x + 5x - 15 = x^2 + 2x - 15$$

◇ **Example 2.3(f):** Multiply $(2x-4)(x+5)$. Another Example

$$(2x-4)(x+5) = 2x^2 + 10x - 4x - 20 = 2x^2 + 6x - 20$$

◇ **Example 2.3(g):** Multiply $(3x-4)(3x+4)$.

$$(3x-4)(3x+4) = 9x^2 + 12x - 12x - 16 = 9x^2 - 16$$

This last example illustrates something that will soon be of importance to us. Look at the results of the last three examples, together:

$$(x+5)(x-3) = x^2+2x-15, \qquad (2x-4)(x+5) = 2x^2+6x-20, \qquad (3x-4)(3x+4) = 9x^2-16$$

Notice that in the first two cases there was a middle term containing an x, whereas that term "disappeared" in Example 2.3(g). The result, $9x^2 - 16$, is something that we call a **difference of squares**.

◇ **Example 2.3(h):** Multiply $(x-2)(x^2-7x+4)$. Another Example

$$(x-2)(x^2-7x+4) = x^3 - 7x^2 + 4x - 2x^2 + 14x - 8 = x^3 - 9x^2 + 18x - 8$$

Section 2.3 Exercises

To Solutions

1. Combine the like terms in each expression.

   (a) $7x^2 - 3x + 2x^2 - 4$

   (b) $-5x^2 + 3x - 2 + 2x^2 + 7x + 4$

   (c) $x^3 - 4x^2 + x^2 - 3x - 6x - 1$

   (d) $4x^3 + 5x - 11x^2 + 2x - 3$

2. Add or subtract the polynomials, as indicated.

   (a) $(2x^2 - 5x + 7) + (7x - 3)$

   (b) $(5x + 7) - (x^2 + x - 3)$

   (c) $(-2a^2 + 4a - 5) - (-3a^2 - a - 9)$

3. For each of the following, multiply the polynomials.

(a) $(x+7)(x-2)$

(b) $x(5x^2+7x-2)$

(c) $(2x-5)(x^2+12x+1)$

(d) $(x+3)^2$

(e) $(x+4)(3x+5)$

(f) $(3x-1)(x+5)$

4. Solve each of the following equations.

(a) $3x+5=2$

(b) $4x-3(5x+2)=4(x+3)$

(c) $\frac{2}{3}x-1=\frac{3}{4}x+\frac{1}{2}$

2.4 Factoring Polynomials

> 2. (e) Factor quadratic trinomials and differences of squares. Factor polynomial expressions (quadratic, higher degree) by factoring out common factors, or grouping.

In the last section you saw how to multiply two polynomials together. We will have need to do that on occasion, but more often we will want to take a single polynomial and break it down into factors that could then be multiplied back together to get the original polynomial. One of the main reasons for developing this skill is to be able to solve equations containing polynomials, like the equation $x^2 + 2x = 15$.

◇ **Example 2.4(a):** Solve $x^2 + 2x = 15$.

Solution: We subtract 15 from both sides to get $x^2 + 2x - 15 = 0$, then factor the left side to get $(x+5)(x-3) = 0$. Here we see that if x was -5 we'd have

$$(-5+5)(-5-3) = (0)(-8) = 0,$$

so -5 is a solution to $x^2 + 2x = 15$. We can check this in the original equation:

$$(-5)^2 + 2(-5) = 25 - 10 = 15.$$

By the same reasoning, $x = 3$ is also a solution.

The simplest type of factoring is factoring out a common factor. Although it is very easy to do, students often overlook it later, so make note of it now! The next example shows this type of factoring.

◇ **Example 2.4(b):** Factor $2x^3 - 10x$.

Solution: Here we see that both terms have factors of both 2 and x, so we "remove" them:

$$2x^3 - 10x = (2x)x^2 - (2x)5 = 2x(x^2 - 5)$$

Factoring Quadratic Trinomials

Perhaps the most common type of factoring that you will do throughout all of your math courses is factoring expressions like

$$x^2 + 4x + 3 \qquad \text{and} \qquad 2x^2 - 5x + 3$$

which we call **quadratic trinomials**. The word 'quadratic' refers to the fact that the highest power of x is two, and 'trinomial' means the expressions have three terms. To give us some insight into how to factor the first of these, we see that

$$(x+a)(x+b) = x^2 + bx + ax + ab = x^2 + (a+b)x + ab \tag{1}$$

The x^2 term comes from multiplying the two x's at the fronts of $x+a$ and $x+b$. The constant term, ab, comes from multiplying the numbers a and b. Lastly, the middle term comes from the sum of ax and bx. This process is what many of us think of as FOIL: first, outside, inside, last. The next example shows how we use these ideas by first thinking about F and L, first and last, then O and I, outside and inside..

◇ **Example 2.4(c):** Factor $x^2 + 4x + 3$.

Solution: To get the x^2 first term of x^2+4x+3 we must have factors like $(x \quad)(x \quad)$. Because the constant term 3 only has factors 1 and 3, the factors of x^2+4x+3 must look like $(x \quad 1)(x \quad 3)$. Next we see that the outside and inside products of what we have so far are $3x$ and $1x$. If these were both positive their sum would be $+4x$, the middle term of the trinomial. Let's check what we have concluded:

$$(x+1)(x+3) = x^2 + 3x + x + 3 = x^2 + 4x + 3,$$

the desired result. Thus $x^2 + 4x + 3$ factors to $(x+1)(x+3)$.

In (1) on the previous page we see that the coefficient (number in front) of x is the sum of a and b. Thus when factoring $x^2 + 4x + 3$ we might realize that a and b can't both be negative or we would get a negative coefficient of x. We would then go directly to the correct factoring right away.

◇ **Example 2.4(d):** Factor $x^2 - 9x - 10$.

Solution: Again we must have factors like $(x \quad)(x \quad)$ and, because $(2)(5) = 10$ the factors of $x^2 - 9x - 10$ might be $(x \quad 2)(x \quad 5)$. In this case the outside and inside products are $5x$ and $2x$. The sum of these can't be nine, regardless of the sign of each, so the combination of 2 and 5 to get 10 is not what we want. Choosing 1 and 10 instead, we have $(x \quad 1)(x \quad 10)$, and the outside and inside products are $10x$ and $1x$. If the $10x$ was negative and the $1x$ positive, their sum would be $-9x$, which is what we want. Checking, we see that

$$(x+1)(x-10) = x^2 - 10x + x - 10 = x^2 - 9x - 10$$

This is the desired result, so $x^2 - 9x - 10$ factors to $(x+1)(x-10)$.

In both of the above examples, the value of a in the quadratic trinomials ax^2+bx+c was one, the easiest situation to deal with. When a is not one, the process of factoring becomes yet a little more complicated. There are various strategies available for factoring. If you already have a method you use to *efficiently and correctly* factor, by all means continue using it. However, if your method is slow or not reliable for giving correct factorings, you may wish to try the method I'm about to describe. The method involves some amount of trial and error, which is unavoidable without reducing the process to a huge set of rules. With a bit of practice and some 'number sense' you will be able to reduce the number of trials needed to get a correct factoring. At the top of the next page are the steps for this method; you might wish to not read them, and instead refer to them as you read through Example 2.4(e).

Factoring ax^2+bx+c

(1) Choose two numbers m and n for which $mn=a$ and set up factors $(mx\quad)(nx\quad)$. Do not change these until you have exhausted all possibilities for p and q (see below)!

(2) Choose two numbers p and q for which $pq=|c|$, the absolute value of c. In other words, don't be concerned yet with the sign of c. Set up the factors $(mx\quad p)(nx\quad q)$.

(3) Find the outer and inner products mqx and pnx from the 'FOIL' process. Add them and subtract them to see if you can get the bx term from the original trinomial ax^2+bx+c. If you can, go on to step (7).

(4) If you were not successful your first time through step (7), switch the order of p and q to get the factors $(mx\quad q)(nx\quad p)$ and try steps (3), and perhaps (7) again. If it still doesn't work, go on to step (5).

(5) Go back to step (2) and choose a new p and q. Repeat step (3). If you have exhausted all possibilities for p and q, go to the next step.

(6) Go back to step (1) and choose a different m and n, then repeat the other steps.

(7) Once you have found factors $(mx\quad q)(nx\quad p)$ for which $pq=|c|$ and mqx and pnx add or subtract to give bx, determine the signs within each factor that give the correct bx and the correct sign for c. If this doesn't work and you haven't tried switching p and q yet, go to step (4). If you have already been through (4), go back to step (2) now.

◇ **Example 2.4(e):** Factor $2x^2-5x+3$.

1) The factors $(2x\quad)(x\quad)$ will give us the $2x^2$ term of $2x^2-5x+3$.

2) We now find two factors of the 3 from $2x^2 - 5x + 3$; 1 and 3 are the only choices. This means the product looks like $(2x \quad 1)(x \quad 3)$ or $(2x \quad 3)(x \quad 1)$.

3) In the first case, multiplying the 'outside' and 'inside' gives $6x$ and x, which can be subtracted to get $5x$. The factors $(2x \quad 1)(x \quad 3)$ are then candidates for the factorization. To get the $5x$ to be negative we would have to have $(2x+1)(x-3)$, but if we 'FOIL these out we get $2x^2 - 5x - 3$ rather than $2x^2 - 5x + 3$.

4) In the first case, multiplying the 'outside' and 'inside' gives $6x$ and x, which can be subtracted to get $5x$.

5) Since $(2x \quad 1)(x \quad 3)$ didn't work, we try $(2x \quad 3)(x \quad 1)$. In this case, to get the $5x$ to be negative we would have to have $(2x-3)(x-1)$, and when we 'FOIL these out we get $2x^2 - 5x + 3$, the desired result. $2x^2 - 5x + 3$ then factors to $(2x-3)(x-1)$.

Factoring Differences of Squares

◇ **Example 2.4(f):** Factor $4x^2 - 25$.

Solution: Although we won't usually do this, let's think of this as $4x^2 + 0x - 25$ and apply the same method as used in the previous example. Although the factors could look like $(4x \quad)(x \quad)$, let's try $(2x \quad)(2x \quad)$ first. To get the constant term of 25, we could have factors that look like $(2x \quad 5)(2x \quad 5)$, and to get the 25 to be negative, one sign must be positive and the other negative. We see that

$$(2x+5)(2x-5) = 4x^2 - 10x + 10x - 25 = 4x^2 - 25,$$

so $4x^2 - 25$ factors to $(2x+5)(2x-5)$.

The quadratic expression in the previous example is called a **difference of squares**. The following is a description of how to factor a difference of squares.

> **Factoring a Difference of Squares** $m^2x^2 - n^2$
>
> The difference of squares $m^2x^2 - n^2$ factors to $(mx+n)(mx-n)$.

◇ **Example 2.4(g):** Factor $9x^4 - x^2$.

Solution: We see that there is a common factor of x^2, and after that we have a difference of squares:

$$9x^4 - x^2 = x^2(9x^2 - 1) = x^2(3x+1)(3x-1)$$

Factoring by Grouping

Factoring by grouping is a method that applies to some third degree polynomials with four terms, like

$$x^3 - x^2 + 7x - 7 \qquad \text{and} \qquad x^3 + 2x^2 - 4x - 8.$$

Basically, the method goes like this:

(1) Factor a common factor out of the first two terms, and a different common factor out of the third and fourth terms.

(2) The result of (1) is something of the form $something \times stuff \pm something\ else \times stuff$, where both pieces of '$stuff$' are the same. This expression only has two terms, with the common factor of '$stuff$' in both terms, so it can be factored out to get $(stuff) \times (something \pm something\ else)$.

It is quite likely that the above description makes no sense at all! Let's see what it means by looking at a couple of examples.

◇ **Example 2.4(h):** Factor $x^3 - x^2 + 7x - 7$.

Solution: First we factor x^2 out of the first two terms and 7 out of the third and fourth terms to get $x^2(x-1) + 7(x-1)$. This expression can be thought of as having just two terms, $x^2(x-1)$ and $7(x-1)$. Both of them have a factor of $x-1$, which can then be factored out to get $(x-1)(x^2+7)$.

Sometimes when factoring by grouping we can go one step beyond what we were able to do in this last example. The next example illustrates this.

◇ **Example 2.4(i):** Factor $x^3 + 5x^2 - 4x - 20$ completely.

Solution: Factoring x^2 out of the first two terms and -4 out of the last two gives $x^2(x+5) - 4(x+5)$. As in the last example, we then factor $(x+5)$ out of the two terms to get $(x+5)(x^2-4)$. Finally, we factor the difference of squares x^2-4 to get $(x+5)(x+2)(x-2)$.

Let's summarize the general method for factoring polynomials:

Factoring Polynomials

- Factor out common factors (of both the the numbers and the variable).
- If the polynomial has four terms, try factoring by grouping.
- Factor any quadratic factors.
- Check to see if any of your factors can be factored further.
- (Optional) Check your factoring by multiplying the factors back together.

Section 2.4 Exercises

To Solutions

1. Factor the largest possible common factor out of $8x^4 - 4x^3 + 16x^2$.

2. Factor each quadratic expression.

   (a) $x^2 - 3x - 10$ (b) $x^2 - 7x + 12$ (c) $x^2 - 2x - 15$
   (d) $x^2 + 10x + 9$ (e) $x^2 - 11x + 28$ (f) $x^2 + 4x - 12$

3. Factor each quadratic expression.

   (a) $7x^2 + 3x - 4$ (b) $6x^2 - x - 2$ (c) $6x^2 + 13x + 5$
   (d) $3x^2 - 19x - 14$ (e) $10x^2 + 27x - 9$ (f) $15x^2 - 21x + 6$

4. Factor each difference of squares.

   (a) $x^2 - 9$ (b) $25x^2 - 1$ (c) $9x^2 - 49$

5. Factor each expression completely. Do this by first factoring out any common factors, then factoring whatever remains after that.

   (a) $x^3 + 5x^2 + 4x$ (b) $3x^3 + 15x^2 - 42x$ (c) $x^4 + 6x^3 + 5x^2$

6. Factor each of the following by grouping.

   (a) $x^3 + 2x^2 - 3x - 6$ (b) $2x^3 + 3x^2 - 2x - 3$ (c) $x^3 + 2x^2 - 25x - 50$

7. Factor each of the following **completely**. Some may not be possible.

   (a) $4x^4 + 4x^3 + 4x^2$ (b) $10x^2 - 17x + 3$ (c) $3x^2 - 4x - 5$
   (d) $9x^2 + 11x + 2$ (e) $6x^3 - 4x^2 + 15x - 10$ (f) $16x^2 - 49$
   (g) $3 + 23x - 8x^2$ (h) $4x^2 + 9$ (i) $12x^3 + 24x^2 + 48x$
   (j) $6x^2 - 19x - 7$ (k) $2x^2 - 15x + 7$ (l) $x^3 + 3x^2 - 4x - 12$
   (m) $10y^2 - 5y - 15$ (n) $1 - 25x^2$ (o) $x^2 + 3x + 1$
   (p) $6x^2 - x - 15$ (q) $7x^2 + 11x + 4$ (r) $8 + 2x - 3x^2$

8. Evaluate each of the following algebraic expressions for the given value of the unknown *without using a calculator*.

   (a) $x^2 + 2x + 1, \qquad x = 5$ (b) $4x - x^2, \qquad x = -2$

9. Evaluate each of the following.

   (a) $\left(\frac{3}{2}\right)^{-3}$ (b) 5^{-2} (c) $(-7)^0$ (d) -5^{-2}

10. Add or subtract the polynomials, as indicated.

    (a) $(5x^2 - 8x + 3) - (9x^2 + 5x - 2)$ (b) $(x^2 - 4) - (x^2 + 12x + 3)$

11. For each of the following, multiply the polynomials.

    (a) $(2x - 5)(3x^2 - 7x + 4)$ (b) $(3x + 5)^2$ (c) $3x(x^2 - 7x - 5)$
    (d) $(2x - 1)(3x + 5)$ (e) $(x - 1)^2$ (f) $(x - 6)(5x^2 + x - 1)$

2.5 Solving Polynomial Equations

2. (f) Solve polynomial equations.

Look back at Example 2.4(a), where the equation $x^2 + 2x = 15$ was solved. This equation is a **polynomial equation**, which is simply an equation in which the expressions on both sides of the equal sign are polynomials. (Remember that a single number, even zero, can be thought of as a polynomial.) The highest power of the unknown that appears in the equation is called the **degree** of the equation.

To solve the equation, all terms were put on one side and the left side was factored to get $(x+5)(x-3) = 0$. The critical idea is that the factors $x+5$ and $x-3$ are being multiplied to get zero, and the only way that can happen is if one of them is zero. The only way that $x+5$ can be zero is if $x = -5$, and the only way that $x-3$ can be zero is if $x = 3$. A basic, but very important idea has been used here:

Zero Factor Property

If the product of two or more factors is zero, then at least one of the factors must be zero. Symbolically, if $A \cdot B = 0$, then either $A = 0$ or $B = 0$ (or in some cases, both).

◇ **Example 2.5(a):** Solve $2x^2 + 3x = 20$. Another Example

Solution: We begin by getting zero alone on one side and factoring:

$$\begin{aligned} 2x^2 + 3x &= 20 \\ 2x^2 + 3x - 20 &= 0 \\ (2x-5)(x+4) &= 0 \end{aligned}$$

From this we can easily see that $x = -4$ is a solution, because it will make the factor $x+4$ be zero, and the entire left side of the equation will be zero as a result. It is not as readily apparent what value of x makes $2x-5$ zero, so we set that factor to zero and solve to find out:

$$\begin{aligned} 2x - 5 &= 0 \\ 2x &= 5 \\ x &= \tfrac{5}{2} \end{aligned}$$

The equation $2x^2 + 3x = 20$ therefore has two solutions, $x = -4$ and $x = \frac{5}{2}$.

◇ **Example 2.5(b):** Solve $4x^3 - 8x^2 - 12x = 0$.

Solution: First we factor the left side, beginning with factoring a common factor of $4x$ out of each term:

$$\begin{aligned} 4x^3 - 8x^2 - 12x &= 0 \\ 4x(x^2 - 2x - 3) &= 0 \\ 4x(x-3)(x+1) &= 0 \end{aligned}$$

In this case there are three factors, $4x$, $x-3$ and $x+1$. The x values 0, 3 and -1 make them zero, so those are the solutions to the equation.

◇ **Example 2.5(c):** Solve $4x^2 - 8x - 12 = 0$.

Solution: This equation is very similar to the one in the previous example, but this one is second degree and the only common factor is 4, rather than $4x$.

$$\begin{aligned} 4x^2 - 8x - 12 &= 0 \\ 4(x^2 - 2x - 3) &= 0 \\ 4(x-3)(x+1) &= 0 \end{aligned}$$

The solutions in this case are just $x = 3$ and $x = -1$, because the factor 4 can't be zero, regardless of what value x has.

Note that Examples 2.5(a) and (c) were both second degree (quadratic) equations, and each had two solutions. The equation in Example 2.5(b) is third degree, and has three solutions. We might guess that the number of solutions is the the degree of the equation. This is close to correct, but what is really true is the following:

Solutions of Polynomial Equations

The number of solutions to a polynomial equation is *at most* the degree of the equation. (For example, a third degree polynomial has three *or fewer* solutions.)

◇ **Example 2.5(d):** Solve $x^2 - 6x + 9 = 0$.

$$\begin{aligned} x^2 - 6x + 9 &= 0 \\ (x-3)(x-3) &= 0 \end{aligned}$$

Solution: $x = 3$ is the only solution. This shows that a second degree polynomial need not have two solutions.

◇ **Example 2.5(e):** Solve $4x^2 = 25$.

$$\begin{aligned} 4x^2 &= 25 \\ 4x^2 - 25 &= 0 \\ (2x+5)(2x-5) &= 0 \end{aligned}$$

$$\begin{aligned} 2x+5 &= 0 & \text{or} \qquad 2x-5 &= 0 \\ 2x &= -5 & \text{or} \qquad 2x &= 5 \\ x &= -\tfrac{5}{2} & \text{or} \qquad x &= \tfrac{5}{2} \end{aligned}$$

◇ **Example 2.5(f):** Solve $3x^3 - x^2 - 12x + 4 = 0$.

Solution: Because this is a third degree polynomial equation, we expect that we might have as many as three solutions. We also notice that the left side is in the correct form to apply factoring by grouping.

$$\begin{aligned} 3x^3 - x^2 - 12x + 4 &= 0 \\ x^2(3x-1) - 4(3x-1) &= 0 \\ (3x-1)(x^2-4) &= 0 \\ (3x-1)(x+2)(x-2) &= 0 \end{aligned}$$

Here we can see that two solutions are 2 and -2, and setting $3x-1$ equal to zero and solving gives a third solution of $x = \frac{1}{3}$.

Solving Polynomial Equations

- Note how many solutions you expect to find.
- Eliminate parentheses (except in special cases like the one shown in the example above) by distributing or multiplying. Eliminate fractions by multiplying both sides by the least common denominator.
- Get zero on one side of the equation, all other terms on the other side.
- Factor the non-zero side of the equation completely.
- Find the value that makes each factor zero.
  - Do this by inspection if possible.
  - When you can't determine the value that makes the factor zero by inspection, set the factor equal to zero and solve.

All of the values that make factors equal to zero are solutions to the original equation.

Section 2.5 Exercises To Solutions

1. For each of the following, first eliminate fractions by multiplying both sides by a number that eliminates all the fractions. Then get zero on one side and factor.

   (a) $\frac{1}{6}x^2 = \frac{2}{3}x + 2$ (b) $\frac{1}{3}x^2 + \frac{8}{3} = 2x$ (c) $\frac{1}{4}x^2 = \frac{1}{2}x + 6$

2. For each of the following you will need to factor out a common factor (number, unknonwn, or both), then factor again.

   (a) $x^3 + 3x^2 = 10x$ (b) $4x^2 + 32x + 28 = 0$ (c) $5x^3 = 5x^2 + 30x$

3. Solve each equation. Remember to get all terms on one side, and be sure to look for common factors!

   (a) $x^2 - 13x + 12 = 0$ (b) $x^2 + 15 = 8x$ (c) $x^2 - 16 = 0$

   (d) $5x^2 + x = 0$ (e) $3x^2 = 20x + 7$ (f) $16x + 16 = x^3 + x^2$

   (g) $3x^2 + 24x + 45 = 0$ (h) $\frac{2}{3}x^2 + \frac{7}{3}x = 5$ (i) $x^3 + 3x^2 + 2x = 0$

   (j) $x^2 + 7x + 6 = 0$ (k) $8x^2 = 16x$ (l) $2x^3 + x^2 = 18x + 9$

4. Simplify each of the following exponential expressions. Give all answers without negative exponents (which in theory you are not supposed to know about at this point).

   (a) $(2x^5)^3(7x^4)$ (b) $4y^3y^4$ (c) $\left(\dfrac{15s^5t^9}{12st}\right)^2$ (d) $\dfrac{3u^3v^2}{6v^2}$

5. If the number given is in decimal form, change it to scientific notation form. If the number given is in scientific notation form, change it to decimal form.

   (a) 2.37×10^2 (b) 0.0049 (c) 1.6×10^{-1} (d) $53,000$

6. Solve each of the following equations.

   (a) $\dfrac{3x}{4} - \dfrac{5}{12} = \dfrac{5x}{6}$ (b) $11x + 9 = 3x + 11$ (c) $4(x + 1) - 7(x + 5) = 20$

3 Equations Containing Rational Expressions

3.1 Rational Expressions

3. (a) Give values that an unknown is not allowed to have in a rational expression.
   (b) Simplify rational expressions.

Consider the following:

- $\frac{8}{4} = 2$ because $2 \cdot 4 = 8$
- $\frac{32}{8} = 4$ because $4 \cdot 8 = 32$
- $\frac{18}{3} = 6$ because $6 \cdot 3 = 18$
- $\frac{0}{5} = 0$ because $0 \cdot 5 = 0$

Now what about $\frac{5}{0}$? Let's suppose that it is some number x; that is, $\frac{5}{0} = x$. Like all of the above, it must then be the case that $x \cdot 0 = 5$. But there is no value of x that makes this true, so *we say that* $\frac{5}{0}$ *is undefined*.

Division and Zero

- As long as $a \neq 0$, $\frac{0}{a} = 0$.
- For any number a, $\frac{a}{0}$ is undefined.

Now consider the expression $\frac{x^2 - 16}{x^2 + x - 12}$, which is what we call a **rational expression** or **algebraic fraction**. This is really just a fancy way of saying that it is a fraction containing an unknown value x. It is possible that for some values of x the bottom of this fraction might be zero. The whole expression would then be undefined for those values of x, so x is not allowed to have those values.

◇ **Example 3.1(a):** Give all values that x is not allowed to have in the rational expression $\frac{x^2 - 16}{x^2 + x - 12}$.

Another Example

Solution: Because $\frac{x^2 - 16}{x^2 + x - 12} = \frac{x^2 - 16}{(x+4)(x-3)}$ and $(x+4)(x-3) = 0$ when $x = -4$ or $x = 3$, x is not allowed to be -4 or 3. We indicate this by writing $x \neq -4,\ 3$.

◇ **Example 3.1(b):** Give all values that x is not allowed to have in the rational expression $\dfrac{x^2+4x-5}{x^2-4x+3}$.

Another Example

$$\frac{x^2+4x-5}{x^2-4x+3} = \frac{x^2+4x-5}{(x-3)(x-1)}, \qquad \text{so} \qquad x \neq 1,\ 3$$

You should recognize that a fraction like $\dfrac{12}{18}$ can be reduced:

$$\frac{12}{18} = \frac{6\cdot 2}{6\cdot 3} = \frac{6}{6}\cdot\frac{2}{3} = 1\cdot\frac{2}{3} = \frac{2}{3}$$

Although you probably don't show all the same steps that I have shown here, they show what really happens when you reduce. We can do the same thing for rational expressions:

$$\frac{x^2-16}{x^2+x-12} = \frac{(x+4)(x-4)}{(x+4)(x-3)} = \frac{x+4}{x+4}\cdot\frac{x-4}{x-3} = 1\cdot\frac{x-4}{x-3} = \frac{x-4}{x-3}$$

◇ **Example 3.1(c):** Simplify $\dfrac{x^2+4x-5}{x^2-4x+3}$.

Another Example

Solution: Rather than showing all of the steps shown above, we will usually just show the original, the factoring and canceling, and the final result:

$$\frac{x^2+4x-5}{x^2-4x+3} = \frac{(x+5)\cancel{(x-1)}}{\cancel{(x-1)}(x-3)} = \frac{x+5}{x-3}$$

Section 3.1 Exercises

To Solutions

1. Determine all values that the unknown is not allowed to have in each of the following.

(a) $\dfrac{x^2+x-2}{x^2-4}$ (b) $\dfrac{x^2-2x-15}{x^2-6x+5}$ (c) $\dfrac{5x+25}{x^2-25}$

(d) $\dfrac{x^2-7x+12}{x^2-9x+20}$ (e) $\dfrac{x^3+12x^2+35x}{2x^2+10x}$ (f) $\dfrac{x^2-4}{x^2-4x+4}$

2. Reduce each of the rational expressions from Exercise 1.

3. Factor each difference of squares.

(a) x^2-25 (b) $4x^2-9$ (c) $16x^2-1$

4. Factor each expression completely. Do this by first factoring out any common factors, then factoring whatever remains after that.

   (a) $20x^4 - 5x^2$ (b) $30x^3 + 21x^2 - 36x$

5. Factor each of the following by grouping.

   (a) $3x^3 + x^2 - 12x - 4$ (b) $x^3 - 5x^2 - 9x + 45$ (c) $2x^3 + 7x^2 - 2x - 7$

6. Solve each equation.

   (a) $4x^2 = 25$ (b) $x^3 = 9x^2 + 22x$ (c) $x^3 + 5x^2 - x - 5 = 0$

   (d) $\frac{1}{15}x^2 = \frac{1}{6}x + \frac{1}{10}$ (e) $21 + 4x = x^2$ (f) $2x + x^2 - 15 = 0$

3.2 Multiplying Rational Expressions

> 3. (c) Multiply rational expressions and simplify the results.

Recall that to multiply two fractions we simply multiply their numerators and denominators (tops and bottoms):

$$\frac{3}{4}\cdot\frac{2}{5}=\frac{6}{20}=\frac{3}{10}$$

It is usually more efficient to factor the numerators and denominators and cancel any common factors *BEFORE* multiplying:

◇ **Example 3.2(a):** Multiply $\frac{9}{25}\cdot\frac{10}{27}$.

$$\frac{9}{25}\cdot\frac{10}{27}=\frac{9}{5\cdot 5}\cdot\frac{2\cdot 5}{3\cdot 9}=\frac{5}{5}\cdot\frac{9}{9}\cdot\frac{2}{3\cdot 5}=1\cdot 1\cdot\frac{2}{15}=\frac{2}{15}$$

The above are all the steps that occur in the process of multiplying the two fractions, but we usually won't show all of those steps. Here is what we would usually show:

$$\frac{\overset{1}{\cancel{9}}}{\underset{5}{\cancel{25}}}\cdot\frac{\overset{2}{\cancel{10}}}{\underset{3}{\cancel{27}}}=\frac{1\cdot 2}{5\cdot 3}=\frac{2}{15}$$

We use the same process to multiply rational expressions:

◇ **Example 3.2(b):** Multiply $\frac{x+1}{x^2-4}\cdot\frac{x+2}{3x+3}$

Solution: We first factor the numerators and denominators of both fractions, and see that there are common factors of $x+1$ and $x+2$ that can be cancelled:

$$\frac{x+1}{x^2-4}\cdot\frac{x+2}{3x+3}=\frac{\cancel{x+1}}{\cancel{(x+2)}(x-2)}\cdot\frac{\cancel{x+2}}{3\cancel{(x+1)}}=\frac{1}{3(x-2)}$$

Notice that there is "nothing left" on top after common factors are canceled, but we must put a one there to keep the remaining factors in the bottom, where they belong. Notice also that we do not usually multiply factors back together at the end, after cancelling.

◇ **Example 3.2(c):** Multiply $\frac{y-1}{y^2-y-6}\cdot\frac{y^2+5y+6}{y^2-1}$ Another Example

$$\frac{y-1}{y^2-y-6}\cdot\frac{y^2+5y+6}{y^2-1}=\frac{\cancel{y-1}}{(y-3)\cancel{(y+2)}}\cdot\frac{(y+3)\cancel{(y+2)}}{(y+1)\cancel{(y-1)}}=\frac{y+3}{(y-3)(y+1)}$$

When we want to multiply a polynomial times a rational expression, we simply give the polynomial a denominator of one:

◇ **Example 3.2(d):** Multiply $(4x^2-9)\cdot\dfrac{x+3}{2x+3}$

$$\begin{aligned}(4x^2-9)\cdot\frac{x+3}{2x+3} &= \frac{4x^2-9}{1}\cdot\frac{x+3}{2x+3}\\ &= \frac{\cancel{(2x+3)}(2x-3)}{1}\cdot\frac{x+3}{\cancel{2x+3}}\\ &= (2x-3)(x+3) \quad \text{or} \quad 2x^2+3x-9\end{aligned}$$

Soon we will see situations where we need to multiply a polynomial times a sum or difference of rational expressions. To do this we simply distribute the polynomial to each of the rational expressions, then multiply as in Example 3.2(d). Be careful to distribute negative signs!

◇ **Example 3.2(e):** Multiply $(x+2)(x-1)\cdot\left(\dfrac{4x}{x^2+x-2}-\dfrac{3}{x+2}\right)$.

Solution: Here we begin by putting $(x+2)(x-1)$ over one and distributing to both parts of the expression $\dfrac{4x}{x^2+x-2}-\dfrac{3}{x+2}$.

$$(x+2)(x-1)\left(\frac{4x}{x^2+x-2}-\frac{3}{x+2}\right)$$

$$\begin{aligned}&= \frac{(x+2)\cancel{(x-1)}}{1}\cdot\frac{4x}{(x+2)\cancel{(x-1)}}-\frac{\cancel{(x+2)}(x-1)}{1}\cdot\frac{3}{\cancel{x+2}}\\ &= 4x-3(x-1)=4x-3x+3=x+3\end{aligned}$$

Section 3.2 Exercises To Solutions

1. Multiply each.

(a) $\dfrac{x+1}{x-4}\cdot\dfrac{x-4}{x^2-2x-3}$ (b) $\dfrac{x^2+x-2}{x^2-4}\cdot\dfrac{x+3}{x-1}$

(c) $\dfrac{x^2+2x-15}{x^2-9}\cdot(x+3)$ (d) $\dfrac{x+1}{x^2-4}\cdot\dfrac{x^2+5x+6}{x+1}$

2. Multiply each.

(a) $6x \cdot \left(\frac{5}{2x} - \frac{1}{3}\right)$

(b) $x(x+1) \cdot \left(\frac{x-3}{x+1} - \frac{x+4}{x}\right)$

(c) $(x+5)(x-5) \cdot \left(\frac{3x}{x-5} + \frac{7}{x+5}\right)$

(d) $(x-3)(x+3) \cdot \left(\frac{2x}{x^2-9} - \frac{5}{x+3}\right)$

3. Evaluate each of the following.

(a) 6^{-1}

(b) $\left(\frac{4}{5}\right)^{-2}$

(c) 6^0

(d) $\left(-\frac{2}{3}\right)^2$

4. Perform the indicated operation on the polynomial(s).

(a) $(2x-1)^2$

(b) $(2x^2+5)+(x^3-3x+4)$

(c) $(x+5)(x^2-4x+2)$

(d) $(x+5)-(x^2-4x+2)$

5. Determine all values that the unknown is not allowed to have in each of the following.

(a) $\frac{x^2-3x-10}{x^2-4}$

(b) $\frac{x+2}{x^2-3x-10}$

(c) $\frac{x^2-3x-4}{x^2+3x+2}$

6. Reduce each of the rational expressions from Exercise 5.

3.3 Solving Rational Equations

3. (d) Solve rational equations.

A rational equation is an equation containing rational expressions. The procedure for solving rational equations is as follows:

Solving Rational Equations

- Factor the denominators of all rational expressions, and determine what values the unknown is not allowed to have.
- Multiply both sides of the equation by *JUST ENOUGH* of the factors of the denominators to "kill off" all the denominators. Be sure to distribute carefully and take all signs properly into account.
- Solve the resulting linear or polynomial equation.
- Eliminate any of the solutions you obtained that are also values that the unknown is not allowed to have.

◇ **Example 3.3(a):** Solve $1 - \frac{1}{x} = \frac{12}{x^2}$

$x \neq 0$

$$\begin{aligned} \frac{x^2}{1}\left(1 - \frac{1}{x}\right) &= \left(\frac{12}{x^2}\right)\frac{x^2}{1} \\ x^2 - \frac{x^2}{1}\cdot\frac{1}{x} &= \frac{12}{x^2}\cdot\frac{x^2}{1} \\ x^2 - x &= 12 \end{aligned}$$

$$\begin{aligned} x^2 - x - 12 &= 0 \\ (x-4)(x+3) &= 0 \\ x &= 4, -3 \end{aligned}$$

◇ **Example 3.3(b):** Solve $\frac{5}{y+1} = \frac{4}{y+2}$

$y \neq -1, -2$

$$\begin{aligned} \frac{(y+1)(y+2)}{1}\cdot\frac{5}{y+1} &= \frac{4}{y+2}\cdot\frac{(y+1)(y+2)}{1} \\ 5(y+2) &= 4(y+1) \\ 5y + 10 &= 4y + 4 \\ y &= -6 \end{aligned}$$

The process used in the previous two examples is often called "clearing the denominators" of the fractions. Note carefully how this is done in the next example.

◇ **Example 3.3(c):** Solve $\dfrac{y+3}{y^2-y} = \dfrac{8}{y^2-1}$

Solution: When we factor the denominators of both sides of the equation we get

$$\frac{y+3}{y(y-1)} = \frac{8}{(y+1)(y-1)}$$

Solution: so, clearly, $y \neq 0, 1, -1$. To clear the denominators we only need *ONE* factor of $y-1$ even though it occurs in the denominators of both sides, because anything we multiply one side by, we must multiply the other side by as well.

$$\begin{aligned}
\frac{y(y+1)(y-1)}{1} \cdot \frac{y+3}{y(y-1)} &= \frac{8}{(y+1)(y-1)} \cdot \frac{y(y+1)(y-1)}{1} \\
(y+1)(y+3) &= 8y \\
y^2+4y+3 &= 8y \\
y^2-4y+3 &= 0 \\
(y-1)(y-3) &= 0 \\
y &= \not{1}, 3
\end{aligned}$$

The solution $y=1$ is not valid because y cannot be one in the original equation. We note this by simply crossing it out, as shown above.

◇ **Example 3.3(d):** Solve $\dfrac{x-4}{x^2+2x-15} = 2 - \dfrac{2}{x-3}$

$$\begin{aligned}
\frac{x-4}{(x+5)(x-3)} &= 2 - \frac{2}{x-3} \qquad x \neq -5, 3 \\
\frac{(x+5)(x-3)}{1} \cdot \frac{x-4}{(x+5)(x-3)} &= \left(2 - \frac{2}{x-3}\right) \frac{(x+5)(x-3)}{1} \\
x-4 &= 2(x+5)(x-3) - \frac{2}{x-3} \cdot \frac{(x+5)(x-3)}{1} \\
x-4 &= 2(x^2+2x-15) - 2(x+5) \\
x-4 &= 2x^2+4x-30-2x-10 \\
x-4 &= 2x^2+2x-40 \\
0 &= 2x^2+x-36 \\
0 &= (2x+9)(x-4)
\end{aligned}$$

At this point we can easily see that $x=4$ is one solution. To obtain the other solution we set $2x+9=0$ and solve to get $x=-\frac{9}{2}$. Neither of these causes a problem with the original equation, so both are solutions.

◇ **Example 3.3(e):** Solve $\dfrac{x}{x+5} = \dfrac{x}{x-2}$

Solution: We see that $x \neq -5, 3$. Multiplying both sides by $(x+5)(x-2)$ we get

$$\begin{aligned}
\frac{(x+5)(x-2)}{1} \cdot \frac{x}{x+5} &= \frac{x}{x-2} \cdot \frac{(x+5)(x-2)}{1} \\
x(x-2) &= x(x+5) \\
x^2 - 2x &= x^2 + 5x \\
-2x &= 5x \\
0 &= 7x \\
x &= 0
\end{aligned}$$

Looking back at the original equation, it is clear that $x = 0$ is a solution, but it is *NOT* clear that it is the only solution.

◇ **Example 3.3(f):** Solve $\dfrac{x-2}{x-6} - \dfrac{4}{x} = \dfrac{24}{x^2 - 6x}$

Solution: Factoring the denominator of the right hand side, we get $\dfrac{x-2}{x-6} - \dfrac{4}{x} = \dfrac{24}{x(x-6)}$. From this we can see that $x \neq 0, 6$.

$$\begin{aligned}
\frac{x(x-6)}{1} \cdot \left(\frac{x-2}{x-6} - \frac{4}{x}\right) &= \frac{24}{x(x-6)} \cdot \frac{x(x-6)}{1} \\
x(x-2) - 4(x-6) &= 24 \\
x^2 - 2x - 4x + 24 &= 24 \\
x^2 - 6x &= 0 \\
x(x-6) &= 0 \\
x &= \cancel{0}, \cancel{6}
\end{aligned}$$

Because the procedure for solving the equation leads to only the two values that x is not allowed to have, the equation has no solution.

Section 3.3 Exercises To Solutions

1. For each equation, tell what values x is not allowed to have, then solve the equation.

(a) $\dfrac{x+4}{2x} + \dfrac{x+20}{3x} = 3$

(b) $\dfrac{x+1}{x-3} = \dfrac{x+2}{x+5}$

(c) $1 - \dfrac{4}{x+7} = \dfrac{5}{x+7}$

(d) $\dfrac{10}{x-3} - \dfrac{2}{x} = -1$

(e) $\dfrac{1}{x-1}+\dfrac{1}{x+1}=\dfrac{6}{x^2-1}$

(f) $\dfrac{2x-1}{x^2+2x-8}=\dfrac{1}{x-2}-\dfrac{2}{x+4}$

2. Solve each equation.

(a) $15x^2 = 20x$

(b) $\dfrac{2}{3}x-\dfrac{1}{6}=\dfrac{3}{2}-\dfrac{7}{12}x$

(c) $2x^3+x^2=18x+9$

(d) $5x^2 = 20$

(e) $4-t=15t-20$

(f) $\dfrac{2}{15}x^2+\dfrac{1}{3}x=\dfrac{1}{5}$

(g) $2x^2-x-10=0$

(h) $8(a-2)+3a=9(1-a)$

4 Equations and Roots, The Quadratic Formula

4.1 Roots

4. (a) Find (without a calculator) real roots of numbers when they exist.
   (b) Simplify a square root.

For any operation we have in mathematics, we always want to know if there is another operation that will reverse it. The operations that reverse powers are called **roots**. Let's start with the root that reverses (not perfectly, as we shall see) squaring.

Square Root

For any number $a \geq 0$, the **square root** of a is the *non-negative* number that can be squared to get a. We denote the square root of a by $\sqrt{a}$.

◇ **Example 4.1(a):** Find $\sqrt{9}$, the square root of nine.

Solution: Because $3^2 = 9$, $\sqrt{9} = 3$. Even though $(-3)^2 = 9$ also, $\sqrt{9}$ cannot be -3 because the square root of a number must not be negative.

◇ **Example 4.1(b):** Find $\sqrt{-16}$.

Solution: -16 is not greater than or equal to zero, so the square root of -16 does not exist. (Well, as a real number anyway. More on this below - for now we'll say that the square root of a negative number does not exist.) In the interest of efficiency (laziness?), we will use the abbreviation DNE for "does not exist."

◇ **Example 4.1(c):** Find $-\sqrt{16}$.

Solution: Here it is 16 that we want the square root of, because the negative sign is *outside* the square root. Because $4^2 = 16$, $\sqrt{16} = 4$. $-\sqrt{16}$ means the negative of the square root of 16, so $-\sqrt{16} = -4$.

Some of you may have previously encountered **imaginary numbers**, which allow us to find a "value" for a root like the one in Example 4.1(b) above. However, *until those numbers are introduced later you should assume that we are working only with* **real numbers**, which are the "ordinary numbers" that you are used to. So when we say "does not exist," we really mean "does not exist as a real number."

It is very important that you understand the difference between the last two examples. In example 4.1(b) we are asked for the square root of a negative number, which goes against the statement in the definition above that a must be greater than or equal to zero. In Example 4.1(c) we are asked for the negative of the square root of 16. Since $16 \geq 0$ we can find its square root, 4. We then take the negative of that to get -4.

Square roots are very useful for many operations. Because of this, it is important that you know (memorize) the perfect squares, starting with zero squared and going at least up until ten squared:

$$0,\ 1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81,\ 100$$

for the record, $11^2 = 121$, and $12^2 = 144$, called a *gross*. (you fireworks fans know that one!) Square roots are the kind of root that we will see most often, but there are other roots that reverse other powers also.

nth Root

- For any number $a \geq 0$ and any **even** whole number n, the **nth root** of a is the *non-negative* number whose nth power is a.
- For any number a and any *odd* whole number n, the **nth root** of a is the number (positive or negative, there will only be one) whose nth power is a.

We denote the nth root of a by $\sqrt[n]{a}$.

◇ **Example 4.1(d):** Find $\sqrt[3]{27}$. (We call third roots **cube roots**.)

Solution: Because $3^3 = 27$, $\sqrt[3]{27} = 3$.

◇ **Example 4.1(e):** Find $\sqrt[3]{-27}$.

Solution: $(-3)^3 = -27$, so $\sqrt[3]{-27} = -3$.

◇ **Example 4.1(f):** Find $\sqrt[4]{16}$.

Solution: Because $2^4 = 16$, $\sqrt[4]{16} = 2$.

◇ **Example 4.1(g):** Find $\sqrt[4]{-16}$.

Solution: Any number to the fourth power is positive, so $\sqrt[4]{-16}$ does not exist.

Note that
$$\sqrt{4 \cdot 9} = \sqrt{36} = 6 = 2 \cdot 3 = \sqrt{4} \cdot \sqrt{9}$$
This is an example of the most important property of square roots:

Product of Square Roots

For any numbers $a \geq 0$ and $b \geq 0$,

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b} \qquad \text{(and of course} \quad \sqrt{a}\sqrt{b} = \sqrt{ab}\text{)}$$

One thing we use this property for is to simplify square roots; this is something like reducing fractions. When we simplify a square root we don't change its value, we just change its appearance. The idea is to find a perfect square factor of the number whose root we are simplifying. (One reason to learn those perfect squares!) Here is an example of how we do this:

◇ **Example 4.1(h):** Simplify $\sqrt{20}$.

$$\sqrt{20} = \sqrt{4 \cdot 5} = \sqrt{4} \cdot \sqrt{5} = 2\sqrt{5}$$

Solution: Using a calculator, $\sqrt{20} \approx 4.472$ and $2\sqrt{5} \approx 4.472$, verifying our work. ($\approx$ means 'approximately equal to.' We use it in this case because those two values are rounded versions of the actual values.)

Note that $\sqrt{20} = \sqrt{2 \cdot 10}$, but that is not useful because we don't know the square root of either two *or* ten. This process can be done in multiple steps if you don't recognize the *largest* perfect square factor:

◇ **Example 4.1(i):** Simplify $\sqrt{72}$.

Solution: One might do this as

$$\sqrt{72} = \sqrt{9 \cdot 8} = \sqrt{9} \cdot \sqrt{8} = 3\sqrt{4 \cdot 2} = 3\sqrt{4} \cdot \sqrt{2} = 3 \cdot 2\sqrt{2} = 6\sqrt{2}$$

Solution: We could get the same result in fewer steps if we recognize that $72 = 36 \cdot 2$:

$$\sqrt{72} = \sqrt{36} \cdot \sqrt{2} = 6\sqrt{2}$$

We will usually show fewer steps when simplifying square roots like this. For $\sqrt{72}$ we might show our calculations like this:

$$\sqrt{72} = \sqrt{9} \cdot \sqrt{8} = 3\sqrt{4} \cdot \sqrt{2} = 3 \cdot 2\sqrt{2} = 6\sqrt{2}$$

Section 4.1 Exercises To Solutions

1. For each of the following, determine whether the root exists. If it doesn't exist, say so (write DNE for "does not exist") and you are done. If the root does exist, give its value without using your calculator.

   (a) $\sqrt{25}$ (b) $\sqrt{9}$ (c) $\sqrt[3]{8}$ (d) $\sqrt{-16}$ (e) $\sqrt[4]{1}$

   (f) $-\sqrt{49}$ (g) $\sqrt[3]{-27}$ (h) $-\sqrt[3]{27}$ (i) $\sqrt{\frac{16}{25}}$ (j) $\sqrt{-1}$

2. For each of the following, simplify the root if possible.

   (a) $\sqrt{45}$ (b) $\sqrt{8}$ (c) $\sqrt{15}$ (d) $\sqrt{72}$

   (e) $\sqrt{98}$ (f) $\sqrt{12}$ (g) $\sqrt{-50}$ (h) $\sqrt{75}$

3. Determine all values that the unknown is not allowed to have in each of the following.

   (a) $\dfrac{x^2-2x-3}{x-3}$ (b) $\dfrac{x^2-9}{x^2+5x+6}$ (c) $\dfrac{2x^2-2x-24}{x^2+4x+3}$

4. Reduce each of the rational expressions from Exercise 3.

5. Multiply each.

   (a) $\dfrac{3x}{x^2-25}\cdot(x+5)(x-5)$ (b) $\dfrac{x+5}{x+1}\cdot\dfrac{x^2+5x+4}{x-3}$

6. For each equation, tell what values x is not allowed to have, then solve the equation.

   (a) $x+\dfrac{6}{x}=-7$ (b) $\dfrac{8}{x^2-9}+\dfrac{4}{x+3}=\dfrac{2}{x-3}$

4.2 Adding, Subtracting, and Multiplying Expressions With Roots

> 4. (c) Add, subtract and multiply expressions containing roots.

If we have an expression like $7\sqrt{5} - 4\sqrt{3}$ we can't simplify any further, because the two different roots are unlike terms, like $7x^2$ and $4x$ are. However, $7\sqrt{5} - 4\sqrt{5} = 3\sqrt{5}$. We don't know an exact value for $\sqrt{5}$ but, whatever it is, we start with seven of them and then take away four of them.

◇ **Example 4.2(a):** Simplify $4\sqrt{3} - 2\sqrt{7} + \sqrt{3}$.

Solution: We can combine the terms $4\sqrt{3}$ and $\sqrt{3}$ to get $5\sqrt{3}$, but we cannot combine that with the $-2\sqrt{7}$. The result is then $5\sqrt{3} - 2\sqrt{7}$.

◇ **Example 4.2(b):** Simplify $3\sqrt{2} + 8\sqrt{2} - 4\sqrt{2}$.

Solution: In this case all three terms contain the same root, so they can all be combined to get $7\sqrt{2}$.

Recall that $\sqrt{a}\sqrt{b} = \sqrt{ab}$. Here are some applications of this:

◇ **Example 4.2(c):** Multiply $\sqrt{3}\sqrt{7}$.

$$\sqrt{3}\sqrt{7} = \sqrt{3 \cdot 7} = \sqrt{21}$$

Solution: We need to check $\sqrt{21}$ to see if it can be simplified. Since the only factors of 21 are 3 and 7, neither of which is a perfect square, it *cannot* be simplified.

◇ **Example 4.2(d):** Multiply $3\sqrt{5} \cdot 2\sqrt{7}$.

Solution: Because the only operations are multiplication, all values can be reordered to get the numbers first, followed by the roots. We then multiply the two numbers and the two roots:

$$3\sqrt{5} \cdot 2\sqrt{7} = 3 \cdot 2\sqrt{5}\sqrt{7} = 6\sqrt{35}$$

◇ **Example 4.2(e):** Multiply $\sqrt{6}\sqrt{3}$.

Solution: After multiplying the two roots together, we see that we need to simplify the resulting root:

$$\sqrt{6}\sqrt{3} = \sqrt{6 \cdot 3} = \sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$$

◇ **Example 4.2(f):** Multiply $\sqrt{7}\sqrt{7}$.

$$\sqrt{7}\sqrt{7} = \sqrt{49} = 7$$

Solution: In general, for any positive number a, $\sqrt{a}\sqrt{a} = a$.

◇ **Example 4.2(g):** Multiply $(3+\sqrt{7})(4-\sqrt{2})$.

Solution: Here we just 'FOIL' this out and multiply the two roots in the last term. There are no like terms to combine:

$$(3+\sqrt{7})(4-\sqrt{2}) = 12 - 3\sqrt{2} + 4\sqrt{7} - \sqrt{7}\sqrt{2} = 12 - 3\sqrt{2} + 4\sqrt{7} - \sqrt{14}$$

◇ **Example 4.2(h):** Multiply $(3+\sqrt{7})(5-2\sqrt{7})$.

Solution: This is done in the same way as the previous example, except that there are some like terms to be combined:

$$(3+\sqrt{7})(5-2\sqrt{7}) = 15 - 6\sqrt{7} + 5\sqrt{7} - 2\sqrt{7}\sqrt{7} = 15 - \sqrt{7} - 2(7) = 1 - \sqrt{7}$$

◇ **Example 4.2(i):** Multiply $(3-\sqrt{5})^2$.

Solution: Here we must only remember that $(3-\sqrt{5})^2$ means $3-\sqrt{5}$ times itself, and then proceed in the same way as the previous example:

$$(3-\sqrt{5})^2 = (3-\sqrt{5})(3-\sqrt{5}) = 9 - 3\sqrt{5} - 3\sqrt{5} + \sqrt{5}\sqrt{5} = 9 - 6\sqrt{5} + 5 = 14 - 6\sqrt{5}$$

Section 4.2 Exercises

To Solutions

1. Simplify by combining like terms, *if possible*.

   (a) $9 - 3\sqrt{5} - 3\sqrt{5} + 25$

   (b) $25 + 10\sqrt{3} - 10\sqrt{3} - 12$

   (c) $15 + 5\sqrt{2} - 6\sqrt{3} - 2\sqrt{6}$

   (d) $8 - 4\sqrt{3} + 2\sqrt{7} - \sqrt{21}$

2. Multiply and simplify.

   (a) $\sqrt{6}\cdot\sqrt{3}$

   (b) $\sqrt{5}\cdot\sqrt{5}$

   (c) $\sqrt{5}\cdot\sqrt{7}$

   (d) $\sqrt[3]{4}\cdot\sqrt[3]{20}$

   (e) $3\sqrt{10}\cdot 2\sqrt{5}$

   (f) $\sqrt{2}\cdot 5\sqrt{6}$

   (g) $4(5+\sqrt{7})$

   (h) $(3+\sqrt{5})(2-\sqrt{3})$

   (i) $(1-\sqrt{10})(5-\sqrt{2})$

3. For each equation, tell what values x is not allowed to have, then solve the equation.

(a) $\dfrac{4}{x-3} - \dfrac{3}{x+3} = 1$ (b) $\dfrac{x+3}{x-1} + \dfrac{x+5}{x} = \dfrac{3x+1}{x-1}$

4. For each of the following, determine whether the root exists. If it doesn't exist, say so (write DNE for "does not exist") and you are done. If the root does exist, give its value without using your calculator.

(a) $\sqrt[3]{-1}$ (b) $\sqrt[50]{1}$ (c) $\sqrt{0}$ (d) $\sqrt{100}$ (e) $\sqrt{\frac{64}{25}}$

5. For each of the following, simplify the root if possible.

(a) $\sqrt{32}$ (b) $\sqrt{24}$ (c) $\sqrt{18}$ (d) $\sqrt{50}$

4.3 Solving Quadratic Equations With The Quadratic Formula

4. (d) Solve quadratic equations using the quadratic formula.

Consider the quadratic equation $x^2 - 6x + 4 = 0$. Of course we expect this equation to have perhaps two solutions, but a few minutes of effort will convince us that the left side of the equation cannot be factored. Note that if we substitute $3 - \sqrt{5}$ into the equation for x we get (using the result of Example 4.2(i))

$$(3 - \sqrt{5})^2 - 6(3 - \sqrt{5}) + 4 = 14 - 6\sqrt{5} - 18 + 6\sqrt{5} + 4 = 0,$$

so $x = 3 - \sqrt{5}$ is a solution to the equation.

It turns out that $x = 3 + \sqrt{5}$ is another solution. $3 + \sqrt{5}$ and $3 - \sqrt{5}$ are sometimes called **conjugates**. One might ask how we would find that those are the solutions to the equation $x^2 - 6x + 4 = 0$. To find those solutions we use something called the **quadratic formula**. The two solutions to the equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

If the quantity $b^2 - 4ac$ is negative we will have the square root of a negative number, which does not exist. In that case the equation has no solution. Note that the above two formulas are exactly the same except that one has a minus sign where the other has a plus. We will combine them into one, using the symbol $\pm$ to indicate that both an addition and a subtraction have to take place at that point in order to get both solutions.

Quadratic Formula

If $b^2 - 4ac \geq 0$, the solutions to $ax^2 + bx + c = 0$ are obtained from

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac < 0$ we will say that the equation $ax^2 + bx + c = 0$ has no solution.

◇ **Example 4.3(a):** Solve $x^2 - 4x - 5 = 0$ using the quadratic formula.

Solution: Here $a = 1$, $b = -4$ and $c = -5$, so

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-5)}}{2(1)} = \frac{4 \pm \sqrt{16 + 20}}{2} =$$

$$\frac{4 \pm \sqrt{36}}{2} = \frac{4 \pm 6}{2} = \frac{10}{2}, \frac{-2}{2} = 5, -1$$

Note that the equation from the previous example can be factored to $(x-5)(x+1)=0$, giving us the solutions $x=-1,5$. If an equation can be solved by factoring but we try to use the quadratic formula, we will get the same solutions as we would by factoring. A person should generally try factoring first, and if they can't figure out how to factor fairly quickly, then they should use the quadratic formula. The real advantage of the quadratic formula is that it allows us to get solutions for equations that can't be factored.

◇ **Example 4.3(b):** Solve $x^2+10x+23=0$ using the quadratic formula.

$$x=\frac{-10\pm\sqrt{10^2-4(1)(23)}}{2(1)}=\frac{-10\pm\sqrt{100-92}}{2}=\frac{-10\pm\sqrt{8}}{2}$$

Solution: At this point we simplify $\sqrt{8}$ to get $\sqrt{8}=\sqrt{4}\sqrt{2}=2\sqrt{2}$. This is substituted into what we have so far, the 'big' fraction is split into two 'smaller' fractions, and each fraction is reduced:

$$\frac{-10\pm\sqrt{8}}{2}=\frac{-10\pm2\sqrt{2}}{2}=\frac{-10}{2}\pm\frac{2\sqrt{2}}{2}=-5\pm\sqrt{2},$$

so $x=-5+\sqrt{2},\ -5-\sqrt{2}$.

At one point in this last example the big fraction from the formula was broken apart into two separate fractions. This uses a very useful little mathematical manipulation:

"Un-Adding" (and "Un-Subtracting") Fractions

For any numbers a,b and c, $\dfrac{a\pm b}{c}=\dfrac{a}{c}\pm\dfrac{b}{c}$.

Section 4.3 Exercises

To Solutions

1. (a) Solve $2x^2-7x-15=0$ using the quadratic formula. Then solve it by factoring; your answers should, of course, be the same either way.

   (b) Solve $4x^2+13x+3=0$ using the quadratic formula. Then solve it by factoring to check your answers.

2. For each of the following, use the quadratic formula to solve the given equation. Simplify your answers.

   (a) $x^2-2x-4=0$ (b) $x^2-6x+7=0$

   (c) $x^2+10x+13=0$ (d) $2x^2=2x+1$

   (e) $9x^2-12x-1=0$ (f) $25x^2+10x=62$

3. Solve each equation by factoring.

(a) $x^3 + 5x^2 - 9x - 45 = 0$

(b) $2x^2 = 5x + 12$

(c) $9x^2 - 1 = 0$

(d) $x^3 - 7x^2 + 10x = 0$

4. For each equation, tell what values x is not allowed to have, then solve the equation.

(a) $\dfrac{3}{x-1} - \dfrac{2}{x+4} = \dfrac{x^2+8x+6}{x^2+3x-4}$

(b) $\dfrac{3}{x-4} = \dfrac{5x+4}{x^2-16} - \dfrac{4}{x+4}$

5. Simplify by combining like terms, *if possible*.

(a) $1 + \sqrt{5} - \sqrt{5} - 5$

(b) $12 + 2\sqrt{2} - 18\sqrt{5} - 3\sqrt{10}$

4.4 Solving Equations Containing Roots

> 4. (e) Solve equations containing roots.

Recall that, if $a \geq 0$,

$$(\sqrt{a})^2 = \sqrt{a} \cdot \sqrt{(a)} = \sqrt{a \cdot a} = \sqrt{a^2} = a.$$

We can use this idea to solve and equation when one side of the equation is a square root; we simply square both sides to eliminate the root, then solve the resulting equation:

◇ **Example 4.4(a):** Solve $\sqrt{3x+4} = 5$.

Solution:

$$\begin{aligned} (\sqrt{3x+4})^2 &= 5^2 \\ 3x+4 &= 25 \\ 3x &= 21 \\ x &= 7 \end{aligned} \qquad \begin{aligned} &\text{Check :} \\ \sqrt{3(7)+4} &\stackrel{?}{=} 5 \\ \sqrt{21+4} &\stackrel{?}{=} 5 \\ \sqrt{25} &= 5 \end{aligned}$$

◇ **Example 4.4(b):** Solve $\sqrt{7x-13} = x-1$. Another Example

Solution:

$$\begin{aligned} (\sqrt{7x-13})^2 &= (x-1)^2 \\ 7x-13 &= (x-1)(x-1) \\ 7x-13 &= x^2-2x+1 \\ 0 &= x^2-9x+14 \\ 0 &= (x-7)(x-2) \\ x &= 2, 7 \end{aligned}$$

$$\begin{aligned} \text{Check} \quad & x = 2 : \\ \sqrt{7(2)-13} &\stackrel{?}{=} 2-1 \end{aligned}$$

$$\begin{aligned} \text{Check} \quad & x = 7 : \\ \sqrt{7(7)-13} &\stackrel{?}{=} 7-1 \\ \sqrt{36} &= 6 \end{aligned}$$

When solving equations like these, a funny thing sometimes happens. It is possible that a solution that you find by the procedure we've been using is not actually a valid solution. We'll see this in the next examples.

◇ **Example 4.4(c):** Solve $\sqrt{4x-7} = -3$.

Solution:

$$\begin{aligned} (\sqrt{4x-7})^2 &= (-3)^2 \\ 4x-7 &= 9 \\ 4x &= 16 \\ x &= 4 \end{aligned} \qquad \begin{aligned} &\text{Check :} \\ \sqrt{4(4)-7} &\stackrel{?}{=} -3 \\ \sqrt{16-7} &\stackrel{?}{=} -3 \\ \sqrt{9} &\neq -3 \end{aligned}$$

Since the only possible solution does not check, the equation has no solution.

When the root is not alone on one side, we need to do a little before squaring both sides, as shown in the next example.

◇ **Example 4.4(d):** Solve $x = \sqrt{6x+1} + 1$. Another Example

Solution: In this case we must first get the square root alone on one side, *THEN* square both sides of the equation:

$$\begin{aligned} x - 1 &= \sqrt{6x+1} \\ (x-1)^2 &= (\sqrt{6x+1})^2 \\ x^2 - 2x + 1 &= 6x + 1 \\ x^2 - 8x &= 0 \\ x(x-8) &= 0 \\ x &= 0,\ 8 \end{aligned}$$

Check $x = 0$:

$$0 \stackrel{?}{=} \sqrt{6(0)+1} + 1$$

$0 \neq \sqrt{1} + 1$, so $x = 0$ is *NOT* a solution

Check $x = 8$:

$$8 \stackrel{?}{=} \sqrt{6(8)+1} + 1$$

$8 = \sqrt{49} + 1$, so $x = 8$ *IS* a solution

It is also possible to find two solutions, neither of which checks. In that case there is no solution.

Section 4.4 Exercises

To Solutions

1. Solve each of the following equations.

   (a) $\sqrt{4x+1} = 3$ (b) $\sqrt{5x+10} = x+2$ (c) $\sqrt[3]{4x+5} = -1$

2. The following equations are slightly different than the ones in Exercise 1. Begin by adding or subtracting something to both sides in order to get the root alone on one side, before squaring both sides.

   (a) $\sqrt{3x+13} - 2 = 3$ (b) $\sqrt{3x+15} - 5 = x$

3. Solve each equation. Be sure to check all solutions to see if they are valid.

   (a) $\sqrt[3]{4x+4} + 6 = 7$ (b) $\sqrt{25x-4} = 4$

   (c) $\sqrt{3x+13} = x+3$ (d) $\sqrt[3]{4x-15} + 5 = 2$

   (e) $\sqrt{3x+2} + 7 = 5$ (f) $\sqrt{x-2} = x-2$

4. Multiply each.

   (a) $\dfrac{x+2}{x-3} \cdot \dfrac{x^2-9}{x^2+9x+14}$ (b) $\dfrac{x^2+7x+10}{2x^2+10x} \cdot 2x(x+5)$

   (c) $(x+4)(x-1) \cdot \left(\dfrac{3}{x-1} - \dfrac{5}{x+4}\right)$ (d) $3x^2 \cdot \left(\dfrac{9}{x^2} + \dfrac{1}{3}\right)$

5. For each of the following, use the quadratic formula to solve the given equation. Simplify your answers.

   (a) $x^2 + 8x + 13 = 0$

   (b) $x^2 + 6x + 7 = 0$

5 Applications Of Equations

5.1 Using Formulas

5. (a) Use formulas to solve applied problems.

Geometry

We will be working with three geometric shapes: rectangles, triangles and circles. When we discuss the amount of surface that a shape covers, we are talking about its **area**. Some standard sorts of practical uses of areas are for measuring the amount of carpet needed to cover a floor, or the amount of paint to paint a certain amount of surface. The distance around a shape is usually called its **perimeter**, except in the case of a circle. The distance around a circle is called its **circumference**.

A **formula** is an equation that describes the relationship between several (two or more) values. Here are some formulas you are likely familiar with:

Area and Perimeter/Circumference Formulas

- For a rectangle with width w and length l, the perimeter P and area A are given by

$$P = 2w + 2l \qquad \text{and} \qquad A = lw$$

- For a circle with radius r, the circumference C and area A are given by

$$C = 2\pi r \qquad \text{and} \qquad A = \pi r^2$$

When computing the circumference or area of a circle it is necessary to use a special number called **pi**. Pi is a little over 3, but in decimal form it never ends or repeats. We use the symbol π for it, and its "exact" value is

$$\pi = 3.141592654...$$

Sometimes we round this to 3.14, but to use it most accurately one should use more places past the decimal. We might not care to type in more, but we don't need to - pi is so important that it has its own key on our calculators! Find it on yours.

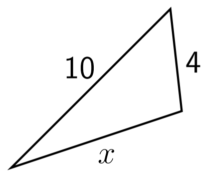
Let's look at some examples that use these formulas.

◇ **Example 5.1(a):** A rectangle that is 7 feet long has an area of 31.5 square feet. (The units of areas are always square somethings.) What is the width of the rectangle?

Solution: $A = lw \quad \Rightarrow \quad 31.5 = 7w \quad \Rightarrow \quad w = \dfrac{31.5}{7} = 4.5 \text{ feet}$

◇ **Example 5.1(b):** The shortest side of a triangle has a length of 4.5 inches, the longest side has a length of 10.0 inches, and the perimeter is 21.0 inches. What is the length of the middle side?

Solution:

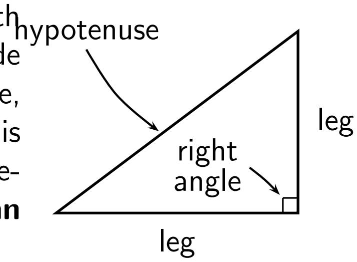


$$\begin{aligned} 4.5 + x + 10 &= 21 \\ x + 14.5 &= 21 \\ x &= 6.5 \text{ inches} \end{aligned}$$

◇ **Example 5.1(c):** A circle has a radius of 6.5 centimeters. What is the area of the circle?

Solution: $A = \pi r^2 = \pi(6.5)^2 = 132.7 \text{ square centimeters}$

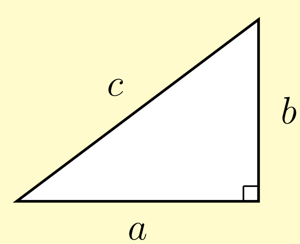
There is a special kind of triangle that comes up often in applications, called a **right triangle**. A right triangle is a triangle with one of its angles being a right angle ("square corner"). The side opposite the right angle is called the **hypotenuse** of the triangle, and the other two sides are called the **legs** of the triangle. There is a special relationship between the sides of the right triangle; this relationship is expressed by a famous "rule" called the **Pythagorean Theorem**.



Pythagorean Theorem

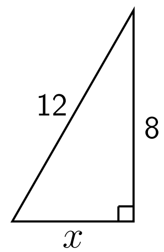
If a right triangle has legs with lengths a and b and hypotenuse of length c, then

$$a^2 + b^2 = c^2$$



◇ **Example 5.1(d):** One of the legs of a right triangle is 8 inches long and the hypotenuse is 12 inches long. How long is the other leg?

Solution:



$$\begin{aligned} x^2 + 8^2 &= 12^2 \\ x^2 + 64 &= 144 \\ x^2 &= 80 \\ x &= \sqrt{80} \\ x &= 8.9 \text{ inches} \end{aligned}$$

Projectile Motion

> **Projectile Motion**
>
> Suppose that an object is projected straight upward from a height of h_0 feet, with an initial velocity of v_0 feet per second. Its height (in feet) at any time t seconds after it is projected (until it hits the ground) is
>
> $$h = -16t^2 + v_0 t + h_0$$

◇ **Example 5.1(e):** A ball is thrown upward from a height of 5 feet, with an initial velocity of 60 feet per second. How high is the ball after 1.5 seconds?

Solution: The values of v_0 and h_0 are 60 and 5, respectively, so the equation becomes

$$h = -16t^2 + 60t + 5.$$

At time $t = 5$ seconds,

$$h = -16(1.5)^2 + 60(1.5) + 5 = 59 \text{ feet}$$

◇ **Example 5.1(f):** For the same ball, at what time or times is the ball 50 feet above the ground? **Round your answers to the nearest hundredth of a second.**

Solution: Here we know that $h = 50$ and we want to find t. This gives us the equation below and to the left. Adding $16t^2$ to both sides and subtracting $60t$ and 5 from both sides gives the equation below and to the right.

$$50 = -16t^2 + 60t + 5 \qquad \Rightarrow \qquad 16t^2 - 60t + 45$$

We need to solve the second equation for t. That equation looks difficult to factor, so let's just use the quadratic formula to solve it:

$$t = \frac{-(-60) \pm \sqrt{(-60)^2 - 4(16)(45)}}{2(16)} = \frac{60 \pm \sqrt{720}}{32} = 1.04,\ 2.71 \text{ seconds}$$

We get two answers because the ball will reach a height of 50 feet on both the way up *and* the way down.

Percents

Many quantities of interest are obtained by taking a certain percentage of other quantities. For example, some salespeople earn a certain percentage of their sales. (In this case the percentage that they earn is called their **commission**.) Various taxes are computed by a percentage of the item bought (or owned, in the case of property tax).

Percent means "out of one hundred," so seven percent means seven out of 100. If a person bought something for $200 and they had to pay 7% sales tax, they would pay seven dollars for every $100, or $14. When doing math with a percent, it is necessary to change a percent to its decimal form. Since seven percent means seven out of one hundred,

$$7\% = \frac{7}{100} = 0.07$$

This last value is called the **decimal form of the percent**. Note that the decimal point in $7 = 7.0$ has been moved two places to the left.

◇ **Example 5.1(g):** Change 4.25% to decimal form.

Solution:

$$4.25\% = 0.0425$$

> **Working With Percents**
>
> To find p percent of an amount A, change p to decimal form and multiply it times A.

◇ **Example 5.1(h):** The standard tip for waiters and waitresses is 15-20% of the cost of the meal. If you and some friends go out to eat and the total bill is $87.50, how much of a tip should you give your waitress if you wish to give 15%?

15% of $\$87.50$ is $(0.15)(87.50) = \$13.13$, so maybe give a tip of $\$13.25$ or $\$14$.

Simple Interest

Suppose that a **principal** of P dollars is invested or borrowed at an annual interest rate of r percent (in decimal form) for t years. The amount A that is then had or owed at the end of t years is found by

$$A = P + Prt$$

◇ **Example 5.1(i):** You borrow \$1000 at 8.5% interest for five years. How much money do you owe at the end of the five years?

Solution: $A = 1000 + 1000(0.085)(5) = \$1425$ owed

Note that what is owed at the end of the period is the original \$1000 borrowed *and* \$425 of interest.

◇ **Example 5.1(j):** You are going to invest \$400 for 5 years, and you would like to have \$500 at the end of the five years. What percentage rate would you need to have, to the nearest hundredth of a percent?

Solution:

$$\begin{aligned} 500 &= 400 + 400(5)r \\ 100 &= 2000r \\ \tfrac{100}{2000} &= r \\ r &= 0.05 = 5\% \end{aligned}$$

Section 5.1 Exercises To Solutions

Give correct units with all answers!

1. A rectangle has a length of 19 inches and a perimeter of 50 inches. What is the width of the rectangle?

2. Find the area of a circle with a radius of 5.3 inches. Round your answer to the tenths place.

3. The hypotenuse of a right triangle has length 13 inches and one of the legs has length 12 inches. What is the length of the other leg?

4. The circumference of a circle is 16.4 inches. Find the radius of the circle, to the nearest tenth of an inch.

5. Sales tax in a particular city is 5.5%, and you buy an item with a *pre-tax* price of \$19.95.

   (a) How much sales tax will you have to pay for the item?

   (b) How much will you have to pay for the item, including tax?

6. A salesperson in an art gallery gets a monthly salary of \$1000 plus 3% of all sales over \$50,000. How much do they make in a month that they sell \$112,350 worth of art?

7. You invest \$1200 at 4.5% simple interest.

   (a) How much money will you have if you take it out after 4 years?

   (b) How many years, to the nearest tenth, would it take to "double your money?"

8. A baseball is hit upward from a height of four feet and with an initial velocity of 96 feet per second.

   (a) When is it at a height of 84 feet?

   (b) When is it at a height of 148 feet?

   (c) When is it at a height of 57 feet? Use your calculator and the quadratic formula, and round your answer(s) to the nearest hundredth of a second. (The hundredth's place is two places past the decimal.)

   (d) When is the ball at a height of 200 feet, to the nearest hundredth of a second?

   (e) When does the ball hit the ground, to the nearest hundredth of a second?

9. A company that manufactures ink cartridges for printers knows that the number x of cartridges that it can sell each week is related to the price per cartridge p by the equation $x = 1200 - 100p$. The weekly revenue (money they bring in) is the price times the number of cartridges: $R = px = p(1200 - 100p)$. What price should they set for the cartridges if they want the weekly revenue to be \$3200?

10. Suppose that a cylinder has a height of h inches and a radius of r inches. The surface area S (in square inches) of the cylinder is given by $S = 2\pi rh + 2\pi r^2$. Find the height of a cylinder that has a a radius of 4.3 inches and a surface area of 465 square inches. **Round your answer to the tenth's place.**

11. For each of the following, determine whether the root exists. If it doesn't exist, say so (write DNE for "does not exist") and you are done. If the root does exist, give its value without using your calculator.

    (a) $\sqrt[4]{-16}$ (b) $\sqrt{-16}$ (c) $-\sqrt{16}$ (d) $\sqrt[3]{1000}$ (e) $\sqrt{36}$

12. Simplify by combining like terms, *if possible*.

(a) $12 - 9\sqrt{5} + 4\sqrt{5} - 15$

(b) $25 + 10\sqrt{3} + 10\sqrt{3} + 12$

13. Multiply and simplify.

(a) $(7 + 2\sqrt{3})(1 - \sqrt{5})$

(b) $(4 + \sqrt{7})(3 - 2\sqrt{7})$

(c) $(2 - \sqrt{3})^2$

14. Solve each equation. Be sure to check all solutions to see if they are valid.

(a) $\sqrt[3]{6x + 9} = 3$

(b) $\sqrt{-2x + 1} = -3$

(c) $\sqrt{2x + 11} - 4 = x$

(d) $\sqrt[3]{x - 3} = -2$

5.2 Solving Formulas

> 5. (b) Solve formulas for given unknowns.

Recall the simple interest formula $A = P + Prt$. Suppose that we wanted to find out how long \$300 must be invested at 4.5% in order to have \$500. We would replace P, r and A with 300, 0.045 and 500, respectively, and solve for t:

$$\begin{aligned} 500 &= 300 + 300(0.045)t \\ 200 &= 300(0.045)t \\ t &= \frac{200}{300(0.045)} = 14.8 \text{ years} \end{aligned}$$

Now suppose instead we wanted to know how long \$2000 must be invested at 7.25% in order to have \$7000. We'd find out like this:

$$\begin{aligned} 7000 &= 2000 + 2000(0.0725)t \\ 5000 &= 2000(0.0725)t \\ t &= \frac{5000}{2000(0.0725)} = 34.5 \text{ years} \end{aligned}$$

What if we wanted to know how long \$800 must be invested at 5% in order to have \$1000? We'd simply repeat the process that we just went through twice already, which is getting a bit boring! If we wanted to find things like this out many times over, it would be simpler to do all of the above steps with the equation $A = P + Prt$, *THEN* substitute the values of A, r and P into the resulting equation to calculate t. The next example shows us how to do this sort of thing.

◇ **Example 5.2(a):** Solve $A = P + Prt$ for t, then use the result to find out how long \$800 must be invested at 5% in order to have \$1000.

Solution:

$$\begin{aligned} A &= P + Prt \\ A - P &= Prt \\ \frac{A-P}{Pr} &= t \end{aligned}$$

$$t = \frac{1000 - 800}{800(0.05)} = 5 \text{ years}$$

We will now look at an example of a computation that we will find valuable later.

◇ **Example 5.2(b):** Solve $-2x-7y=14$ for y. Give your answer in $y=mx+b$ form.

Solution:

$$\begin{aligned} -2x-7y &= 14 && \text{original equation} \\ -7y &= 2x+14 && \text{add } 2x \text{ to both sides} \\ y &= \frac{2x+14}{-7} && \text{divide both sides by } -7 \\ y &= \frac{2x}{-7}+\frac{14}{-7} && \text{split the fraction into two fractions} \\ y &= -\tfrac{2}{7}x-2 && \text{simplify fractions} \end{aligned}$$

Suppose now that we wish to solve the equation $ax-8=bx+3$ for x. The problem here is that there are two terms with x in them, one on each side of the equation. The key is to get both those terms on one side of the equation, then factor the x out:

◇ **Example 5.2(c):** Solve $ax-8=bx+3$ for x.

Solution:

$$\begin{aligned} ax-8 &= bx+3 && \text{original equation} \\ ax-bx &= 3+8 && \text{subtract } bx \text{ from, and add } 8 \text{ to, both sides} \\ (a-b)x &= 11 && \text{add } 3 \text{ and } 8, \text{ factor } x \text{ out of the left side} \\ x &= \frac{11}{a-b} && \text{divide both sides by } a-b \end{aligned}$$

Section 5.2 Exercises

To Solutions

1. (a) Use the result of the example to determine how long \$1000 must be invested at 3.5% to have \$1800.

   (b) Solve the equation $A=P+Prt$ for r.

2. Solve $PV=nRT$ for R.

3. Solve $ax+3=bx-5$ for b.

4. Solve $P=2w+2l$ for l.

5. Solve $C=2\pi r$ for r. Here the symbol π is for the special number pi. When solving for r you can treat π just as you would any letter.

6. Solve each equation for y. Give your answers in $y=mx+b$ form.

   (a) $3x+4y=-8$ (b) $5x+2y=-10$ (c) $3x-2y=-5$

   (d) $3x-4y=8$ (e) $3x+2y=5$

7. (a) Solve $8x + 3 = 5x - 7$ for x.

   (b) Try solving $ax + 7 = bx + 3$ for x. You may have some trouble getting x alone on one side (and not on the other side). *If you can't figure out how to do this, take a look at Example 5.2(c).*

8. Solve each equation for x.

   (a) $ax + b = cx + d$

   (b) $x + 1.065x = 8.99$

   (c) $ax + bx = c$

   (d) $ax + 3 = cx - 7$

9. Solve $A = P + Prt$ for P.

10. For each of the following, simplify the root if possible.

    (a) $\sqrt{17}$

    (b) $\sqrt{\frac{20}{9}}$

    (c) $-\sqrt{72}$

11. For each of the following, use the quadratic formula to solve the given equation. Simplify your answers.

    (a) $x^2 - 19 = 6x$

    (b) $x^2 = 2x + 17$

12. A projectile is shot upward from ground level with an initial velocity of 48 feet per second.

    (a) When is the projectile at a height of 20 feet?

    (b) When does the projectile hit the ground?

    (c) What is the height of the projectile at 1.92 seconds? **Round your answer to the nearest tenth of a foot.**

    (d) When is the projectile at a height of 32 feet?

    (e) When is the projectile at a height of 36 feet? Your answer here is a bit different than your answers to (a) and (d). What is happening physically with the projectile?

    (f) When is the projectile at a height of 40 feet?

5.3 Applications of Equations

5. (c) Create equations whose solutions are answers to applied problems.

In this section we will attempt to solve "word problems." The process we will use is the following:

- Get to know the situation by guessing a value for the thing we are supposed to find, then doing the appropriate arithmetic to check our guess. (If we are supposed to find several quantities, we just guess the value of one of them.) Do this a few times if necessary.
- Let x be the correct value that we are looking for. Do everything to x that we did to our guess in order to check it, but set the result equal to what it is really supposed to be. This gives us an equation.
- Solve the equation to get the value asked for. If asked for more than one value, find the others as well.

Let's look at some examples to see how this process works.

⋄ **Example 5.3(a):** The longest side of a triangle is five more than the medium side, shortest side is half the medium side. The perimeter is $27\frac{1}{2}$. Find the lengths of the sides of the triangle. Another Example

Solution: Let's guess the medium side has length 8. Then the longest side is five more than the medium side, or $8 + 5 = 13$. The shortest side is half the medium side, or $\frac{1}{2}(8) = 4$. The perimeter (distance around, remember) is then $4 + 8 + 13 = 25$. Since the perimeter is supposed to be 27.5, our guess is close but incorrect.

Rather than adjusting our guess and trying again, let's say that the medium side has length x. The longest side is then five more, or $x + 5$. Notice that this repeats what we did above, but with x instead of 8. The shortest side is half the medium side, or $\frac{1}{2}x$. The perimeter is then $\frac{1}{2}x + x + (x + 5)$, and must equal 27.5. We set these equal and solve:

$\frac{1}{2}x + x + (x + 5)$	$=$	27.5	original equation
$x + 2x + 2(x + 5)$	$=$	55	multiply both sides by two
$3x + 2x + 10$	$=$	55	add first two terms, distribute 2
$5x$	$=$	45	combine x terms, subtract 10 from both sides
x	$=$	9	divide both sides by 5

x represents the length of the medium side, so the length of the longest side is $9 + 5 = 14$ and the length of the shortest side is $\frac{1}{2}(9) = 4\frac{1}{2}$.

◇ **Example 5.3(b):** In a certain city sales tax is 4.5%. If the sales tax on an item was $1.80, what was the price of the item?

Solution: Guess the price was $30: 4.5% of $30 = (0.045)(30) = \$1.35$. Now that we see what to do with the price, let's suppose that the correct price is x:

$$4.5\% \text{ of } x = 0.045x = \$1.80 \qquad \Rightarrow \qquad x = \frac{1.80}{0.045} = \$40$$

The price of the item was $\$40$.

Read the next example, and make sure you see the difference between what it is asking, versus the last example.

◇ **Example 5.3(c):** In the same city you buy something for $78.37, including tax. What was the price of the thing you bought?

Solution: Guess the price was $\$70$. Then the tax was $(0.045)(70) = \$3.15$. The cost, with tax, is then $70 + 3.15 = \$73.15$. Now let x be the correct price, so it takes the place of 70 in the above two calculations. The tax is then $0.045x$ and the cost with tax is $x + 0.045x$. Note that x represents the price of the item, which must be paid, and $0.045x$ represents the tax paid. So we set $x + 0.045x$ equal to the total cost of $\$78.37$ and solve:

$$\begin{aligned} x + 0.045x &= 78.37 && \text{the equation} \\ (1 + 0.045)x &= 78.37 && \text{factor } x \text{ out of the left side} \\ 1.045x &= 78.37 && \text{add } 1 + 0.045 \\ x &= \$75.00 && \text{divide both sides by } 1.045 \end{aligned}$$

◇ **Example 5.3(d):** The length of a rectangle is one more than three times the width, and the area is 520. Write an equation and use it to find the length and width of the rectangle.

Solution: Suppose that the width of the rectangle is $w = 5$. Then the length is one more than three times the width, or $l = 1 + 3(5) = 16$. The area would then be $lw = (16)(5) = 80$. This is far too low, but it doesn't matter! We've used our guess to see what needs to be done. Now suppose that the width is just w units. Then the length is $l = 1 + 3w$ and the area is $lw = (1 + 3w)w$. We set this equal to 520 and solve:

$$\begin{aligned} (1 + 3w)w &= 520 \\ w + 3w^2 &= 520 \\ 3w^2 + w - 520 &= 0 \end{aligned}$$

It turns out that the left side of the last equation can be factored, but not in a way that is obvious to most of us! Let's use the quadratic formula instead:

$$w = \frac{-1 \pm \sqrt{1^2 - 4(3)(-520)}}{2(3)} = \frac{-1 \pm \sqrt{6241}}{6} = \frac{-1 \pm 79}{6} = \frac{78}{6}, -\frac{80}{6} = 13, -\frac{40}{3}$$

The width cannot be negative, so the only possible value of the width is $13,$ giving a corresponding length of 40 units.

Section 5.3 Exercises

To Solutions

For exercises one through five, write an equation that can be used to solve the problem, then solve.

1. The length of a rectangle is 3.2 inches less than twice its width, and its perimeter is 57.1 inches. Find the length and width of the rectangle.

2. Sales tax in a state is 5.5%. If the tax on an item was $10.42, what was the price of the item?

3. One leg of a right triangle is twice as long as the other leg, and the hypotenuse has a length of 15 feet. How long are the legs?

4. After a 2% raise your hourly wage is $7.37 per hour. What was it before the raise?

5. The length of the shortest side of a triangle is half the length of the longest side. The middle side is 1.4 inches longer than the shortest side, and the perimeter is 42 inches. Find the lengths of all three sides.

6. Solve each equation. Be sure to check all solutions to see if they are valid.

   (a) $\sqrt{2x+7}-6=-2$ (b) $\sqrt{5x+9}=x-1$

   (c) $\sqrt[3]{6x+5}=2$

7. Solve each equation for x.

   (a) $ax+3=bx-5$ (b) $ax+7=b(x+c)$

6 Equations Relating Two Variables

6.1 Graphs of Equations in Two Unknowns

6. (a) Graph the solution set of an equation in two unknowns.

Consider the equation $5x+2=17$. The solution to this equation is $x=3$ because if we substitute three for x in the expression on the left side of the equation, the equation becomes a true statement. *Remember that we don't have to actually solve an equation to determine whether something is a solution - it suffices to show that the value makes the statement true.* As we know, this equation has only one solution, so three is the only solution.

Now suppose that we have the equation $3x-y=8$, which contains two unknowns, x and y. (**NOTE:** When we use two different letters for unknowns it is implied that they likely have different numerical values, but *they can be the same*.) What do we mean by a solution to such an equation? In this case, where the equation contains two unknowns, a solution consists of a *pair* of numbers that make the equation true when substituted in. One solution to the equation is $x=3$ and $y=1$, because $3(3)-1=8$.

◇ **Example 6.1(a):** Is $x=-1,\ y=-11$ a solution to $3x-y=8$? Is $x=5,\ y=3$? Is $x=2,\ y=-2$?

Solution: $3(-1)-(-11)=-3+11=8$, so $x=-1,\ y=-11$ is a solution.

$3(5)-(3)=15-3=12\neq 8$, so $x=5,\ y=3$ is not a solution.

$3(2)-(-2)=6+2=8$, so $x=2,\ y=-2$ is a solution.

As you might guess, the equation $3x-y=8$ has infinitely many solutions.

At some point (maybe already!) we will tire of writing $x=$ and $y=$ for every solution pair. To eliminate this annoyance, mathematicians have developed the following convention. In order to indicate $x=5$, $y=-4$, we write $(5,-4)$. This is called an **ordered pair**. Note three important things:

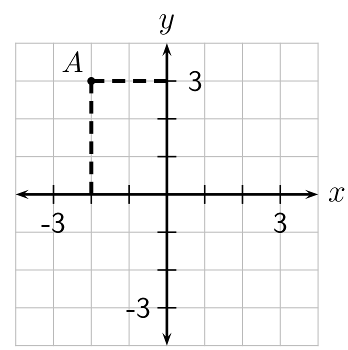
- We write the value of x, *then* y, always in that order.
- The two values are separated by a comma.
- The values are enclosed by parentheses. *Do not write* $\{5,-4\}$ *or* $[5,-4]$*; those mean something else in mathematics.*

◇ **Example 6.1(b):** Give some solutions to the equation $y=x^2-3$ in the form just shown.

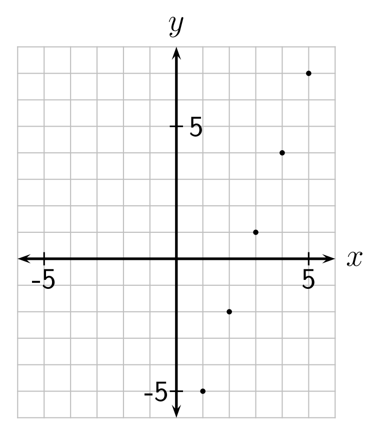
Solution: Here we can see that if $x=1$, $y=1^2-3=-2$, so $(1,-2)$ is a solution pair. Substituting 0, -1 and 5 for x, we get the solution pairs $(0,-3)$, $(-1,-2)$, $(5,22)$.

For the last example you may have noticed that to find solutions for the equation $y = x^2 - 3$ you can simply choose any value for x, substitute it into the right hand side of the equation and evaluate, giving you the value that y must have for that particular choice of x.

At this point you have three solution pairs for the equation $3x - y = 8$, $(3,1), (-1,-11)$ and $(2,-2)$. It should be clear to you that there are many more **solution pairs**, as we call them. Rather than try to find all of them, we'll draw a 'picture' of them. We will 'plot' the points we have on what we call a **coordinate grid**, or **Cartesian plane**, named after the philosopher and mathematician René Descartes. We then try to guess (correctly) where *all* other points would be.



The Cartesian plane consists of two number lines, placed at right angles to each other, with zero on one line placed on zero on the other. The picture to the right shows such a coordinate grid, with a point A on it. The point represents two numbers, an x and a y. To get the x value represented by A we read from the point straight down to the the horizontal line, which we call the x-**axis**. So for A we get -2, which we call the x-**coordinate** of the point. The value on the y-**axis** associated A is called its y-**coordinate**; in this case it is $y = 3$. In this manner, every point on the Cartesian plane represents an ordered pair, and every ordered pair has a spot on the Cartesian plane.

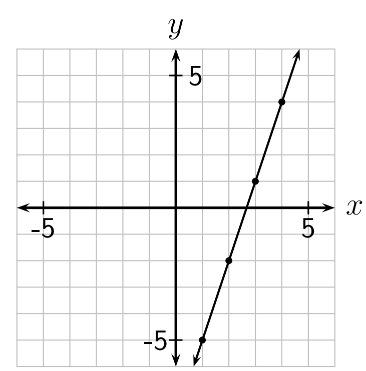


Going back now to the equation $3x - y = 8$ with solution pairs including $(3,1)$ and $(2,-2)$, we would find that $(4,4), (1,-5)$ and $(5,7)$ are also solutions. We can plot those points on a coordinate grid, as shown to the right. Note that it appears that all five points lie on a line. (In mathematics, when we talk about a line it means a *straight* line.) That line is significant in that

- every (x,y) that is a solution to the equation, when plotted on the coordinate grid, will lie on that line,
- the x- and y-coordinates of any point on the line are a solution pair for the equation.

We call the line the **solution set** of the equation. We will often just call it the graph of the equation, which is slightly incorrect, but we all know what we mean.

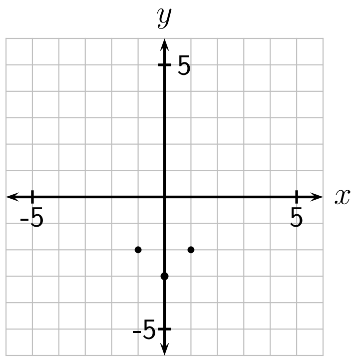
◇ **Example 6.1(c):** Graph the equation $3x - y = 8$.

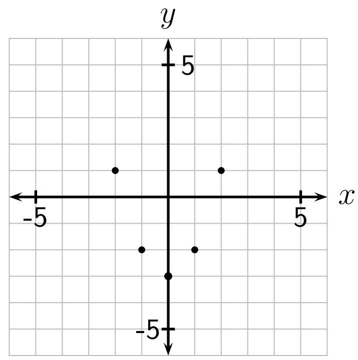


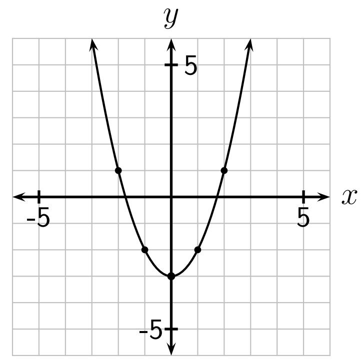
Solution: Once we have found four or five solution pairs we plot each one as a point, as shown above. We then draw a curve or line that smoothly connects all the points, as shown to the right. The arrowheads on each end of the line indicates that the line keeps going in the direction of the arrows.

◇ **Example 6.1(d):** Graph the equation $y = x^2 - 3$. Another Example

Solution: In Example 6.1(b) we found the solution pairs $(1, -2)$, $(0, -3)$, $(-1, -2)$ and $(5, 22)$. Plotting the first three of those points gives us the graph shown below and to the left. Those are not enough to give a clear indication of what the graph would look like, so we let $x = -2$ and $x = 2$ to get two more ordered pairs $(-2, 1)$ and $(2, 1)$, which are added to the graph shown in the middle below. We can then see that the graph is a U-shape, called a **parabola**, as shown in the third graph below.

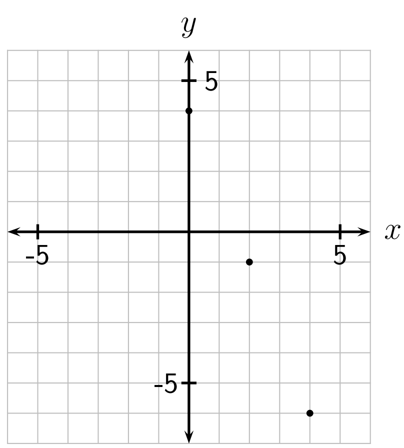


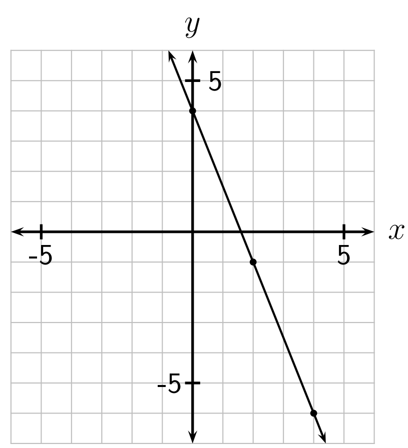




◇ **Example 6.1(e):** Graph the equation $y = -\frac{5}{2}x + 4$.

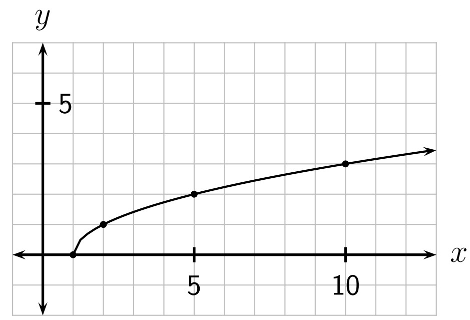
Solution: As in the previous example, we want to choose some values for x and substitute them into the equation to find the corresponding y value for each x. It should be easily seen that one solution pair is $(0, 4)$. If we substitute one for x we get $y = -\frac{5}{2}(1) + 4 = \frac{3}{2} = 1\frac{1}{2}$. Now the ordered pair $(1, 1\frac{1}{2})$ can be plotted, but it is not as easy to work with as a pair in which both numbers are integers. Note that if instead of choosing $x = 1$ we use $x = 2$, we get $y = -\frac{5}{2}(2) + 4 = -5 + 4 = -1$, giving us the ordered pair $(2, -1)$. What made this work out better is that the value $x = 2$ cancels with the two in the bottom of $-\frac{5}{2}$. The same sort of this happens if we choose x to be other multiples of two, like four, which gives us the ordered pair $(4, -6)$. Plotting the three points we now have gives us the graph shown below and to the left. The three points appear to be in a line, and in fact they are; the graph is shown below and to the right.





◇ **Example 6.1(f):** Graph the equation $y = \sqrt{x-1}$.

Solution: As before, we want to substitute values for x and find the corresponding y values. It is nicest if we choose values like $x = 5$, because then $y = \sqrt{5-1} = \sqrt{4} = 2$, an integer. This shows that we want to choose values of x resulting in finding square roots of $0, 1, 4, 9, 16$, ..., the perfect squares. From this we get the solution pairs $(1, 0)$, $(2, 1)$, $(5, 2)$, $(10, 3)$, $(17, 4)$, etc. Plotting the few of those that can fit on the grid to the right, we get the graph shown there.



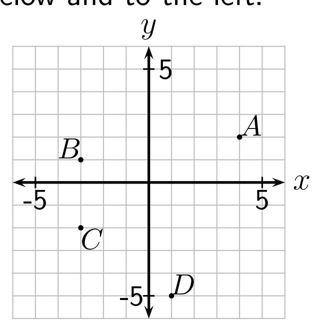
Section 6.1 Exercises

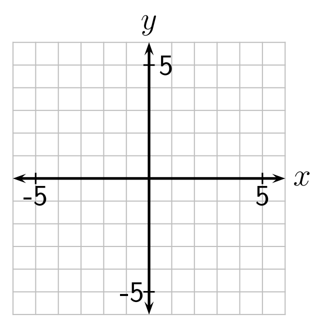
To Solutions

1. Consider again the equation $3x - y = 8$. Solve for y and find five solutions pairs, including some with negative values of x.

2. Solve the equation $x^2 + y = 5$ for y, then find five solution pairs. **Use some negative values for x as well as positive values.** Once you find one solution, it should be easy to get another - explain.

3. (a) Give the coordinates for each of the points A through D plotted on the coordinate grid below and to the left.





   (b) Plot and label (with letters) the points $A(4, -1)$, $B(-3, -2)$, $C(1, 5)$, $D(5, 1)$ on the coordinate grid above and to the right.

4. Plot the solution pairs that you found for the equation from Exercise 2. *They should not lie on a line*, but there should be a pattern in their locations. If you can't see it, find a few more. Then draw a curve that you think goes through all of them.

5. There is no reason that we need to solve for y and substitute values for x. Sometimes it will be much easier to solve for x and substitute values for y to get solution pairs.

   (a) Do this to find three solutions for $x - y^2 = 1$. *Make sure that you give solution pairs with x first, y second!*

(b) Can you use the idea from Exercise 3 to get more solutions from the three that you found? You should be able to get at least one more.

(c) Plot your solution pairs and draw a curve that you think contains all solutions to the equation.

6. Consider the equation $2x + 3y = 15$.

   (a) Solve the equation for y.

   (b) To find solution pairs, we should choose values of x that are multiples of what number?

   (c) Find four solution pairs.

   (d) Graph the equation.

7. Find three solution pairs to $y = \frac{3}{5}x - 2$ and plot them. Do you think you might know what the graph of the equation is? Draw it.

8. Use the equation $x = \sqrt{y+5}$ for the following.

   (a) Find four integer solutions to the equation. This will require values of y that result in $\sqrt{0}$, $\sqrt{1}$, $\sqrt{4}$, $\sqrt{9}$, etc.

   (b) Is there a largest or smallest value that you can use for y? What is the solution pair in that case?

   (c) There is no reason that we have to use values of y that give perfect square values for $y + 5$. Give the solution pair corresponding to $y = 7$, *in exact form*.

   (d) Give the solution pair corresponding to $y = 7$, *in decimal form, rounded to the hundredth's place*.

   (e) Graph the equation.

9. Consider the equation $y + 3 = \sqrt{1-x}$

   (a) Solve the equation for y.

   (b) Is there a largest or smallest value that you can use for x? What is the solution in that case?

   (c) Find three more solutions and graph the equation.

10. Multiply and simplify.

    (a) $(1-\sqrt{2})(1+\sqrt{2})$ (b) $(3-5\sqrt{6})^2$ (c) $(5+3\sqrt{10})(5-3\sqrt{10})$

11. A retailer adds 40% of her cost for an item to get the price she sells it at in her store. Find her cost for an item that she sells for $\$59.95$. **Write an equation that can be used to solve this problem, and solve the equation.**

12. The hypotenuse of a right triangle is two more than one of the legs, and the other leg has length 8. What are the lengths of the sides of the triangle? **Write an equation that can be used to solve this problem, and solve the equation.**

6.2 x- and y-Intercepts

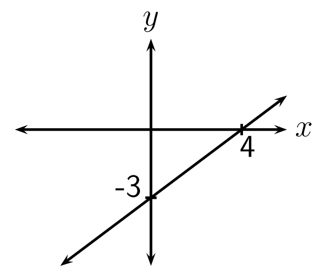
6. (b) Find x- and y-intercepts of an equation in two variables.

In this section we begin with one simple observation: zero is usually the easiest number with which to do computations!

◇ **Example 6.2(a):** For the equation $3x-4y=12$, let $x=0$ and $y=0$ to get two solution pairs. Given that the graph of the equation is a line, give the graph. Another Example

Solution: Letting each of x and y be zero, we get

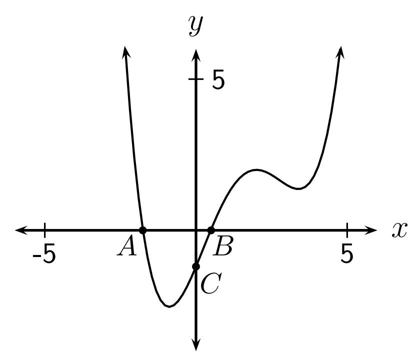
$$\begin{aligned} 3(0)-4y &= 12 \\ -4y &= 12 \\ y &= -3 \end{aligned} \qquad \begin{aligned} 3x-4(0) &= 12 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$



This gives us the two points $(0,-3)$ and $(4,0)$. Having been told that the graph is a line, the graph must be as shown to the right.

We call point $(4,0)$ the x-**intercept** of the equation and $(0,-3)$ the y-**intercept** of the equation. One or both of these two points are often of interest to us, so this section is devoted entirely to finding the intercepts.

Consider the graph shown to the right, for some unknown equation. Points A and B are *both* x-intercepts, and point C is a y-intercept. *It is possible to have more than one intercept on each axis.* The points are called intercepts because they are where the graph of the equation intercepts the axes.



Since I put no scale on either axis, you don't know the x-coordinates of A and B, but it should be clear that *they both have y-coordinates of zero.* Similarly, C has an x-coordinate of zero. This indicates the following.

Finding Intercepts

- To find the x-intercepts for an equation, let $y=0$ and solve for x.
- To find the y-intercepts for an equation, let $x=0$ and solve for y.
- We sometimes give the intercepts as single numbers rather than ordered pairs, because we know that the y-coordinates of x-intercepts must be zero, and vice-versa.

◇ **Example 6.2(b):** Find the intercepts of $x = 4 - y^2$

Solution: As just discussed, to find the x-intercept(s) we set $y = 0$ and solve, and vice-versa for the y-intercept(s):

$$\begin{aligned} x &= 4 - (0)^2 \\ x &= 4 \end{aligned} \qquad\qquad \begin{aligned} 0 &= 4 - y^2 \\ y^2 &= 4 \\ y &= \pm 2 \end{aligned}$$

The x-intercept is $(4, 0)$ and there are two y-intercepts, $(0, 2)$ and $(0, -2)$.

Let's take a moment to note a couple of things:

- If you can remember that the intercepts are found by letting x and y be zero and solving for the other variable, you can just find all ordered pairs you can this way. Then think about where each would be plotted in order to determine whether it is an x-intercept or y-intercept.
- For an x-intercept of $(a, 0)$ we will often just say the x-intercept is a, since we know that the y-value at an x-intercept must be zero, and similarly for a y-intercept. So for the above example, we would say that there is an x-intercept of 4 and y-intercepts of 2 and -2.

Section 6.2 Exercises

To Solutions

1. Find the intercepts of $-5x + 3y = 15$.

2. Find the intercepts of $y = x^2 - 2x - 3$.

3. This exercise will illustrate that intercepts are not always integers! (Remember that integers are positive or negative whole numbers, or zero.) Find the intercepts of $2x + 3y = 9$.

4. This exercise illustrates that the an x-intercept can also be a y-intercept.

   (a) Find the x-intercepts of $y = x^2 - 5x$. *Give your answers as ordered pairs.*

   (b) Find the y-intercepts of $y = x^2 - 5x$. *Give your answers as ordered pairs.*

5. Find just the y-intercept of each of the following. y-intercepts of equations like these will be important to us soon.

   (a) $y = \frac{3}{5}x - 2$ (b) $y = -\frac{4}{3}x + 5$ (c) $y = -3x + 1$

6. Find the intercepts for each of the following equations.

   (a) $3x - 2y = 2$ (b) $x = y^2 - 2y$ (c) $y = \sqrt{x + 4}$

   (d) $\sqrt{x + 4} = y + 1$ (e) $3x + 5y = 30$ (f) $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

7. For each equation, tell what values x is not allowed to have, then solve the equation.

(a) $\dfrac{3}{2} + \dfrac{5}{x-3} = \dfrac{x+9}{2x-6}$ (b) $\dfrac{4x}{x+2} = 4 - \dfrac{2}{x-1}$

8. For each of the following, use the quadratic formula to solve the given equation. Simplify your answers.

(a) $x^2 + 4x = 41$ (b) $25x^2 + 7 = 30x$

9. Solve each equation for y. Give your answers in $y = mx + b$ form.

(a) $2x - 3y = 6$ (b) $3x + 5y + 10 = 0$

10. A manufacturer of small calculators knows that the number x of calculators that it can sell each week is related to the price per calculator p by the equation $x = 1300 - 100p$. The weekly revenue (money they bring in) is the price times the number of calculators: $R = px = p(1300 - 100p)$. What price should they set for the cartridges if they want the weekly revenue to be \$4225?

11. For each of the following equations, find at least five solution pairs, then draw a graph of the solution set for the equation.

(a) $y = x^2 - 2x$ (b) $y = (x+3)^2$ (c) $x + 2y = 4$

6.3 Slopes of Lines

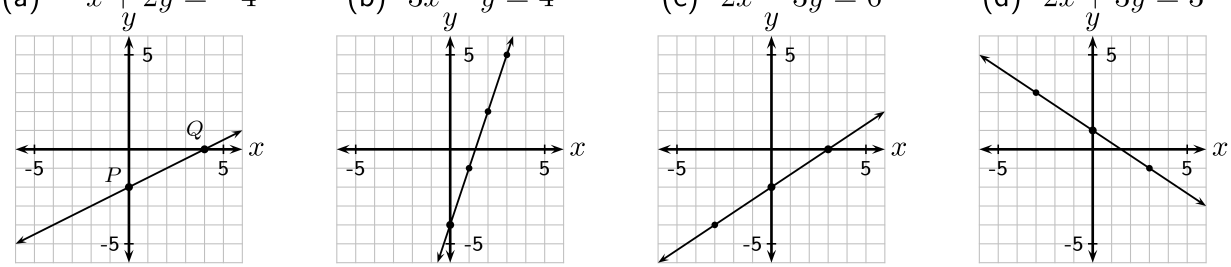
6. (c) Find the slope of a line, including vertical and horizontal.
   (d) Know and apply the relationship between slopes of parallel and perpendicular lines.

Consider the following equations. Note that each has the form $Ax + By = C$, where any of A, B or C could be negative. For example, for equation (a) we have $A = -1$, $B = 2$ and $C = -4$.

(a) $-x + 2y = -4$ (b) $3x - y = 4$ (c) $2x - 3y = 6$ (d) $2x + 3y = 3$

Because neither the x or the y is squared, under a square root, or in the bottom of a fraction, the graphs of all of these equations will be lines. Here are their graphs:

(a) $-x + 2y = -4$ (b) $3x - y = 4$ (c) $2x - 3y = 6$ (d) $2x + 3y = 3$



When dealing with lines, it will be convenient to define a concept called the **slope** of a line. For any line, the slope is a number that should describe the "steepness" of the line, with steeper lines having larger numbers for their slopes. So, for example, the slope of the line in (a) above should be less than the slope of the line in (b). A horizontal line has no steepness, so its slope should be zero.

Take a close look at the lines in (c) and (d) above - both have the same steepness even though they look quite different. We want to distinguish those two from each other, so we will say that the line in (c) has a positive slope, and the line in (d) has a negative slope. *Any line going upward from left to right has a positive slope*, and any line going downward from left to right has a negative slope.

This gives us a general idea what slope is about, but we now need a way to determine the actual slope of a line. The general idea is this: We find two points on the line and consider the vertical difference between the two points, which we call the **rise**, and the horizontal difference between the two points, which we call the **run**. We then define the slope to be the *rise over the run*, with the appropriate sign as described above.

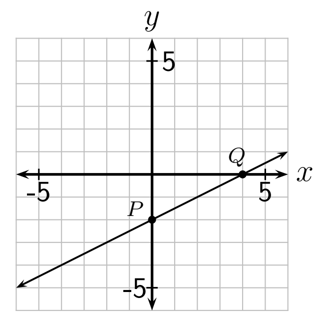
◇ **Example 6.3(a):** Find the slope of the line in graph (a) above.

Solution: The slope is positive, with a rise of two (from point P to point Q) and a run of four. The slope is therefore $\frac{2}{4} = \frac{1}{2}$.

◇ **Example 6.3(b):** Find the slope of the line in graph (d).

Solution: The slope is negative in this case, with a rise of two and a run of three, so the slope is $-\frac{2}{3}$.

Graph (a) is shown again to the right. Note that point A has coordinates $(0,-2)$ and point B has coordinates $(4,0)$. Now the y-coordinate of each point tells its "height" in some sense, so if we subtract the y-coordinates of point A from the y-coordinate of point B we get the rise between the two points: rise $=2=0-(-2)$. If we subtract the x-coordinate of point A from the x-coordinate of point B we get the run between the two points: run $=4=4-0$. We can then take the rise over the run to get a slope of $\frac{2}{4}=\frac{1}{2}$.



The upshot of all this is that to find the slope of the line through two points, we can simply subtract the y-coordinates of the points, subtract the x-coordinates, and divide the results. Letting the letter m represent slope, we can summarize this as follows. If we have two points (x_1,y_1) and (x_2,y_2), the slope of the line through the two points is

$$m=\frac{y_2-y_1}{x_2-x_1}$$

Given two ordered pairs, it doesn't matter which pair we call (x_1,y_1) and which pair we call (x_2,y_2), but x_1 and y_1 have to come from the same ordered pair, as do x_2 and y_2.

◇ **Example 6.3(c):** Find the slope of the line through $(2,3)$ and $(-4,5)$. Another Example

Solution: Subtracting the y's we get $5-3=2$ and subtracting the x's *in the same order* we get $-4-2=-6$. the slope is then

$$m=\frac{\text{rise}}{\text{run}}=\frac{5-3}{-4-2}=\frac{2}{-6}=-\frac{1}{3}$$

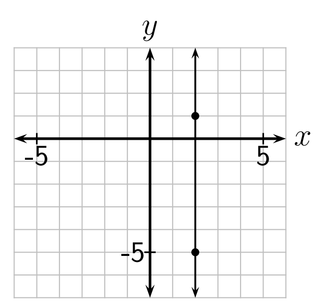
Note that if we were to subtract in the other order we would get the same result in the end:

$$m=\frac{3-5}{2-(-4)}=\frac{-2}{6}=-\frac{1}{3}\ \text{again}$$

The next example illustrates an important idea.

◇ **Example 6.3(d):** Find the slope of the line through $(2,1)$ and $(2,-5)$.Another Example

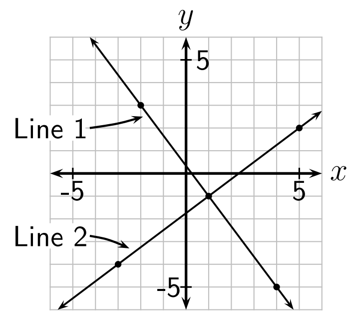
Solution: $m=\dfrac{-5-1}{2-2}=\dfrac{-6}{0}$, which is not defined. When we plot the two points and the line containing them, shown to the right, we see that the line is vertical.



The last example shows us that *the slope of a vertical line is undefined*. With a little thought it should be clear that a horizontal line has a rise of zero for any value of run we want, resulting in a fraction of the form $\frac{0}{a} = 0$. Therefore *horizontal lines have slopes of zero*.

It should also be clear that if two lines are parallel, they have the same slope. What about if the two lines are perpendicular?

◇ **Example 6.3(e):** The two lines shown to the right are perpendicular. Find each of their slopes.



Solution: The slope of Line 1 is $m = -\dfrac{4}{3}$ and the slope of Line 2 is $m = \dfrac{3}{4}$.

The above example seems to indicate that if two lines are perpendicular, their slopes have opposite signs and are reciprocal fractions of each other. This is in fact always the case.

Let's summarize what we now know about slopes of lines.

Slopes of Lines

- The slope of a line is a single number that indicates how steep the line is.
- Lines that go up from left to right have positive slopes and lines that go down from left to right have negative slopes.
- The slope of a horizontal line is zero and the slope of a vertical line is undefined, or is said to not exist.
- The slope of a line is $\dfrac{\text{rise}}{\text{run}}$
- **Slope Formula:** The slope of the line through two points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
- Parallel lines have the same slope.
- Slopes of perpendicular lines are negative reciprocals of each other.

Section 6.3 Exercises To Solutions

1. Find the slope of the line in graph (b) at the start of the section.

2. Find the slope of the line in graph (c) at the start of the section.

3. Find the slope of the line through the given pair of points.

   (a) $(1,0)$ and $(5,2)$ (b) $(3,-4)$ and $(7,-4)$

4. The slope of Line 1 is $-\frac{3}{4}$.

   (a) Line 2 is parallel to Line 1. What is the slope of Line 2?

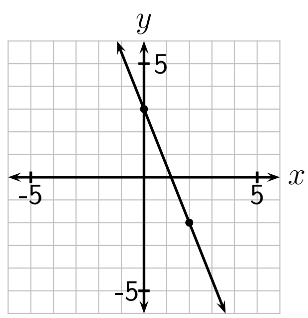
   (b) Line 3 is perpendicular to Line 1. What is the slope of Line 3?

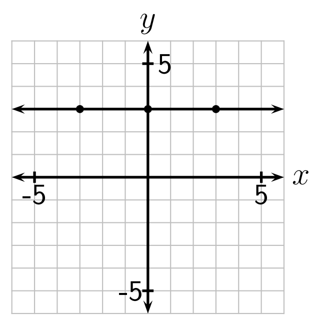
5. Find the slopes of the lines through the following pairs of points.

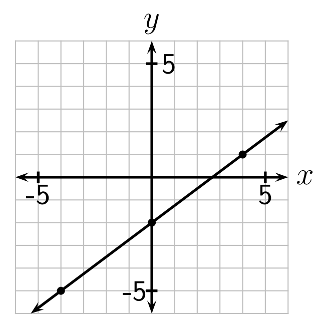
   (a) $(5,-2)$ and $(3,2)$ (b) $(-3,7)$ and $(7,11)$ (c) $(3,2)$ and $(-5,2)$

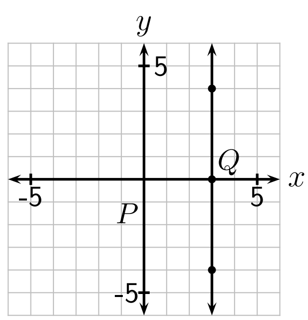
   (d) $(1,7)$ and $(1,-1)$ (e) $(-4,-5)$ and $(-1,4)$ (f) $(-1,3)$ and $(2,-3)$

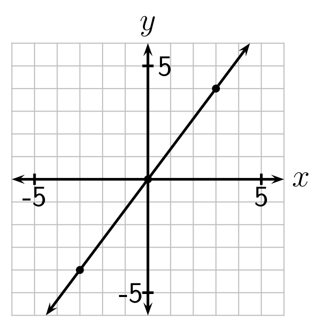
6. Find the slope of each line.

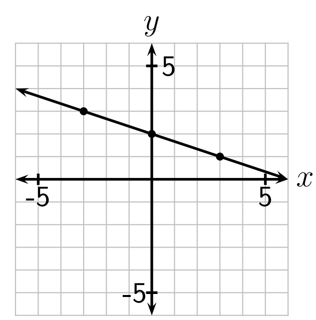
(a)


(b)


(c)


(d)


(e)


(f)


7. Line 1 has slope $-\frac{1}{3}$, and Line 2 is perpendicular to Line 1. What is the slope of Line 2?

8. Line 1 has slope $\frac{1}{2}$. Line 2 is perpendicular to Line 1, and Line 3 is perpendicular to Line 2. What is the slope of Line 3?

9. The width of a rectangle is three more than half the length. The perimeter is 39. How long are the sides of the rectangle? **Write an equation that can be used to solve this problem, and solve the equation.**

10. The cost of a compact disc, with 6% sales tax, was \$10.55. What was the price of the compact disc? **Write an equation that can be used to solve this problem, and solve the equation.**

11. Find the intercepts for each of the following equations.

    (a) $y = x^2 - 2x$ (b) $y = (x+3)^2$ (c) $x + 2y = 4$

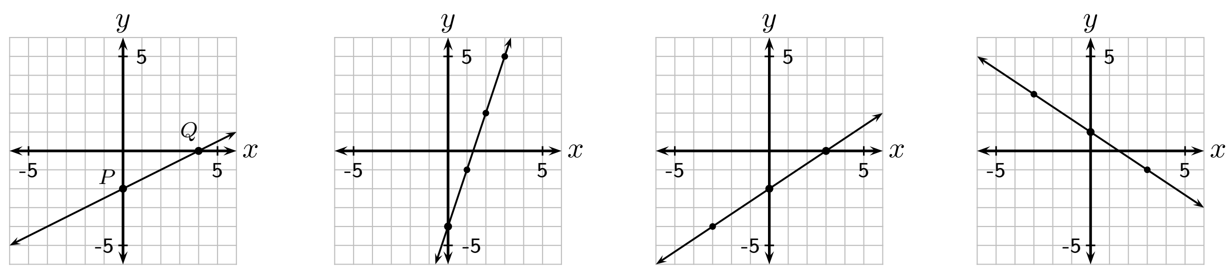
6.4 Equations of Lines

> 6. (e) Graph a line, given its equation; give the equation of a line having a given graph.
>
>   (f) Determine the equation of a line through two points.

We'll begin with three examples, all based on the same four graphs. You may wish to treat the examples as exercises; you should already know how to do all of them yourself.

◇ **Example 6.4(a):** Find the slopes of the four lines graphed below: Another Example

(a) $-x + 2y = -4$ (b) $3x - y = 4$ (c) $2x - 3y = 6$ (d) $2x + 3y = 3$



Solution: The slopes of the lines are (a) $\frac{1}{2}$, (b) 3 or $\frac{3}{1}$, (c) $\frac{2}{3}$, (d) $-\frac{2}{3}$.

◇ **Example 6.4(b):** Give the y-intercepts of the lines whose graphs are shown in Example 6.4(a).

Solution: The y-intercepts of the lines are (a) -2, (b) -4, (c) -2, (d) 1.

◇ **Example 6.4(c):** The equations of the lines graphed in Example 6.4(a) are given above each of the graphs. Solve each equation for y.

Solution:

(a)
$$\begin{aligned} -x + 2y &= -4 \\ 2y &= x - 4 \\ y &= \tfrac{1}{2}x - 4 \end{aligned}$$

(b)
$$\begin{aligned} 3x - y &= 4 \\ -y &= -3x + 4 \\ y &= 3x - 4 \end{aligned}$$

(c)
$$\begin{aligned} 2x - 3y &= 6 \\ -3y &= -2x + 6 \\ y &= \tfrac{2}{3}x - 2 \end{aligned}$$

(d)
$$\begin{aligned} 2x + 3y &= 3 \\ 3y &= -2x + 3 \\ y &= -\tfrac{2}{3}x + 1 \end{aligned}$$

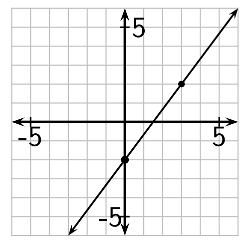
We want to look at the results of the above three examples and see how the equations, when solved for y, are related to the slopes and y-intercepts. First, we note that each equation is of the form $y = mx + b$ for some numbers m and b, with either or both of them perhaps being negative. From the above, the number m always seems to be the slope of the line, and the number b seems to be the y-intercept. Both of these things are in fact true:

Slope-Intercept Form of a Line

A line with slope m and y-intercept b has equation $y = mx + b$.

This can be used two ways, to get the equation of a line from a graph, and to graph a line whose equation is given.

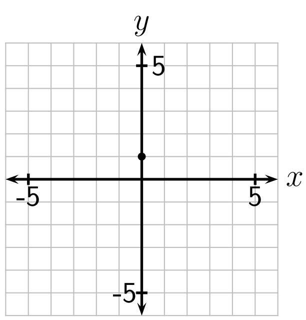
◇ **Example 6.4(d):** Give the equation of the line with the graph shown to the right. Another Example

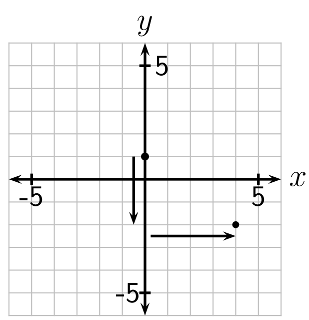


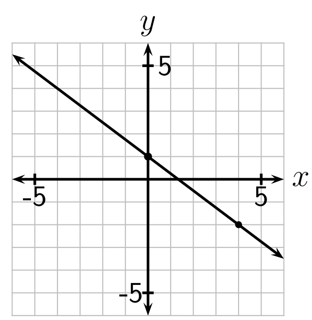
Solution: We can see that the line has a slope of $\frac{4}{3}$ and a y-intercept of -2, so the equation of the line is $y = \frac{4}{3}x - 2$.

◇ **Example 6.4(e):** Graph the line with equation $y = -\frac{3}{4}x + 1$. Another Example Another Example

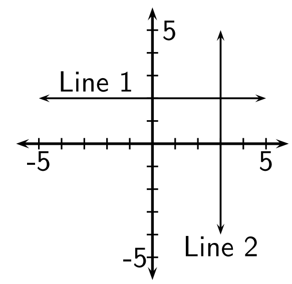
Solution: First we plot the y-intercept $(0, 1)$, as shown on the graph below and to the left. From there we go down three units and to the right four units, based on the slope of $-\frac{3}{4}$, and plot another point. This is shown o the middle graph below. Finally we draw a line through those two points, as shown on the graph below and to the right.







Consider a horizontal line, like the Line 1 shown to the right. Note that the line has a slope of zero, since it is horizontal, and its y-intercept is 2. Therefore the equation of the line is $y = 0x + 2$, which is really just $y = 2$. Now the equation of a line gives us conditions on how the values of x and y have to be related. In this case nothing is said about x, so we can take it to be anything. y, on the other hand, has to be 2 no matter what x is. So the graph is all the points for which y is 2, which is the horizontal line at $y = 2$.



We can't use $y = mx + b$ to get the equation of a vertical line because the slope of a vertical line is undefined, so we have nothing to put in for m! For Line 2 on the graph above, however, we can note that every point on Line 2 has an x-coordinate of 3, so the equation of the line is $x = 3$. These observations lead us to the following:

Equations of Horizontal and Vertical Lines

A horizontal line has equation $y = a$ and a vertical line has equation $x = b$.

All that we need in order to plot the graph of a line is two points that are on the line. Now we will use two given points and a bit of algebra to find the *equation* of the line through those points. The procedure is given first, but it may not make sense until you follow along with it as you read Example 6.4(f).

Equation of a Line Through Two Points

To find the equation of a line through two points you need to find values for m and b in $y = mx + b$. To do this,

- use the two points to find the slope of the line,
- put the slope into $y = mx + b$,
- put *either* point into $y = mx + b$ (with the slope you found for m) and solve for b,
- write the equation of the line with the m and b that you have found.

◇ **Example 6.4(f):** Find the equation of the line through the points $(-3, 3)$ and $(6, -3)$ algebraically. Another Example

Solution: The slope of the line is $m = \dfrac{3-(-3)}{-3-6} = \dfrac{6}{-9} = -\dfrac{2}{3}$, so the equation of the line must look like $y = -\frac{2}{3}x + b$. We then insert the coordinates of either of the two points on the line for x and y, and solve for b:

$$\begin{aligned} 3 &= -\tfrac{2}{3}(-3)+b \\ 3 &= 2+b \\ b &= 1 \end{aligned}$$

We insert this value into $y=-\frac{2}{3}x+b$ to get the equation of the line as $y=-\frac{2}{3}x+1$.

◇ **Example 6.4(g):** Find the equation of the line through the points $(-3,3)$ and $(-3,-1)$ algebraically. Another Example

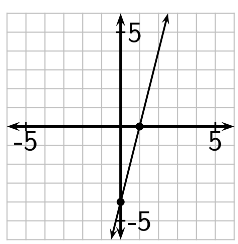
Solution: $m=\dfrac{3-(-1)}{-3-(-3)}=\dfrac{3+1}{-3+3}=\dfrac{4}{0}$. The slope is undefined, so this is a vertical line. We can see that both points have x-coordinates of three, so the equation of the line is $x=3$.

Section 6.4 Exercises

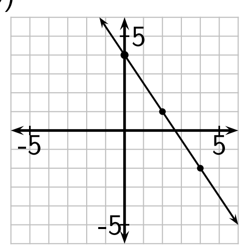
To Solutions

1. Give the equation of each line.

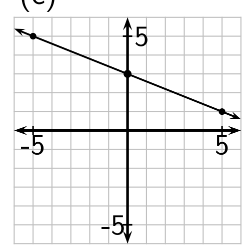
(a)



(b)



(c)



2. Sketch the graph of each line.

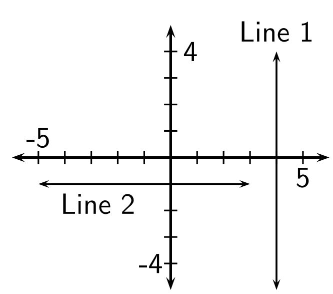
(a) $y=-\frac{1}{2}x+3$ (b) $y=\frac{4}{5}x-1$ (c) $y=3x-4$

3. (a) Give the equation of Line 1 to the right.

   (b) Give the equation of Line 2 to the right.

   (c) Sketch the graph of $x=-2$.

   (d) Sketch the graph of $y=4$.



4. Find the equation of the line through the two points **algebraically**.

(a) $(3,6),\ (12,18)$ (b) $(-3,9),\ (6,3)$ (c) $(3,2),\ (-5,2)$

5. For each of the following equations, find at least five solution pairs, then draw a graph of the solution set for the equation.

   (a) $3x - 2y = 2$ (b) $x = y^2 - 2y$ (c) $y = \sqrt{x + 4}$

6. (a) Give the slope of a line that is parallel to the line graphed in Exercise 1(a).
   (b) Give the slope of a line that is perpendicular to the line graphed in Exercise 1(b).
   (c) Give the slope of the line graphed in Exercise 1(c).
   (d) Give the slope of a line that is parallel to the line whose equation is given in Exercise 2(a).
   (e) Give the slope of the line whose equation is given in Exercise 2(b).
   (f) Give the slope of a line that is perpendicular to the line whose equation is given in Exercise 2(c).

6.5 Applications of Linear Equations

6. (g) Use a given linear model of a "real" situation and a given value of either of the two linearly related quantities to find the value of the other.
   (h) Given a linear model of a "real" situation, interpret the values of the slope and intercept.

Using a Linear Model to Solve Problems

An insurance company collects data on amounts of damage (in dollars) sustained by houses that have caught on fire in a small rural community. Based on their data they determine that the expected amount D of fire damage (in dollars) is related to the distance d (in miles) of the house from the fire station. (Note here the importance of distinguishing between upper case variables and lower case variables!) The equation that seems to model the situation well is

$$D = 28000 + 9000d$$

This tells us that the damage D is a function of the distance d of the house from the fire station. Given a value for either of these variables, we can find the value of the other.

◇ **Example 6.5(a):** Determine the expected amount of damage from house fires that are 3.2 miles from the fire station, 4.2 miles from the fire station and 5.2 miles from the fire station.

Solution: For convenience, let's rewrite the equation using function notation, and in the slope-intercept form: $D(d) = 9000d + 28000$. Using this we have

$$D(3.2) = 9000(3.2) + 28000 = 56800, \qquad D(4.2) = 65800, \qquad D(5.2) = 74800$$

The damages for distances of 3.2, 4.2 and 5.2 miles from the fire station are $\$56,800$, $\$65,800$ and $\$74,800$.

Note that in the above example, for each additional mile away from the fire station, the amount of damage increased by $9000, which is the slope of the line with equation $D = 9000d + 28000$.

◇ **Example 6.5(b):** If a house fire caused $47,000 damage, how far would you expect that the fire might have been from the fire station, *to the nearest tenth of a mile*?

Solution: Here we are given a value for D and asked to find a value of d. We do this by substituting the given value of D into the equation and solving for d:

$$\begin{aligned} 47000 &= 9000d + 28000 \\ 19000 &= 9000d \\ 2.1 &= d \end{aligned}$$

We would expect the house that caught fire to be about 2.1 miles from the fire station.

◇ **Example 6.5(c):** How much damage might you expect if your house was right next door to the fire station?

Solution: A house that is right next door to the fire station is essentially a distance of zero miles away. We would then expect the damage to be

$$D(0) = 9000(0) + 28000 = 28000.$$

The damage to the house would be $\$28,000$.

Interpreting the Slope and Intercept of a Linear Model

There are a few important things we want to glean from the above examples.

- When we are given a mathematical relationship between two variables, if we know one we can find the other.
- Recall that slope is rise over run. In this case rise is damage, measured in units of dollars, and the run is distance, measured in miles. Therefore the slope is measured in $\frac{\text{dollars}}{\text{miles}}$, or dollars per mile. The slope of 9000 dollars per mile tells us that for each additional mile farther from the fire station that a house fire is, the amount of damage is expected to increase by $9000.
- The amount of damage expected for a house fire that is essentially right at the station is $28,000, which is the D-intercept for the equation.

In general we have the following.

> **Interpreting Slopes and Intercepts of Lines**
>
> When an "output" variable depends linearly on another "input" variable,
>
> - the slope has units of the output variable units over the input variable units, and it represents the amount of increase (or decrease, if it is negative) in the output variable for each one unit increase in the input variable,
> - the output variable intercept ("y"-intercept) is the value of the output variable when the value of the input variable is zero, and its units are the units of the output variable. *The intercept is not always meaningful.*

The first of the above two items illustrates what was pointed out after 3.2(a). As the distance increased by one mile from 3.2 miles to 4.2 miles the damage increased by $65800 - 56800 = 9000$ dollars, and when the distance increased again by a mile from 4.2 miles to 5.2 miles the damage again increased by $\$9000$.

When dealing with equations we often call the "input" variable (which is *ALWAYS* graphed on on the horizontal axis) the **independent variable**, and the "output" variable (which is always graphed on the vertical axis) we call the **dependent variable**. Using this language we can reword the items in the previous box as follows.

Slope and Intercept in Applications

For a linear model $y = mx + b$,

- the slope m tells us the amount of increase in the dependent variable for every one unit increase in the independent variable
- the vertical axis intercept tells us the value of the dependent variable when the independent variable is zero

Section 6.5 Exercises

To Solutions

1. The weight w (in grams) of a certain kind of lizard is related to the length l (in centimeters) of the lizard by the equation $w = 22l - 84$. This equation is based on statistical analysis of a bunch of lizards between 12 and 30 cm long.

   (a) Find the weight of a lizard that is 3 cm long. Why is this not reasonable? What is the problem here?

   (b) What is the w-intercept, and why does it have no meaning here?

   (c) What is the slope, with units, and what does it represent?

2. A salesperson earns \$800 per month, plus a 3% commission on all sales. Let P represent the salesperson's gross pay for a month, and let S be the amount of sales they make in a month. (Both are of course in dollars. Remember that to compute 3% of a quantity we multiply by the quantity by 0.03, the decimal equivalent of 3%.)

   (a) Find the pay for the salesperson when they have sales of \$50,000, and when they have sales of \$100,000.

   (b) Find the equation for pay as a function of sales, given that this is a linear relationship.

   (c) What is the slope of the line, and what does it represent?

   (d) What is the P-intercept of the line, and what does it represent?

3. The cost y (in dollars) of renting a car for one day and driving x miles is given by the equation $y = 0.24x + 30$. Of course this is the equation of a line. Explain what the slope and y-intercept of the line represent, *in terms of renting the car*.

4. Solve each equation for y. Give your answers in $y = mx + b$ form.

   (a) $-5x + 3y = 9$ (b) $5x + 2y = 20$

5. For each of the following equations, find at least five solution pairs, then draw a graph of the solution set for the equation.

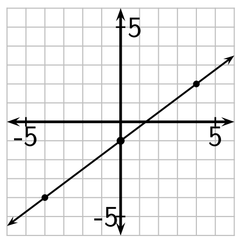
   (a) $x^2 + y = 1$ (b) $x - y^2 = 1$ (c) $y = x^3$

6. Find the intercepts for each of the following equations.

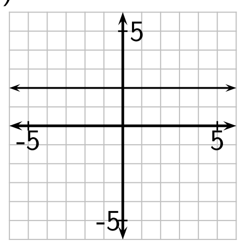
   (a) $4x - 5y = 20$ (b) $x^2 + y^2 = 25$ (c) $x = \sqrt{1 - y}$

7. Give the equation of each line.

   (a)



   (b)



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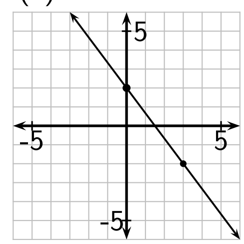


8. Sketch the graph of each line.

   (a) $y = -x + 2$ (b) $3x + 5y = 10$ (solve for y, then graph)

9. Find the equation of the line through the two points **algebraically**.

   (a) $(-4, -2),\ (-2, 4)$ (b) $(1, 7),\ (1, -1)$ (c) $(-1, -5),\ (1, 3)$

7 Systems of Two Linear Equations

7.1 Solving Systems of Two Linear Equations

7. (a) Solve a system of two linear equations by addition.
   (b) Solve a system of two linear equations by substitution.

Consider the two equations

$$\begin{aligned} 2x - 4y &= 18 \\ 3x + 5y &= 5 \end{aligned}$$

Taken together, we call them a **system of equations**. In particular, this is a system of *linear* equations. From past experience it should be clear that each equation has infinitely many solution pairs. When faced with a system of linear equations, our goal will be to *solve the system*. This means find a value of x and a value of y which, when taken together, make *BOTH* equations true. In this case you can easily verify that the values $x = 5,\ y = -2$ make both equations true. We say that the ordered pair $(5, -2)$ is a solution to the system of equations; it turns out it is the *only* solution.

You may be wondering how that solution pair could be obtained from the equations themselves. There are two methods for solving a system of equations, the **addition method** and the **substitution method**.

The Addition Method

We'll begin with the addition method, going from the easiest scenario to the most difficult (which still isn't too hard).

◇ **Example 7.1(a):** Solve the system $\begin{aligned} 3x - y &= 5 \\ 2x + y &= 15 \end{aligned}$.

Solution: The basic idea of the addition method is to add the two equations together so that one of the unknowns goes away. In this case, as shown below and to the left, nothing fancy need be done. The remaining unknown is then solved for and placed back into *either* equation to find the other unknown as shown below and to the right.

$$\begin{aligned} 3x - y &= 5 \\ 2x + y &= 15 \\ \hline 5x &= 20 \\ x &= 4 \end{aligned} \qquad \longrightarrow \qquad \begin{aligned} 3(4) - y &= 5 \\ 12 - y &= 5 \\ 12 &= y + 5 \\ 7 &= y \end{aligned}$$

The solution to the system is $(4, 7)$.

What made this work so smoothly is the $-y$ in the first equation and the $+y$ in the second; when we add the two equations, the sum of these is zero and y "has gone away." In the next example we see what to do in a slightly more difficult situation.

◇ **Example 7.1(b):** Solve the system $\begin{aligned} 3x+4y &= 13 \\ x+2y &= 7 \end{aligned}$.

Solution: We can see that if we just add the two equations together we get $4x+6y=20$, which doesn't help us find either of x or y. The trick here is to multiply the second equation by -3 so that the the first term of that equation becomes $-3x$, the opposite of the first term of the first equation. When we then add the two equations the x terms go away and we can solve for y:

$$\begin{aligned} 3x+4y &= 13 \\ x+2y &= 7 \end{aligned} \quad \begin{aligned} &\Longrightarrow \\ &\underset{\text{times } -3}{\Longrightarrow} \end{aligned} \quad \begin{aligned} 3x+4y &= 13 \\ -3x-6y &= -21 \\ \hline -2y &= -8 \\ y &= 4 \end{aligned} \quad \longrightarrow \quad \begin{aligned} x+2(4) &= 7 \\ x+8 &= 7 \\ x &= -1 \end{aligned}$$

The solution to the system of equations is $(-1,4)$. Note that we could have eliminated y first instead of x:

$$\begin{aligned} 3x+4y &= 13 \\ x+2y &= 7 \end{aligned} \quad \begin{aligned} &\Longrightarrow \\ &\underset{\text{times } -2}{\Longrightarrow} \end{aligned} \quad \begin{aligned} 3x+4y &= 13 \\ -2x-4y &= -14 \\ \hline x &= -1 \end{aligned} \quad \longrightarrow \quad \begin{aligned} -1+2y &= 7 \\ 2y &= 8 \\ y &= 4 \end{aligned}$$

The solution to the system of equations is of course the same, $(-1,4)$.

◇ **Example 7.1(c):** Solve the system $\begin{aligned} 2x-4y &= 18 \\ 3x+5y &= 5 \end{aligned}$.

Solution: This is the same system as last time, but we will show how we can eliminate y instead of x if we want. We'll multiply the second equation by -2 so that the the y term of that equation becomes $-4y$, the opposite of the y term of the first equation. When we then add the two equations the y terms go away and we can solve for x:

$$\begin{aligned} 2x-4y &= 18 \\ 3x+5y &= 5 \end{aligned} \quad \begin{aligned} &\overset{\text{times } 5}{\Longrightarrow} \\ &\underset{\text{times } 4}{\Longrightarrow} \end{aligned} \quad \begin{aligned} 10x-20y &= 90 \\ 12x+20y &= 20 \\ \hline 22x &= 110 \\ x &= 5 \end{aligned} \quad \longrightarrow \quad \begin{aligned} 2(5)-4y &= 18 \\ 10-4y &= 18 \\ -4y &= 8 \\ y &= -2 \end{aligned}$$

The solution to the system of equations is $(5,-2)$.

Let's summarize the steps for the addition method, which you've seen in the above examples.

The Addition Method

To solve a system of two linear equations by the addition method,

1) Multiply each equation by something as needed in order to make the coefficients of either x or y the same but opposite in sign.

2) Add the two equations and solve the resulting equation for whichever unknown remains.

3) Substitute that value into either original equation and solve for the other unknown.

The Substitution Method

We will now describe the substitution method, then give an example of how it works.

The Substitution Method

To solve a system of two linear equations by the substitution method,

1) Pick one of the equations in which the coefficient of one of the unknowns is either one or negative one. Solve that equation for that unknown.

2) Substitute the expression for that unknown into *the other* equation and solve for the unknown.

3) Substitute that value into the equation from (1), or into either original equation, and solve for the other unknown.

◇ **Example 7.1(d):** Solve the system of equations $\begin{aligned} x - 3y &= 6 \\ -2x + 5y &= -5 \end{aligned}$ using the substitution method.

Solution: Solving the first equation for x, we get $x = 3y + 6$. We now replace x in the second equation with $3y + 6$ and solve for y. Finally, that result for y can be substituted into $x = 3y + 6$ to find x:

$$\begin{aligned} -2(3y+6) + 5y &= -5 \\ -6y - 12 + 5y &= -5 \\ -y - 12 &= -5 \\ -y &= 7 \\ y &= -7 \end{aligned}$$

$$\begin{aligned} x - 3(-7) &= 6 \\ x + 21 &= 6 \\ x &= -15 \end{aligned}$$

The solution to the system of equations is $(-15, -7)$.

Section 7.1 Exercises To Solutions

1. Solve each of the following systems by the addition method.

(a) $\begin{aligned} 7x - 6y &= 13 \\ 6x - 5y &= 11 \end{aligned}$ (b) $\begin{aligned} 5x + 3y &= 7 \\ 3x - 5y &= -23 \end{aligned}$ (c) $\begin{aligned} 5x - 3y &= -11 \\ 7x + 6y &= -12 \end{aligned}$

2. Solve each of the following systems by the substitution method.

(a) $\begin{aligned} x - 3y &= 6 \\ -2x + 5y &= -5 \end{aligned}$ (b) $\begin{aligned} 2x - 3y &= -6 \\ -3x + y &= -5 \end{aligned}$ (c) $\begin{aligned} 4x - y &= 9 \\ 2x + 3y &= -27 \end{aligned}$

3. Consider the system of equations $\begin{aligned} 2x - 3y &= 4 \\ 4x + 5y &= 3 \end{aligned}$.

   (a) Solve for x by using the addition method to eliminate y. Your answer should be a fraction.

   (b) Ordinarily you would substitute your answer to (a) into either equation to find the other unknown. However, dealing with the fraction that you got for part (a) could be difficult and annoying. Instead, use the addition method again, but eliminate x to find y.

4. Consider the system of equations $\begin{aligned} \tfrac{1}{2}x - \tfrac{1}{3}y &= 2 \\ \tfrac{1}{4}x + \tfrac{2}{3}y &= 6 \end{aligned}$. The steps below indicate how to solve such a system of equations.

   (a) Multiply both sides of the first equation by the least common denominator to "kill off" all fractions.

   (b) Repeat for the second equation.

   (c) You now have a new system of equations without fractional coefficients. Solve that system by the addition method.

5. Attempt to solve the following two systems of equations. With each, something will go wrong at some point. This illustrates that there is a little more to this subject than we have seen so far! We will find out soon what is going on here.

(a) $\begin{aligned} 2x - 5y &= 3 \\ -4x + 10y &= 1 \end{aligned}$ (b) $\begin{aligned} 2x - 5y &= 3 \\ -4x + 10y &= -6 \end{aligned}$

6. Solve each system of equations by the addition method.

(a) $\begin{aligned} 3x - 5y &= -2 \\ 2x - 3y &= 1 \end{aligned}$ (b) $\begin{aligned} 3x - 5y &= 11 \\ 2x - 6y &= 2 \end{aligned}$ (c) $\begin{aligned} 7x - 6y &= 13 \\ 6x - 5y &= 11 \end{aligned}$

7. The length of a rectangle is three more than twice the width. The area is 44. Find the length and width. **Write an equation that can be used to solve this problem, and solve the equation.**

8. The sales tax in a particular city is 6.5%. You pay $41.37 for an electric can opener, including tax. What was the price of the can opener? **Write an equation that can be used to solve this problem, and solve the equation.**

9. Give the slope of each line whose equation is given.

   (a) $y = -\frac{4}{7}x + 2$ (b) $x = 2$ (c) $3x + 5y = 2$ (d) $y = -1$

10. The equation $F = \frac{9}{5}C + 32$ gives the Fahrenheit temperature F corresponding to a given Celsius temperature C. This equation describes a line, with C playing the role of x and F playing the role of y.

    (a) What is the F-intercept of the line, and what does it tell us?

    (b) What is the slope of the line, and what does it tell us?

7.2 More on Systems of Linear Equations

> 7. (c) Recognize when a system of two linear equations has no solution or infinitely many solutions.
>   (d) Solve a system of two linear equations by graphing.

Let's begin with two examples:

◇ **Example 7.2(a):** Solve the system $\begin{array}{rcl} 2x-5y &=& 3 \\ -4x+10y &=& 1 \end{array}$.

Solution:

$$\begin{array}{rclcrcl} 2x-\ \ 5y &=& 3 & \stackrel{\text{times } 2}{\Longrightarrow} & 4x-10y &=& 6 \\ -4x+10y &=& 1 & \Longrightarrow & -4x+10y &=& 1 \\ \hline & & & & 0 &=& 7 \end{array}$$

We are not able to find a solution as we did before.

The next example is just like the last, except that we will change the right side of the second equation to -6.

◇ **Example 7.2(b):** Solve the system $\begin{array}{rcl} 2x-5y &=& 3 \\ -4x+10y &=& -6 \end{array}$.

Solution:

$$\begin{array}{rclcrcl} 2x-\ \ 5y &=& 3 & \stackrel{\text{times } 2}{\Longrightarrow} & 4x-10y &=& 6 \\ -4x+10y &=& -6 & \Longrightarrow & -4x+10y &=& -6 \\ \hline & & & & 0 &=& 0 \end{array}$$

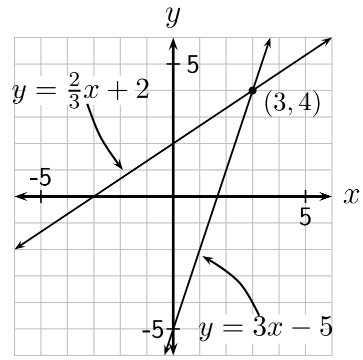
We are still not able to find a solution.

You might recognize these as being the systems in Exercise 5 of the previous section, and clearly something has "gone wrong" here. Before going into the above examples further, lets consider the system of equations $\begin{array}{rcl} 2x-3y &=& -6 \\ 3x-y &=& 5 \end{array}$ from Exercise 1(b) of the previous section, which you should have found to have the solution $(3,4)$. If we solve each of those equations for y we get the equations

$$y=\tfrac{2}{3}x+2 \qquad \text{and} \qquad y=3x-5$$

If we graph the two equations on the same graph, we get the result shown at the top of the next page.

Note that the two lines cross at the point $(3,4)$, which is the solution to the system. This is because every point on a line is a solution to the equation of the line, so the point where the two lines cross is a solution to both equations!



Now consider the system $\begin{aligned} 2x - 5y &= 3 \\ -4x + 10y &= 1 \end{aligned}$ from Example 7.2(a) of the previous section. When we tried using the addition method to solve this, we ended up with $0 = 7$, which is clearly not true! This is trying to tell us something; let's see what it is. If we solve each of the two equations for y we get

$$y = \tfrac{2}{5}x - \tfrac{3}{5} \qquad \text{and} \qquad y = \tfrac{2}{5}x + \tfrac{1}{10}$$

It would be fairly hard to graph these two equations, but we can see from their equations that they have the same slope, but different y-intercepts. This means that *they are parallel lines*, so they don't intersect. Therefore they have no point in common, so the system has no solution.

In the case of the other system from Exercise 5, $\begin{aligned} 2x - 5y &= 3 \\ -4x + 10y &= -6 \end{aligned}$, when trying to solve the equation we arrived at $0 = 0$. This is also trying to tell us something, but it is not the same thing as before. This time, if we solve each of the two equations for y we get

$$y = \tfrac{2}{5}x - \tfrac{3}{5} \qquad \text{and} \qquad y = \tfrac{2}{5}x - \tfrac{3}{5}$$

In this case the two equations are for the same line, so every point on that line is a solution to both equations. Thus there are infinitely many solutions.

In summary, think about taking two infinitely long pieces of dry spaghetti and throwing them randomly on the ground. They could land in any one of three ways: they could cross each other in one place, they could be parallel, or they could land exactly on top of each other. These illustrate the three possibilities when solving a system of two linear equations: the system can have one solution, no solution, or infinitely many solutions. To find out which, we simply try solving the system with the addition method or the substitution method. If everything goes smoothly you will arrive at one solution. If something funny happens, it means there is either no solution or infinitely many solutions. Let's summarize this, along with the graphical interpretation of each.

Systems of Two Linear Equations

- If $a = b$ is obtained when attempting to solve a system of equations, where a and b are different numbers, then the system has no solution. The graphs of the two equations are parallel lines.
- If $a = a$ is obtained, for some number a, when attempting to solve a system of equations, then the system has infinitely many solutions. The graphs of the two equations are the same line.
- If a system of two linear equations has a solution, it is the point where the graphs of the two equations intersect.

Section 7.2 Exercises

To Solutions

1. *Use the addition method* to solve each of the following systems of equations if possible. If it is not possible, tell whether the system has no solution or infinitely many solutions.

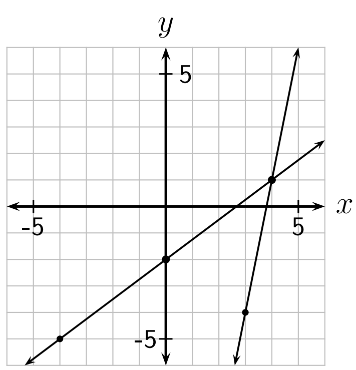
   (a) $\begin{aligned} -4x + 6y &= 5 \\ 6x - 9y &= 7 \end{aligned}$ (b) $\begin{aligned} 3x - 4y &= 18 \\ 6x + 4y &= 0 \end{aligned}$ (c) $\begin{aligned} 2x - 3y &= 5 \\ -4x + 6y &= 8 \end{aligned}$

2. *Use the substitution method* to solve each of the following systems of equations if possible. If it is not possible, tell whether the system has no solution or infinitely many solutions.

   (a) $\begin{aligned} 10x - 2y &= 14 \\ 5x - y &= 7 \end{aligned}$ (b) $\begin{aligned} x + 3y &= 5 \\ 4x + 12y &= 20 \end{aligned}$ (c) $\begin{aligned} 3x + 2y &= 3 \\ x - 5y &= -16 \end{aligned}$

3. Each graph below shows the graphs of two linear equations. For each one, give the solution to the system if there is one. If there is no solution or if there are infinitely many solutions, say so.

   (a)



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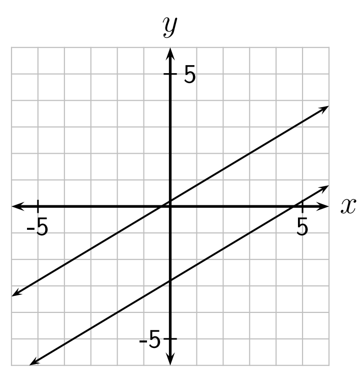

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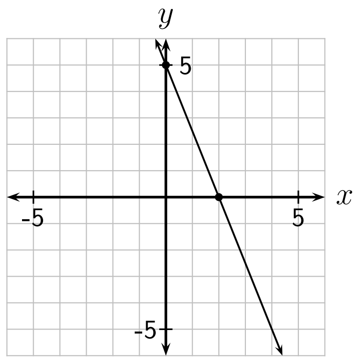

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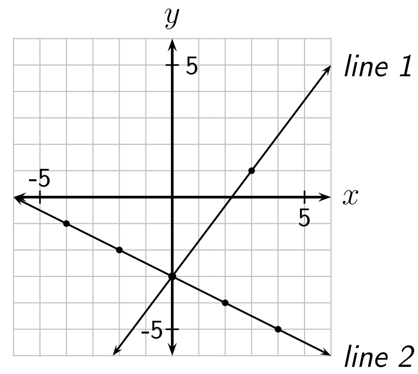


4. Find the intercepts for each of the following equations.

   (a) $x^2 + y = 1$ (b) $x - y^2 = 1$ (c) $y = x^3$

5. (a) Give the slope of a line that is parallel to *line 1* from Exercise 3(d).

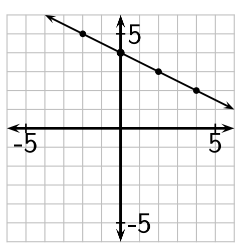
   (b) Give the slope of a line that is perpendicular to *line 2* from Exercise 3(d).

6. Sketch the graph of each line.

   (a) $y = -3x + 2$ (b) $-x + y = 4$ (c) $y = 2$

7. Give the equation of each line.

   (a)

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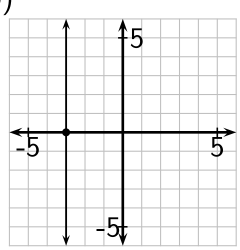
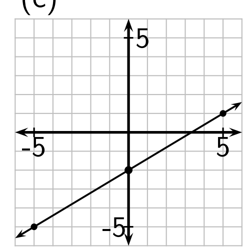
   

   (c)

8. Find the equation of the line through the two points **algebraically**.

   (a) $(1,7),(3,11)$ (b) $(-6,-2),\ (5,-3)$ (c) $(-2,-5),\ (2,5)$

9. We again consider the manufacture of Widgets by the Acme Company. The costs for one week of producing Widgets is given by the equation $C = 7x + 5000$, where C is the costs, in dollars, and x is the number of Widgets produced in a week. This equation is clearly linear.

   (a) What is the C-intercept of the line, and what does it represent?

   (b) What is the slope of the line, and what does it represent?

   (c) If they make 1,491 Widgets in one week, what is their total cost? What is the cost for each individual Widget made that week? (The answer to this second question should *NOT* be the same as your answer to (b).)

8 Functions

8.1 Introduction To Functions

8. (a) Evaluate a function for a given numerical or algebraic value.
   (b) Find all numerical ("input") values for which a function takes a certain ("output") value.

◇ **Example 8.1(a):** Find some solutions to $y = \frac{2}{3}x - 5$ and sketch the graph of the equation.

Solution: Let x be some multiples of 3:

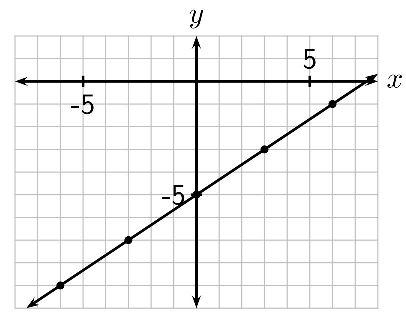
$x = 0: \quad y = \frac{2}{3}(0) - 5 = -5$

$x = 3: \quad y = \frac{2}{3}(3) - 5 = -3$

$x = 6: \quad y = \frac{2}{3}(6) - 5 = -1$

$x = -3: \quad y = \frac{2}{3}(-3) - 5 = -7$

$x = -6: \quad y = \frac{2}{3}(-6) - 5 = -9$



◇ **Example 8.1(b):** Find some solutions to $y = x^2 - 3x + 1$ and sketch the graph of the equation.

$x = 0: \quad y = 0^2 - 3(0) + 1 = 1$

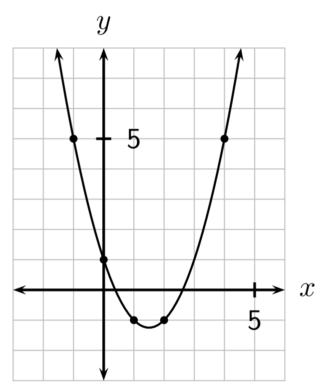
$x = 1: \quad y = 1^2 - 3(1) + 1 = -1$

$x = 2: \quad y = 2^2 - 3(2) + 1 = -1$

$x = 3: \quad y = 3^2 - 3(3) + 1 = 1$

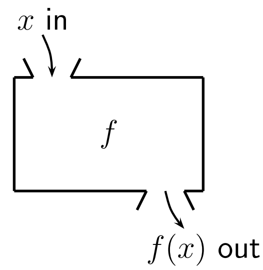
$x = 4: \quad y = 4^2 - 3(4) + 1 = 5$

$x = -1: \quad y = (-1)^2 - 3(-1) + 1 = 5$



Notice that in both of the above examples we chose values for x, put them into the equations, and 'got out' values for y. A mathematical machine that takes in an 'input' and gives us an 'output" is called a function; the equations $y = \frac{2}{3}x - 5$ and $y = x^2 - 3x + 1$ are functions.

There is an alternative notation for functions that is very efficient once a person is used to it. Suppose we name the function $y = \frac{2}{3}x - 5$ with the letter f. We then think of f as a sort of 'machine' that we feed numbers into and get numbers out of. We denote by $f(3)$ the value that comes out when 3 is fed into the machine, so $f(3) = -3$. This process can be shown pictorially in the manner seen to the right. f is the name of the machine, with input x and output $f(x)$. The notation $f(x)$ represents the result when 3 is put into the machine. In general, $f(x) = \frac{2}{3}x - 5$ for any value of x.



◇ **Example 8.1(c):** For $f(x) = \frac{2}{3}x - 5$, find $f(-6)$ and $f(1)$.

Solution: When $x = -6$, $\frac{2}{3}(-6) - 5 = -4 - 5 = -9$, so $f(-6) = -9$. We can think of replacing every x in $f(x) = \frac{2}{3}x - 5$ with -6 to get

$$f(-6) = \frac{2}{3}(-6) - 5 = -4 - 5 = -9.$$

Similarly,

$$f(1) = \frac{2}{3}(1) - 5 = \frac{2}{3} - \frac{15}{3} = -\frac{13}{3}.$$

◇ **Example 8.1(d):** Let $g(x) = x^2 - 3x + 1$, and find $g(4)$ and $g(-1)$.

$$g(4) = (4)^2 - 3(4) + 1 = 16 - 12 + 1 = 5$$
$$g(-1) = (-1)^2 - 3(-1) + 1 = 1 = 3 = 1 = 5$$

What we did in the previous two examples is find values of the function for given *numerical* inputs. We can also find values of the function for inputs that contain unknown values. This is shown in the following two examples.

◇ **Example 8.1(e):** For $g(x) = x^2 - 3x + 1$, find $g(a)$.

Solution: To find $g(a)$ we simply replace x with a:

$$g(a) = (a)^2 - 3(a) + 1 = a^2 - 3a + 1$$

◇ **Example 8.1(f):** For $g(x) = x^2 - 3x + 1$, find and simplify $g(a - 2)$.

Solution: To find $g(a - 2)$ we replace x with $a - 2$. To simplify, we then 'FOIL' $(a - 2)^2$, distribute and combine like terms:

$$\begin{aligned} g(a-2) &= (a-2)^2 - 3(a-2) + 1 \\ &= (a-2)(a-2) - 3a + 6 + 1 \\ &= a^2 - 4a + 4 - 3a + 7 \\ &= a^2 - 7a + 11 \end{aligned}$$

Therefore $g(a - 2) = a^2 - 7a + 11$.

Here is an important fact: *We will usually use x for the "input" value of a function, but there is no reason that we couldn't use some other letter instead.* For example, the function g could be written as

$$g(x) = 2x^2 + x - 5 \quad \text{or} \quad g(t) = 2t^2 + t - 5 \quad \text{or} \quad g(a) = 2a^2 + a - 5$$

The letter used has no impact on how we compute the output for a given input.

So far we have found outputs of functions for given inputs. Sometimes we would like to find an input that gives a particular output. For example, we might wish to find a value of x that gives an output of 7 for the function $f(x) = 3x - 8$; that is, we wish to find a value of x such that $f(x) = 7$. To do this we simply replace $f(x)$ with 7 and solve:

$$\begin{aligned} 7 &= 3x - 8 \\ 15 &= 3x \\ 5 &= x \end{aligned}$$

Thus $f(5) = 7$, and we have found the desired value of x.

◇ **Example 8.1(g):** Let $f(x) = 2x^2 - x$. Find all values of x for which $f(x) = 3$.

Solution: Note that $\underline{x}$ is not three, $\underline{f(x)}$ is! (One clue that we shouldn't put 3 in for x is that we are asked to find x.) We replace $f(x)$ with 3 and find x:

$$\begin{aligned} 3 &= 2x^2 - x \\ 0 &= 2x^2 - x - 3 \\ 0 &= (2x - 3)(x + 1) \end{aligned} \quad \longrightarrow \quad x = -1 \quad \text{or} \quad \begin{aligned} 2x - 3 &= 0 \\ 2x &= 3 \\ x &= \tfrac{3}{2} \end{aligned}$$

$f(x) = 3$ when $x = -1$ or $x = \frac{3}{2}$.

Section 8.1 Exercises

To Solutions

1. For $f(x) = x^2 - 3x$, find each of the following, writing each of your answers in the form $f(5) = number$.

   (a) $f(5)$ (b) $f(-1)$ (c) $f(-3)$ (d) $f(0)$

2. Still letting $f(x) = x^2 - 3x$, find $f(\frac{1}{2})$, giving your answer in fraction form.

3. We don't always use the letter f to name a function; sometimes we use g, h or another letter. Consider the function $g(x) = \sqrt{3 - x}$.

   (a) Find $g(2)$ (b) Find $g(-6)$

   (c) Find $g(-5)$, giving your answer in simplified square root form.

(d) Find $g(-2)$, using your calculator. Your answer will be in decimal form and you will have to round it somewhere - round to the hundredth's place. This is two places past the decimal - the first place past the decimal is the tenth's, the next place is the hundredth's, the next is the thousandth's, and so on.

4. Let the function h be given by $h(x) = \sqrt{25 - x^2}$.

   (a) Find $h(-4)$.

   (b) Find $h(2)$, giving your answer in decimal form rounded to the thousandth's place.

   (c) Why can't we find $h(7)$?

   (d) For what values of x *CAN* we find $h(x)$? Be sure to consider negative values as well as positive.

5. Let $f(x) = x^2 - 3x$.

   (a) Find $f(s)$. (b) Find and simplify $f(s+1)$.
   (c) Find all values of x for which $f(x) = 10$.

6. For the function $h(x) = \frac{2}{3}x - 5$, find and simplify

   (a) $h(a-3)$ (b) $h(a+2)$

7. Still for $h(x) = \frac{2}{3}x - 5$, find all values of x for which $h(x) = 2$. Give your answer(s) in exact form - no decimals!

8. For the function $g(x) = \sqrt{3 - x}$, find and simplify $g(s+1)$. You will not be able to get rid of the square root.

9. Consider again the function $h(x) = \sqrt{25 - x^2}$.

   (a) Find and simplify $h(a-2)$.

   (b) Find all values of x for which $h(x) = 2$. Give your answer(s) in exact form - no decimals!

10. Solve each of the following systems by the substitution method.

    (a) $\begin{aligned} x + y &= 3 \\ 2x + 3y &= -4 \end{aligned}$ (b) $\begin{aligned} 2x + y &= 13 \\ -5x + 3y &= 6 \end{aligned}$ (c) $\begin{aligned} 2x - 3y &= -6 \\ 3x - y &= 5 \end{aligned}$

11. Solve each of the systems from Exercise 10 using the addition method.

8.2 Compositions Of Functions

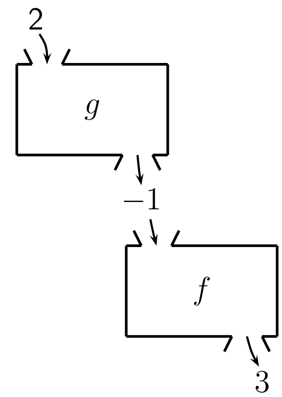
> 8. (c) Evaluate the composition of two functions for a given "input" value; determine a simplified composition function for two given functions.

Recall that we can think of a function as a "machine" that takes in a number, works with it, and outputs another number that depends on the input number. For example, when we input the number five to the function $f(x) = x^2 - 2x$ we get out

$$f(5) = 5^2 - 2(5) = 25 - 10 = 15.$$

Similarly, if we input $x = -1$ we get

$$f(-1) = (-1)^2 - 2(-1) = 1 + 2 = 3.$$



Now suppose that we have a second function $g(x) = x - 3$ and we wish to give a number to g as input, then use the output as an input for f. The picture you can think of is shown to the right. If we put two into g we get $g(2) = 2 - 3 = -1$ out; we then put that result into f to get three out, as demonstrated at the end of the previous paragraph.

For now, the notation that we will use for what we did is $f[g(2)]$. To think about this, start with the number inside the parentheses, then apply the function that is closest to the number; in this case it is g. Take the result of that computation and apply the function f to that number.

◇ **Example 8.2(a):** For the same two functions $f(x) = x^2 - 2x$ and $g(x) = x - 3$, find $g[f(4)]$.

Solution: First $f(4) = 4^2 - 2(4) = 16 - 8 = 8$. Now we apply g to the result of 8 to get $g(8) = 8 - 3 = 5$, so $g[f(4)] = 5$. We will often combine both of these computation into one as follows:

$$\begin{aligned} g[f(4)] &= g[4^2 - 2(4)] \\ &= g[16 - 8] \\ &= g(8) \\ &= 8 - 3 \\ &= 5 \end{aligned}$$

Think carefully about the above example. $f(4)$ becomes $4^2 - 2(4)$ because of how f is defined, and that simplifies to 8. Then g acts on 8, resulting in $8 - 3 = 5$.

◇ **Example 8.2(b):** Again using $f(x) = x^2 - 2x$ and $g(x) = x - 3$, find $f[g(4)]$.

$$\begin{aligned} f[g(4)] &= f[4-3] \\ &= f(1) \\ &= 1^2 - 2(1) \\ &= 1 - 2 \\ &= -1 \end{aligned}$$

Note that $f[g(4)] \neq g[f(4)]$; this shows that *when we apply two functions one after the other, the result depends on the order in which the functions are applied.*

In Section 8.1 we saw how to apply functions to algebraic quantities. With a little thought we can see that for $f(x) = x^2 - 2x$ and $g(x) = x - 3$,

$$\begin{aligned} f[g(x)] &= f[x-3] \\ &= (x-3)^2 - 2(x-3) \\ &= x^2 - 6x + 9 - 2x + 6 \\ &= x^2 - 8x + 15 \end{aligned}$$

This final expression $x^2 - 8x + 15$ can thought of as a function in its own right. It was built by putting f and g together in the way that we have been doing, with g acting first, then f. We call this function the **composition** of f and g, and we denote it by $f \circ g$. So

$$(f \circ g)(x) = x^2 - 8x + 15$$

◇ **Example 8.2(c):** For $g(x) = 2x - 3$ and $h(x) = x^2 - 5x + 2$, find $(g \circ h)(x)$.

$$\begin{aligned} (g \circ h)(x) &= g[h(x)] \\ &= g[x^2 - 5x + 2] \\ &= 2(x^2 - 5x + 2) - 3 \\ &= 2x^2 - 10x + 1 \end{aligned}$$

Therefore $(g \circ h)(x) = 2x^2 - 10x + 1$.

Section 8.2 Exercises To Solutions

1. Consider again the functions $f(x) = x^2 - 2x$ and $g(x) = x - 3$.

   (a) Find $g[f(1)]$. (b) Find $g[f(-3)]$.

2. Once again consider the functions $f(x) = x^2 - 2x$ and $g(x) = x - 3$.

   (a) Use the same kind of process to find $f[g(1)]$, noting that **the order in which** f **and** g **are applied has been reversed**.

(b) Find $f[g(-3)]$.

3. For the functions $f(x) = x^2 - 2x$ and $g(x) = x - 3$ we found that $(f \circ g)(x) = x^2 - 8x + 15$. Find the value of $(f \circ g)(-3)$ and compare the result with the answer to Exercise 2(b).

4. Continue to let $f(x) = x^2 - 2x$ and $g(x) = x - 3$

   (a) Find the other composition function $(g \circ f)(x)$, noting that it is just $g[f(x)]$.

   (b) Find $(g \circ f)(1)$ and make sure it is the same as the answer to Exercise 1(a).

5. Consider the functions $f(x) = x^2 + 3x$ and $g(x) = x - 7$.

   (a) Find $f[g(2)]$.
   (b) Find $(g \circ f)(2)$.
   (c) Find $(g \circ f)(x)$.
   (d) Find $(f \circ g)(x)$.

6. (a) Give the equation of the horizontal line graphed in Exercise 8(a).

   (b) Give the equation of the vertical line graphed in Exercise 8(a).

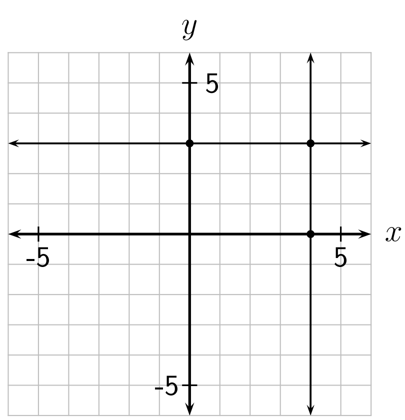
   (c) The lower of the two lines graphed in Exercise 8(b) passes through the point $(-6, 3)$. Give the equation of the line.

7. Solve each of the following systems of equations if possible. If it is not possible, tell whether the system has no solution or infinitely many solutions.

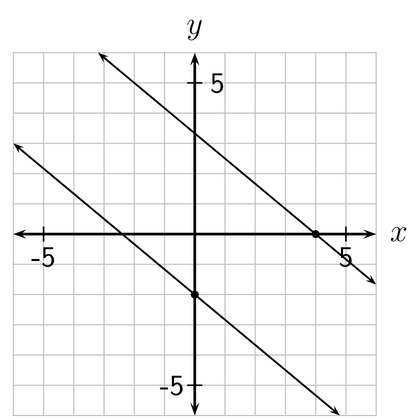
   (a) $\begin{aligned} 5x + y &= 12 \\ 7x - 2y &= 10 \end{aligned}$
   (b) $\begin{aligned} 3x - 9y &= 12 \\ -2x + 6y &= -8 \end{aligned}$
   (c) $\begin{aligned} x - 3y &= -2 \\ 5x - 15y &= 3 \end{aligned}$

8. Each graph below shows the graphs of two linear equations. For each one, give the solution to the system if there is one. If there is no solution or if there are infinitely many solutions, say so.

(a)



(b)



9. For the function $h(x) = 2x - 7$, find all values of x such that $h(x) = 12$.

10. Let $g(x) = x - \sqrt{3x+1}$. Find all values of x for which $g(x) = 3$.

11. Let $f(x) = 2x^2 - 2x$.

    (a) Find $f(-3)$.

    (b) Find all values of x for which $f(x) = 60$.

    (c) Find all values of x for which $f(x) = 1$. *You will need the quadratic formula for this.*

8.3 Sets of Numbers

8. (d) Describe sets of numbers using inequalities or interval notation.

Consider the function $h(x) = \sqrt{25 - x^2}$, from Exercise 4 of Section 8.1. In part (d) of that exercise you were to find the values of x for which the function can be evaluated. For example,

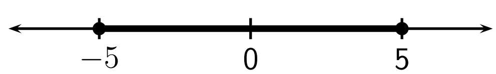
$$h(-4) = \sqrt{25-16} = \sqrt{9} = 3, \qquad h(2) = \sqrt{25-4} = \sqrt{21}, \qquad h(7) = \sqrt{25-49} = \sqrt{-24}$$

Clearly the function can be evaluated for $x = -4$, and for $x = 2$ as well; even though $\sqrt{21}$ is not a "nice" number, we could use our calculators to find an approximate decimal value for it. On the other hand, we cannot take the square root of a negative number, so h cannot be evaluated for $x = 7$.

After a bit of thought, one should realize that h can be evaluated for any number between -5 and 5, including both of those. Taken together, all of those numbers constitute something we call a **set** of numbers. Clearly there are infinitely many of them because we can use numbers that are not whole numbers, as long as whatever we get under the square root is positive. So far we have described the set of values that the function can be evaluated for verbally, using words: 'all the numbers between -5 and 5, including both of those.' We will now see a way top describe this set graphically, and two ways to describe the set symbolically.

First let us recall inequalities. When we write $a < b$ for two numbers a and b, it means that the number a is less than the number b. $b > a$ says the same thing, just turned around. $a \leq b$ means that a is less than, or possibly equal to, b. This, of course, is equivalent to $b \geq a$. Using this notation, we know that we want x to be less than or equal to 5, so we write $x \leq 5$. At the same time we want $x \geq -5$, which is equivalent to $-5 \leq x$. We can combine the two statements into one inequality: $-5 \leq x \leq 5$. This is one of our two ways to describe the set symbolically.

The statement $-5 \leq x \leq 5$ then describes a set of numbers, the numbers that are allowed for x when considering the function $h(x) = \sqrt{25 - x^2}$. This set can be described graphically by shading it in on a number line:



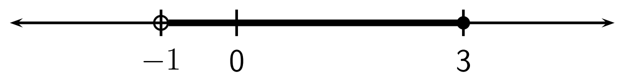
The solid dots at either end of the set indicate that those numbers are included. If we wanted to show all the numbers less than three, we would do it like this:

In this case the small circle at three indicates that three is not included, but every number up to three is.

Another way to describe a set symbolically is to use something called **interval notation**. The interval notation for the set $-5 \leq x \leq 5$ is formed by first writing the number at the smaller end of the set, then a comma, then the number at the larger end of the set: $-5, 5$ Then, if the number at the smaller end is to be included we put a square bracket to its left: $[-5, 5$. *If the* *number was not to be included we would use a parenthesis*. We then do the same thing for the larger number to get the final result $[-5, 5]$.

◇ **Example 8.3(a):** Give the interval notation for the set of all numbers greater than negative one and less than or equal to three.

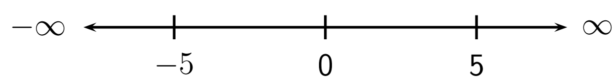
Solution: For those of us who process visually, it might be helpful to graph this set on the number line:



The two endpoints of our interval are -1 and 3, so we put them in that order, separated by a comma: $-1, 3$ Next, because the set is for numbers *greater* than -1, we enclose the -1 of our pair with a parenthesis: $(-1, 3$. Finally, because the number 3 *is* in the set, we put a bracket on the right side of our pair: $(-1, 3]$.

Consider the set with this graph:

The smallest number in this set is clearly -5, but there is no largest number in the set. To deal with this kind of thing we invent two symbols, ∞ and $-\infty$, **infinity** and **negative infinity**. These are not really numbers, but can be thought of as "places" beyond the two ends of the number line:



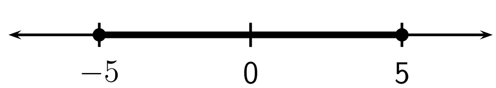
The symbols $-\infty$ and ∞ allow us to use interval notation to describe sets like the one shown above, all numbers greater than or equal to negative five. The interval notation for this set is $[-5, \infty)$. *We always use a parenthesis with either of the infinities, since they are not numbers that can be included in a set.*

◇ **Example 8.3(b):** Give the interval notation for the set of all numbers less than four.

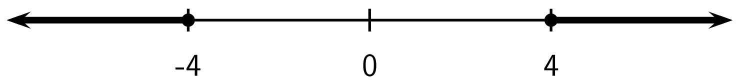
Solution: The graph of this set on the number line is

The interval notation for this set is $(-\infty, 4)$.

Returning again to our function $h(x) = \sqrt{25 - x^2}$, the set of numbers that x is allowed to be can be shown graphically by



If instead we consider the function $f(x) = \sqrt{x^2 - 16}$, a little thought would show us that the set of numbers for which f can be computed is the following:



The right hand part of this graph can be described using interval notation as $[4, \infty)$, and the left hand side as $(-\infty, -4]$. The whole set, consisting of the two parts together, is then described by the notation $(-\infty, -4] \cup [4, \infty)$. The symbol $\cup$ is "union," which means the two sets put together.

Section 8.3 Exercises To Solutions

1. For each part below, a set is described in words. Give the same set using inequalities. Sometimes you will need two inequalities combined, like you just saw, sometimes one will be enough. Then give the same set with a graph of a number line.

   (a) All numbers x less than ten.

   (b) All numbers t greater than or equal to zero.

   (c) All numbers x between -3 and 3, not including either.

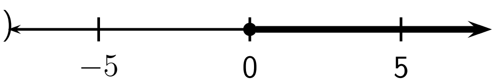
   (d) All numbers y greater than -2 and less than or equal to 4.

2. A set can be described with (1) words, (2) inequalities, (3) a graph on a number line, (4) interval notation. For each of the following, a set of numbers is described in one of these four ways. Describe the set in each of the other three ways. When using words, make it clear whether "boundary" numbers are included. When using interval notation and more than one interval is needed, use union, $\cup$.

   (a) All numbers greater than five.

   (b) $(-\infty, 3]$

   (c) $x < -4$ or $x \geq 1$

   (d) 

   (e) $(-\infty, -7) \cup [1, \infty)$

   (f) $x < 17$

   (g) All numbers greater than -10 and less than or equal to -3.

3. Find the intercepts for each of the following equations algebraically.

   (a) $4x - 5y = 20$ (b) $x^2 + y^2 = 25$ (c) $x = \sqrt{1 - y}$

4. Give the slope of a line that is perpendicular to the line with equation $5x - 2y = 7$.

5. Find the equation of the line through the two points **algebraically**.

   (a) $(-8, -7)$, $(-4, -6)$ (b) $(2, -1)$, $(6, -4)$ (c) $(-2, 4)$, $(1, 4)$

6. Solve each of the following systems of equations if possible. If it is not possible, tell whether the system has no solution or infinitely many solutions.

(a) $\begin{aligned} 4x - 5y &= 20 \\ -3x + 6y &= -15 \end{aligned}$ (b) $\begin{aligned} 3x - 9y &= -3 \\ 4x - 12y &= -4 \end{aligned}$ (c) $\begin{aligned} -6x + 3y &= -1 \\ 10x - 5y &= 2 \end{aligned}$

7. For $g(x) = 2x - 3$ and $h(x) = x^2 - 5x + 2$, find $(h \circ g)(x)$.

8. Consider the functions $f(x) = 3 - x^2$ and $g(x) = 2x + 1$.

(a) Find $f[g(-1)]$.

(b) Find $(g \circ f)(4)$.

(c) Find $(g \circ f)(x)$.

(d) Find $(f \circ g)(x)$.

8.4 Domains of Functions

8. (e) Determine the domains of rational and radical functions.

In the previous section we discussed at some length the fact that the only values for which we can compute the function $h(x) = \sqrt{25 - x^2}$ are the numbers in the interval $[-5, 5]$. The set of numbers for which a function can be computed is called the **domain** of the function. When finding the domain of a function we need to make sure that

1. We don't try to compute the square root of a negative number.
2. We don't have zero in the denominator (bottom) of a fraction.

◇ **Example 8.4(a):** Find the domain of $f(x) = \dfrac{x-1}{x+4}$.

If $x = -4$ we would get zero in the denominator, which can't be allowed. The domain is all real numbers *except* -4. We usually just say the domain is $x \neq -4$; because the only number we say x can't be is -4, this implies that x can be any other number.

◇ **Example 8.4(b):** Find the domain of $g(x) = \sqrt{3 - x}$.

Solution: Here there is a problem if x is greater than three, so we must require that x is less than or equal to three. The domain is then $x \leq 3$ or, using interval notation, $(-\infty, 3]$.

◇ **Example 8.4(c):** Find the domain of $h(x) = \dfrac{1}{\sqrt{3-x}}$.

Solution: There is no problem computing $\sqrt{3-x}$ as long as x is less than or equal to three. However, we can't allow x to be three, because that would cause a zero in the denominator. The domain is therefore $(-\infty, 3)$.

Section 8.4 Exercises To Solutions

1. Give the domain of each of the following functions.

(a) $y = \dfrac{1}{x+4}$ (b) $g(x) = x^2 - 5x + 6$ (c) $h(x) = \dfrac{1}{x^2 - 5x + 6}$

(d) $y = \sqrt{x+5}$ (e) $f(t) = \dfrac{1}{\sqrt{t-3}}$ (f) $g(x) = \dfrac{1}{x^2 - 16}$

2. Let $h(x) = x^2 - 2x - 3$.

   (a) Find all values of x such that $h(x) = 5$.

   (b) Find all values of x such that $h(x) = -3$

   (c) Find $h(-3)$.

   (d) Find all values of x such that $h(x) = 2$. Give your answers in both in decimal form to the nearest hundredth and simplified exact form.

8.5 Graphs of Functions

> 8. (f) Graph quadratic, polynomial and simple root and rational functions.

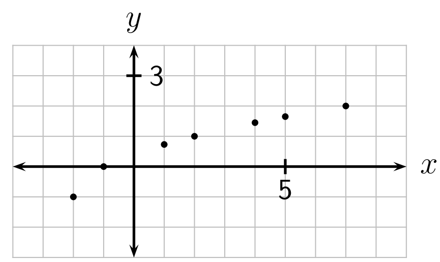
Look back at Examples 8.1(a) and 8.1(b). There we found some solutions to $y = \frac{2}{3}x - 5$ and $y = x^2 - 3x + 1$, then graphed two equations. Later we saw how to think of these two equations a little differently, as functions $f(x) = \frac{2}{3}x - 5$ and $g(x) = x^2 - 3x + 1$. This shows us that functions have graphs; for a function f we simply evaluate the function for a number of x values, and plot the corresponding $f(x)$ values as y's corresponding to those x's.

◇ **Example 8.5(a):** Graph the function $h(x) = \sqrt{x+2} - 1$.

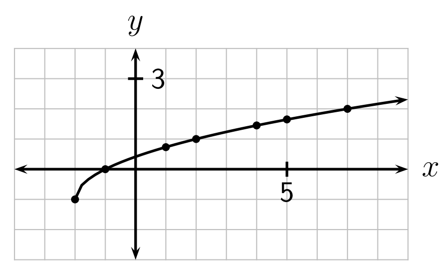
Solution: First we note the domain of this function. x *can* be negative, like $x = -1$, but it can't be less than -2 or we'll get a negative under the square root. Therefore the domain is $x \geq 2$, or $[-2, \infty)$. We can let x take a few values for which it is easy to calculate the square root, as shown in the table below and to the left. We can even use a calculator to evaluate $h(x)$ for other values of x, shown in the center below.

x	$h(x)$
-2	-1
-1	0
2	1
7	2

x	$h(x)$
1	0.4
4	1.45
5	1.65



The graph above and to the right shows the points found and recorded in the tables. We now need to connect them to form the graph of the function. Before doing so, lets return again to the domain of the function, $[-2, \infty)$. This tells us that the smallest value of x that we can have is -2, so the graph should not extend to the left of -2. The domain also tells us that x can be as large as we want, so the graph extends forever in the positive x direction. The final graph is shown to the right.

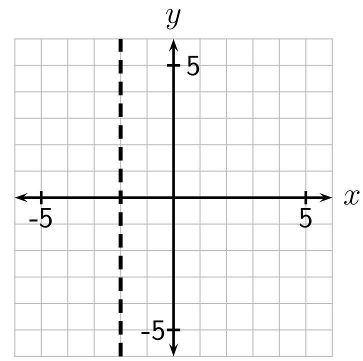


Let's reiterate: When graphing something like $h(x) = \sqrt{x+2} - 1$, the process is exactly the same as graphing $y = \sqrt{x+2} - 1$. The only difference is that the y values now represent the function values $h(x)$.

Now we'll graph a kind of function that you didn't see until this chapter, called a **rational function**.

◇ **Example 8.5(b):** Graph the function $f(x) = \dfrac{4}{x+2}$.

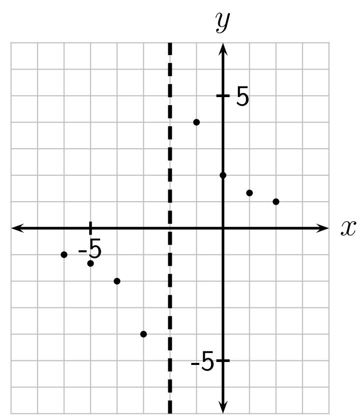
Solution: The domain of the function is $x \neq -2$, so the graph cannot have any points with x-coordinates of $x = -2$. All such points are indicated by the vertical dashed line shown on the grid to the right. Even though those points are not on the graph of the function, it is customary to show them as a vertical dashed line. Such a line is called a **vertical asymptote**. The asymptote causes the graph to appear as two pieces, one to the left of the asymptote and one to the right, with neither piece crossing the asymptote.

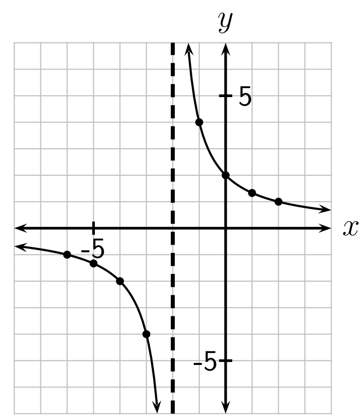


To find what the two pieces of the graph look like, we need to find some input-output pairs for the function. The first x values chosen should be on either side of the value that x is not allowed to have; in this case we begin with $x = -3$ and $x = -1$, as shown in the first table to the left below. I've included a DNE for the value of $f(x)$ when $x = -2$ to give our table a central point to work out from. The next table to the right shows some additional x values and their corresponding function values $f(x)$.

x	$f(x)$
-3	-4
-2	DNE
-1	4

x	$f(x)$
-6	-1
-5	$-\frac{4}{3}$
-4	-2
-3	-4
-2	DNE
-1	4
0	2
1	$\frac{4}{3}$
2	1





Solution: The leftmost of the two graphs above shows the points we found, plotted. We can see that they arrange themselves into two groups, one on either side of the asymptote. The graph above and to the right shows how each set of dots is connected to get the two parts of the graph.

Section 8.5 Exercises

To Solutions

1. Graph each of the following functions. See this example.

   (a) $y = \dfrac{4}{x+4}$ (b) $g(x) = x^2 - 5x + 6$ (c) $h(x) = \dfrac{6}{2-x}$

   (d) $y = \sqrt{x+5}$ (e) $f(x) = \dfrac{8}{x^2}$ (f) $h(x) = \sqrt{9 - x^2}$

2. Let $h(x) = x^2 - 2x - 3$ and let $f(x) = 3x - 1$. Find $(f \circ h)(x)$ and $(h \circ f)(x)$.

3. Give the domain of each of the following functions.

   (a) $h(x) = \sqrt{x^2 - 9}$

   (b) $y = \sqrt{9 - x^2}$

   (c) $f(x) = \dfrac{1}{\sqrt{9 - x^2}}$

8.6 Quadratic Functions

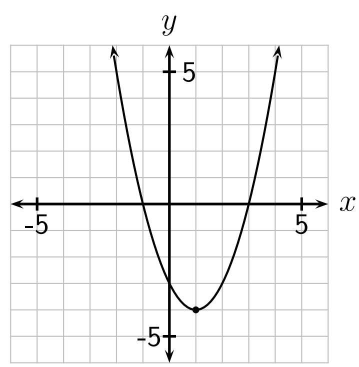
8. (g) Determine whether the graph of a quadratic function is a parabola opening up or down, and find the coordinates of the vertex of the parabola.

Quadratic functions are functions of the form

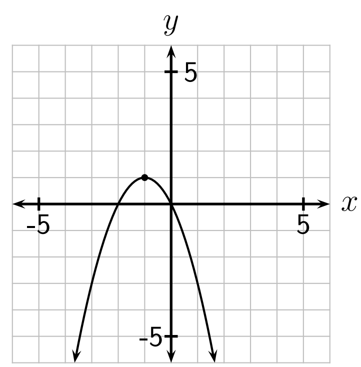
$$f(x) = ax^2 + bx + c\ ,$$

where a, b, and c are real numbers and $a \neq 0$. You have seen a number of these functions already, of course, including some in 'real world' situations. We will refer to the value a, that x^2 is multiplied by, as the **lead coefficient**, and c will be called the **constant term**. From previous work you should know that the graph of a quadratic function is a U-shape that we call a **parabola**. In this section we will investigate the graphs of quadratic functions a bit more.

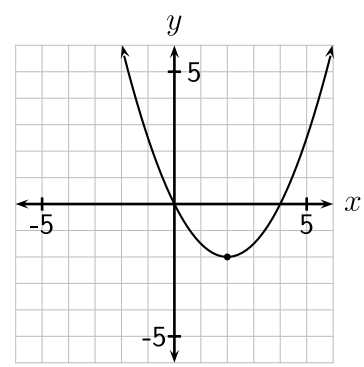
Let's begin by looking at some graphs. Below are the graphs of three quadratic functions, with the equation for each below its graph.



$f(x) = x^2 - 2x - 3$



$g(x) = -x^2 - 2x$



$h(x) = \frac{1}{2}x^2 - 2x$

The graphs and corresponding equations illustrate the following:

The effect of a on the graph of $f(x) = ax^2 + bx + c$ is as follows: If a is positive, the graph of the function is a parabola that "opens upward." If a is negative, the graph of the function is a parabola that opens downward.

The **vertex** of a parabola is the point at the bottom or top of the parabola, depending on whether it opens upward or downward. For the three parabolas graphed above, the coordinates of the vertices (plural of vertex) are $(1, -4)$, $(-1, 1)$ and $(2, -2)$, respectively. *The vertex of a parabola is the most important point on its graph*. In fact, if we can find the vertex of a parabola and we know whether it opens upward or downward, we have a pretty good idea what the graph of the parabola looks like. The following will help us find the coordinates of the vertex of a parabola:

Vertex of a Parabola

The x-coordinate of the vertex of any parabola is always given by $x = \dfrac{-b}{2a}$.

The y-coordinate is then found by substituting this value of x into the equation of the parabola.

◇ **Example 8.6(b):** Find the vertices of $f(x) = -3x^2+6x-10$ and $g(x) = \frac{1}{2}x^2-2x+5$.

Solution: The x-coordinate of the vertex of $f(x)$ is given by $x = \dfrac{-6}{2(-3)} = \dfrac{-6}{-6} = 1$. The y-coordinate is found by substituting this value in for x:

$$f(1) = -3(1)^2 + 6(1) - 10 = -3 + 6 - 10 = -7$$

The vertex of $f(x)$ is $(1, -7)$.

Solution: The x-coordinate of the vertex of $g(x)$ is $x = \dfrac{-(-2)}{2(\frac{1}{2})} = \dfrac{2}{1} = 2$. The y-coordinate is

$$g(2) = \frac{1}{2}(2)^2 - 2(2) + 5 = 2 - 4 + 5 = 3$$

The vertex of $g(x)$ is $(2, 3)$.

Section 8.6 Exercises To Solutions

1. Use the above method to find the coordinates of the vertices for each of the parabolas whose graphs are shown on the previous page. Make sure your answers agree with what you see on the graphs!

2. For each of the following quadratic equations, first determine whether its graph will be a parabola opening upward or opening downward. Then find the coordinates of the vertex.

   (a) $f(x) = 3x^2 + 6x + 2$ (b) $y = x^2 - 6x - 7$

   (c) $g(x) = \frac{1}{2}x^2 + 2x + 5$ (d) $g(x) = 10 + 3x - x^2$

3. Solve each of the following systems by the addition method.

   (a) $\begin{aligned} 2x - 3y &= -7 \\ -2x + 5y &= 9 \end{aligned}$ (b) $\begin{aligned} 2x - 3y &= -6 \\ 3x - y &= 5 \end{aligned}$ (c) $\begin{aligned} 4x + y &= 14 \\ 2x + 3y &= 12 \end{aligned}$

4. Two of the systems from Exercise 4 can easily be solved using the substitution method. Determine which they are, and solve them with that method.

5. Graph each of the following functions.

   (a) $y = \dfrac{6}{x-1}$ (b) $f(x) = -\dfrac{2}{3}x + 4$

9 More Exponents and Roots, Complex Numbers

9.1 Fractional Exponents

9. (a) Evaluate numerical expressions containing fractional exponents.

When we first saw exponents, the exponents we encountered were the counting numbers 1, 2, 3, 4, and so on. Later we gave meaning to negative and zero exponents. Here we will briefly introduce fractional exponents, which are useful in certain situations. Here is how we define fractional exponents:

Fractional Exponents

- We define $a^{\frac{1}{n}}$ to mean $\sqrt[n]{a}$
- $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

◇ **Example 9.1(a):** Find $16^{\frac{1}{4}}$

Solution: $16^{\frac{1}{4}} = \sqrt[4]{16} = 2,$ since $2^4 = 16$

◇ **Example 9.1(b):** Find $(-25)^{\frac{1}{2}}$ and $-25^{\frac{1}{2}}$ if possible. If not possible, say so.

Solution: $(-25)^{\frac{1}{2}} = \sqrt{-25},$ which does not exist. $-25^{\frac{1}{2}} = -\sqrt{25} = -5$

◇ **Example 9.1(c):** Find $(-27)^{\frac{1}{3}}$ and $-27^{\frac{1}{3}}$ if possible. If not possible, say so.

Solution:

$(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3$ because $(-3)^3 = -27.$ $-27^{\frac{1}{3}} = -\sqrt[3]{27} = -3$ also.

◇ **Example 9.1(d):** Find $9^{\frac{3}{2}}$.

Solution: $9^{\frac{3}{2}} = (\sqrt{9})^3 = 3^3 = 27$

We could have done this as $9^{\frac{3}{2}} = \sqrt{9^3}$, but 9^3 is hard to find without a calculator, and we probably wouldn't know the square root of the resulting large number either!

◇ **Example 9.1(e):** Find $(-27)^{\frac{2}{3}}$ and $27^{-\frac{2}{3}}$ if possible. If not possible, say so.

Solution: $$(-27)^{\frac{2}{3}} = (\sqrt[3]{-27})^2 = (-3)^2 = 9$$

To compute $27^{-\frac{2}{3}}$ we use the definition of a negative exponent first:

$$27^{-\frac{2}{3}} = \frac{1}{27^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{27})^2} = \frac{1}{3^2} = \frac{1}{9}$$

Section 9.1 Exercises

To Solutions

1. Find each of the following, if possible. If not, write DNE for "does not exist."

   (a) $4^{\frac{1}{2}}$ (b) $1000^{\frac{1}{3}}$ (c) $-4^{\frac{1}{2}}$ (d) $(-16)^{\frac{1}{4}}$

2. Find each of the following, if possible. If not, write DNE for "does not exist."

   (a) $(-8)^{\frac{1}{3}}$ (b) $27^{\frac{2}{3}}$ (c) $(-9)^{\frac{3}{2}}$ (d) $16^{\frac{1}{4}}$

3. Give the domain of each of the following functions.

   (a) $s(t) = \dfrac{t+2}{7-t}$ (b) $h(x) = \dfrac{x^2}{5} + 2$ (c) $y = \dfrac{1}{(x-3)^2}$

4. For each of the following quadratic equations, first determine whether its graph will be a parabola opening upward or opening downward. Then find the coordinates of the vertex.

   (a) $h(x) = -x^2 - 2x - 3$ (b) $s(t) = \frac{1}{2}t^2 + 3t + \frac{9}{2}$

9.2 Rationalizing Denominators

> 9. (b) Rationalize denominators of rational expressions with roots in their denominators.

When working with fractions, we always expect final results to be given in lowest terms; that is, reduced as much as possible. A fraction not in lowest terms is considered "undesirable" in a sense. When working with roots it is considered undesirable to have a root in the denominator of a fraction. There is a simple method called **rationalizing the denominator** for solving this problem. Before going into it, let's examine how we *really* reduce a fraction. (We usually don't show all the steps I'm going to show, but they really take place.) We'll use $\frac{15}{20}$ as an example.

$$\frac{15}{20} = \frac{3 \cdot 5}{4 \cdot 5} = \frac{3}{4} \cdot \frac{5}{5} = \frac{3}{4} \cdot 1 = \frac{3}{4}$$

Note how we essentially 'take out' the fraction $\frac{5}{5}$, which is really just the number one.

To rationalize the denominator of a fraction we instead 'put in' an extra fraction that is really equal to one. When considering square roots the fraction to be used is the root in the denominator over itself. Let's demonstrate with the fraction $\frac{5}{\sqrt{3}}$:

$$\frac{5}{\sqrt{3}} = \frac{5}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{\sqrt{9}} = \frac{5\sqrt{3}}{3}$$

If we have an expression like $\frac{10x}{15}$ we can reduce it to $\frac{2x}{3}$. We do something similar when rationalizing the denominator, as shown in the next example.

◇ **Example 9.2(a):** Rationalize the denominator: $\frac{15}{\sqrt{6}}$.

Solution: $$\frac{15}{\sqrt{6}} = \frac{15}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{15\sqrt{6}}{\sqrt{36}} = \frac{\overset{5}{\cancel{15}}\sqrt{6}}{\underset{2}{\cancel{6}}} = \frac{5\sqrt{6}}{2}$$

Sometimes there will be a root in the numerator as well, and we might have to simplify a square root, as shown in the next example.

◇ **Example 9.2(b):** Rationalize the denominator: $\frac{3\sqrt{2}}{\sqrt{10}}$.

Solution: $$\frac{3\sqrt{2}}{\sqrt{10}} = \frac{3\sqrt{2}}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{3\sqrt{20}}{\sqrt{100}} = \frac{3\sqrt{4}\sqrt{5}}{10} = \frac{\overset{3}{\cancel{6}}\sqrt{5}}{\underset{5}{\cancel{10}}} = \frac{6\sqrt{5}}{10} = \frac{3\sqrt{5}}{5}$$

Occasionally we run into numbers like $\frac{5}{3+\sqrt{2}}$. This of course has a square root in the denominator, so we need to rationalize the denominator. If we multiply both the numerator and denominator by $\sqrt{2}$ the denominator will become $3\sqrt{2}+2$, so there will still be a root in the denominator. The trick here is to multiply both the numerator and denominator by $3-\sqrt{2}$, which we call the **conjugate** of $3+\sqrt{2}$.

◇ **Example 9.2(c):** Rationalize the denominator of $\frac{5}{3+\sqrt{2}}$.

Solution: $$\frac{5}{3+\sqrt{2}} = \frac{5}{3+\sqrt{2}} \cdot \frac{3-\sqrt{2}}{3-\sqrt{2}} = \frac{15-5\sqrt{2}}{9-3\sqrt{2}+3\sqrt{2}-\sqrt{2}\sqrt{2}} = \frac{15-5\sqrt{2}}{7}$$

NOTE: If the denominator in the above example had been $7-3\sqrt{2}$, we would have multiplied the numerator and denominator each by $7+3\sqrt{2}$.

Section 9.2 Exercises

To Solutions

1. Rationalize each denominator, and don't forget to reduce and/or simplify.

   (a) $\frac{5}{\sqrt{6}}$ (b) $\frac{10}{\sqrt{15}}$ (c) $\frac{2}{1-\sqrt{5}}$ (d) $\frac{12}{\sqrt{3}}$

   (e) $\frac{3\sqrt{2}}{\sqrt{6}}$ (f) $\frac{\sqrt{3}}{7+\sqrt{3}}$

2. Consider the functions $f(x)=3-x$ and $g(x)=x^2-3x$.

   (a) Find $f[g(3)]$ without actually finding $(f \circ g)(x)$.

   (b) Find $(f \circ g)(x)$.

   (c) Use your answer to (b) to find $(f \circ g)(3)$. It should give the same result as you got for (a) - if not, check your work.

   (d) Find $(g \circ f)(x)$.

3. Graph each of the following functions.

   (a) $g(x)=x+1$ (b) $y=\frac{8}{(x-3)^2}$

4. Find each of the following, if possible. If not, write DNE for "does not exist." Use the facts that $a^{\frac{1}{n}}=\sqrt[n]{a}$ and $a^{\frac{m}{n}}=(\sqrt[n]{a})^m=\sqrt[n]{a^m}$.

   (a) $4^{-\frac{5}{2}}$ (b) $9^{\frac{3}{2}}$ (c) $-25^{\frac{3}{2}}$ (d) $(-1)^{\frac{5}{3}}$

9.3 Complex Numbers and Their Arithmetic

9. (c) Add, subtract and multiply complex numbers.
   (d) Divide complex numbers numbers.
   (e) Simplify square roots of negative numbers.

Consider trying to solve the equation $x^2 - 4x + 13 = 0$. The first thing that one would be inclined to try is to factor the left side, but that won't go anywhere. Next we might try the quadratic formula:

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2}\,.$$

But since $\sqrt{-36}$ is undefined, we can't solve the equation using the quadratic formula. The equation therefore has no solution.

At one point in the past this was not satisfactory to everyone, so some people tried to find a way around the problem here. In doing so, they invented a 'number' i that has the special property that $i^2 = -1$. Now we know that any number like the kind we are used to (what are called **real numbers**) gives a number greater than or equal to zero when squared. Therefore this thing i is not a real number.

A number made up of a real number times i, like $3i$, $-5i$, $\frac{2}{3}i$ or even $i\sqrt{2}$ is called an **imaginary number**. A real number plus or minus an imaginary number, like $2+3i$, $-7+3i$, $7-i\sqrt{5}$ or $\frac{2}{3}-\frac{1}{3}i$ is called a **complex number**. Note that *it is customary to put the real number first, then the imaginary number*. We sometimes refer to this as $a+bi$ form.

We can add, subtract or multiply complex numbers by simply treating i as we would x:

◇ **Example 9.3(a):** Combine like terms: $3-5i-2+7i+6$.

Solution: $$3-5i-2+7i+6 = (3-2+6)+(-5i+7i) = 7+2i$$

◇ **Example 9.3(b):** Subtract $(2-6i)-(4-3i)$.

Solution: $$(2-6i)-(4-3i) = 2-6i-4+3i = -2-3i$$

When multiplying complex numbers we again treat i as we would x, but replace i^2 with -1 everywhere that it occurs. The following examples demonstrate this.

◇ **Example 9.3(c):** Multiply $(5i)(1-3i)$.

Solution:

$$(5i)(1-3i) = (5i)(1)-(5i)(3i) = 5i-15i^2 = 5i-15(-1) = 5i+15 = 15+5i$$

Note that the answer was put in the standard form $a+bi$ - you should always do this.

◇ **Example 9.3(d):** Multiply $(-3+5i)(2-i)$.

Solution: $(-3+5i)(2-i) = -6+3i+10i-5i^2 = -6+13i+5 = -1+13i$

The next example shows that when we multiply two complex numbers the result is not necessarily a complex number.

◇ **Example 9.3(e):** Multiply $(5-2i)(5+2i)$.

Solution: $(5-2i)(5+2i) = 25+10i-10i-4i^2 = 25-4(-1) = 25+4 = 29$

The two numbers $5-2i$ and $5+2i$ are called **complex conjugates**. Computations like the one in this last example are important when dividing complex numbers, as the next example shows.

◇ **Example 9.3(f):** Divide $4+3i$ by $5-2i$, giving your answer in $a+bi$ form.

Solution: The key here is to write the division as a fraction, and multiply the fraction by one, in the form of the complex conjugate of the denominator over itself:

$$(4+3i)\div(5-2i) = \frac{4+3i}{5-2i} = \frac{4+3i}{5-2i}\cdot\frac{5+2i}{5+2i} = \frac{20+8i+15i+6i^2}{25+10i-10i-4i^2} = \frac{14+23i}{29} = \tfrac{14}{29}+\tfrac{23}{29}i$$

Now think about this: $\sqrt{4}=2$ because $2^2=4$, $\sqrt{9}=3$ because $3^2=9$. So since $i^2=-1$, it must be the case that $\sqrt{-1}=i$. This allows us to take square roots of negative numbers. We do this in the same way that we simplify square roots of positive numbers, using the fact that $\sqrt{a\cdot b}=\sqrt{a}\cdot\sqrt{b}$. Here are some examples:

◇ **Example 9.3(g):** Simplify $\sqrt{-16}$, allowing complex numbers.

Solution: $$\sqrt{-16} = \sqrt{(16)(-1)} = \sqrt{16}\sqrt{-1} = 4i$$

◇ **Example 9.3(h):** Simplify $\sqrt{-18}$, allowing complex numbers.

Solution: $$\sqrt{-16} = \sqrt{(9)(-1)(2)} = \sqrt{9}\sqrt{-1}\sqrt{2} = 3i\sqrt{2}$$

As shown here, it is customary to write the number, then the i, then the square root.

Section 9.3 Exercises

To Solutions

1. Add, subtract or just combine like terms as directed.

   (a) $(7+10i)-(3-8i)$ (b) $3i+(5-i)$

   (c) $(1+i)-(1-i)$ (d) $(2-3i)+(-5+13i)$

2. Multiply and simplify each of the following. *Give your answers in the form* $a+bi$, *where either of* a *or* b *might be negative.*

   (a) $(3i)(5i)$ (b) $(-2i)(7i)$ (c) $3(6i)$

   (d) $3(5-2i)$ (e) $2i(4-i)$ (f) $(7+i)(2+5i)$

   (g) $(a+bi)(a-bi)$ (h) $(9+15i)(2-i)$ (i) $(c+di)^2$

3. Divide each, giving your answers in $a+bi$ form.

   (a) $3 \div (2+5i)$ (b) $(1+i) \div (6-3i)$

   (c) $(2+5i) \div (3+2i)$ (d) $(c+di) \div (c-di)$

4. Simplify each of the following, allowing complex numbers.

   (a) $\sqrt{-25}$ (b) $\sqrt{-24}$ (c) $-\sqrt{8}$

   (d) $\sqrt{-100}$ (e) $\sqrt{-75}$ (f) $-\sqrt{64}$

5. For each of the following quadratic equations, first determine whether its graph will be a parabola opening upward or opening downward. Then find the coordinates of the vertex.

   (a) $y=-\frac{1}{4}x^2+\frac{5}{2}x-\frac{9}{4}$ (b) $y=3x^2+6x-2$

6. Rationalize each denominator, and don't forget to reduce and/or simplify.

   (a) $\dfrac{3\sqrt{2}}{\sqrt{10}}$ (b) $\dfrac{2+\sqrt{5}}{2-\sqrt{5}}$

9.4 Solving Quadratic Equations, Again

9. (f) Solve quadratic equations with complex solutions.

We now return to the equation $x^2 - 4x + 13 = 0$. When we first looked at it, it appeared that it had no solution. But

$$(2+3i)^2 = 4 + 6i + 6i + 9i^2 = 4 + 12i - 9 = -5 + 12i,$$

so if we then substitute $x = 2 + 3i$ into $x^2 - 4x + 13$ we get

$$(2+3i)^2 - 4(2+3i) + 13 = -5 + 12i - 8 - 12i + 13 = 0.$$

This shows that $2 + 3i$ is a solution to $x^2 - 4x + 13 = 0$! Two questions should occur to you at this point:

1) How do we get that solution?

2) What is the other solution, if there is one?

To answer the first question above, recall the following:

Quadratic Formula

The solutions to $ax^2 + bx + c = 0$ are obtained from $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Let's see what happens when we try using the quadratic formula to solve $x^2 - 4x + 13 = 0$:

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2}$$

At this point we conclude that there is no solution *if we are restricting ourselves to the real numbers*, which we often do. However, if we allow complex numbers we can continue to get

$$x = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm \sqrt{(36)(-1)}}{2} = \frac{4 \pm 6i}{2} = \frac{4}{2} \pm \frac{6i}{2} = 2 \pm 3i$$

Thus the equation has the two solutions, $2+3i$ and $2-3i$. We call two complex numbers like this **complex conjugates**. In Exercise 2(g) of the previous section you showed that when we multiply complex conjugates together the result is always a real number. That is one important thing about complex conjugates.

Another important thing about complex conjugates is that what happened when solving the equation $x^2 - 4x + 13 = 0$ was not a fluke; *when solving a quadratic equation that has complex solutions, the two solutions will always be complex conjugates.*

◇ **Example 9.4(a):** Solve $x^2 + 16 = 6x$, allowing complex solutions.

$$x^2 + 16 = 6x$$
$$x^2 - 6x + 16 = 0$$
$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(16)}}{2}$$
$$x = \frac{6 \pm \sqrt{36 - 64}}{2}$$

$$x = \frac{6 \pm \sqrt{-28}}{2}$$
$$x = \frac{6 \pm 2i\sqrt{7}}{2}$$
$$x = \frac{6}{2} \pm \frac{2i\sqrt{7}}{2}$$
$$x = 3 + i\sqrt{7},\ 3 - i\sqrt{7}$$

◇ **Example 9.4(b):** Solve $4x^2 - 20x + 27 = 0$, allowing complex solutions.

$$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(4)(27)}}{2(4)}$$
$$x = \frac{20 \pm \sqrt{400 - 432}}{8}$$
$$x = \frac{20 \pm \sqrt{-32}}{8}$$

$$x = \frac{20 \pm 4i\sqrt{2}}{8}$$
$$x = \frac{20}{8} \pm \frac{4i\sqrt{2}}{8}$$
$$x = \frac{5}{2} \pm \frac{\sqrt{2}}{2}i$$
$$x = \tfrac{5}{2} + \tfrac{\sqrt{2}}{2}i,\ \tfrac{5}{2} - \tfrac{\sqrt{2}}{2}i$$

Section 9.4 Exercises

To Solutions

1. Solve each of the following quadratic equations, allowing complex solutions.

   (a) $x^2 + 13 = -6x$ (b) $x^2 + 10x + 26 = 0$ (c) $x^2 + x = 6$

   (d) $x^2 + 4 = 0$ (e) $2x^2 + 5 = 2x$ (f) $x^2 + 14x + 53 = 0$

2. Give the domains of $f(x) = \sqrt{3 - x}$ and $g(x) = \dfrac{x - 1}{x^2 - 3x - 10}$.

3. Rationalize each denominator, reduce and/or simplify your answer when possible.

   (a) $\dfrac{15}{\sqrt{6}}$ (b) $\dfrac{5 + \sqrt{3}}{2 - \sqrt{3}}$

4. Add, subtract or just combine like terms as directed.

   (a) $(4 - 7i) - (-23 - 17i)$ (b) $9 + 6i - 6i - 7$

5. Multiply and simplify each of the following. *Give your answers in the form* $a + bi$, *where either of* a *or* b *might be negative*.

   (a) $(2+3i)^2$ (b) $(4-3i)(4+3i)$ (c) $(-2+3i)(5-4i)$

6. Simplify each square root, **allowing complex numbers**.

   (a) $\sqrt{-9}$ (b) $\sqrt{-18}$ (c) $-\sqrt{16}$

10 More With Polynomial and Rational Expressions

10.1 Dividing Polynomials

> 10. (a) Divide polynomial expressions.

Suppose that you were going to divide $252 \div 3$, without using a calculator. You would use "long division," beginning with a setup like (a) below. You would probably check to see if 3 goes into 2 and, since it doesn't, you would see how many times it goes into 25 without going over that value. That would be 8 times, so you would put an 8 above the 5, which signifies the tens place, as shown in (b) below. *There is another way to think about this step.* Rather than asking how many times 3 goes into 25 we could ask the equivalent question "What do we have to multiply 3 by to get as close to 25 as possible, without going over?" This gives the same result, 8, but will soon be a better way of thinking of this computation.

$$\begin{array}{cccccc}
3\,\overline{)\,252} &
\begin{array}{r} 8 \\ 3\,\overline{)\,252} \end{array} &
\begin{array}{r} 8 \\ 3\,\overline{)\,252} \\ 24 \end{array} &
\begin{array}{r} 8 \\ 3\,\overline{)\,252} \\ \underline{24} \\ 1 \end{array} &
\begin{array}{r} 8 \\ 3\,\overline{)\,252} \\ \underline{24}\downarrow \\ 12 \end{array} &
\begin{array}{r} 84 \\ 3\,\overline{)\,252} \\ \underline{24} \\ 12 \\ \underline{12} \\ 0 \end{array} \\
\text{(a)} & \text{(b)} & \text{(c)} & \text{(d)} & \text{(e)} & \text{(f)}
\end{array}$$

At this point we multiply 8 times 3 and put the result, 24, below the 25, as shown in (c) above. We then subtract $25 - 24$ to get the result shown in (d). *Here we can think of* $25 + (-24)$ *instead of* $25 - 24$, *which is also helpful later.* The 2 in the ones place is then brought down, (e), and we then see that $4 \cdot 3 = 12$. This is placed at the bottom of the "stack" and subtracted to give zero at (f) above. Since there is no remainder, we are done.

Now suppose that we want to divide $(x^3 + 8x^2 + 14x - 3) \div (x + 3)$. The method here is much like the process we used above with numbers. We set up as shown to the left below and ask what we need to multiply times x to get x^3. That would be x^2, so we put x^2 above the bar in the x^2 column. Then we multiply x^2 times $x + 3$ and put the result below $x^3 + 8x^2 + 14x - 3$ with like terms aligned. This is shown to the right below.

$$x + 3\,\overline{)\,x^3 + 8x^2 + 14x - 3} \qquad\qquad \begin{array}{l} x^2 \\ x + 3\,\overline{)\,x^3 + 8x^2 + 14x - 3} \\ x^3 + 3x^2 \end{array}$$

At this point we need to subtract $(x^3 + 8x^2) - (x^3 + 3x^2)$, which can also be thought of as $(x^3 + 8x^2) + (-x^3 - 3x^2)$. Here we have used the idea that subtraction is addition of the opposite. The result is $5x^2$, and we then carry the $14x$ down to get $5x^2 + 14x$. This is shown below.

$$
\begin{array}{r}
x^2 \\
x+3\,\overline{)\,x^3+8x^2+14x-3} \\
\underline{x^3+3x^2} \\
5x^2+14x
\end{array}
\qquad\qquad
\begin{array}{r}
x^2+\ 5x-1 \\
x+3\,\overline{)\,x^3+8x^2+14x-3} \\
\underline{x^3+3x^2} \\
5x^2+14x \\
\underline{5x^2+15x} \\
-\,x-3 \\
\underline{-\,x-3} \\
0
\end{array}
$$

We now ask what we need to multiply by x to get $5x^2$. That amount, $+5x$, is put in the x place above the bar. *Note that the sign is included.* $5x$ is then multiplied times $x+3$ to get $5x^2+15x$, which is put below $5x^2+14x$ and subtracted to get $-x-3$ after the -3 is brought down. To get the $-x$ we have to put -1 above the top bar. This is multiplied by $x+3$ to get the $-x-3$ that is subtracted from the $-x-3$ we already have. Of course this can also be thought of as $(-x-3)+(x+3)$ instead. The result is zero, which tells us that $x+3$ goes into $x^3+8x^2+14x-3$ evenly. The final result is shown to the right at the bottom of the previous page.

Many people find it easier, after multiplying and putting the result at the bottom, to change the signs of the second expression and add (instead of subtracting). This works because adding the opposite is equivalent to subtraction. The idea is probably best seen "live," but I've attempted to show it below. Each dotted oval encloses part of the original computation, but then the dashed arrows take you to where the signs of the second expression have been changed, and then the two expressions added.

◇ **Example 10.1(a):** Divide $(8x^4+26x^3-11x^2+9x-2)\div(4x-1)$

Solution:

$$
\begin{array}{r}
2x^3+\ 7x^2-\ \ x+2 \\
4x-1\,\overline{)\,8x^4+26x^3-11x^2+9x-2} \\
\underline{8x^4-\ 2x^3} \\
28x^3-11x^2 \\
\underline{28x^3-\ 7x^2} \\
-\,4x^2+9x \\
\underline{-\,4x^2+\ x} \\
8x-2 \\
\underline{8x-2} \\
0
\end{array}
$$

$$
\begin{array}{r}
8x^4+26x^3 \\
\underline{-\,8x^4+\ 2x^3} \\
0\ +\ 28x^3
\end{array}
\qquad
\begin{array}{r}
28x^3-11x^2 \\
\underline{-28x^3+\ 7x^2} \\
-\,4x^2
\end{array}
\qquad
\begin{array}{r}
-\,4x^2+\ 9x \\
\underline{+\,4x^2-\ \ x} \\
8x
\end{array}
\qquad
\begin{array}{r}
8x-2 \\
\underline{-\,8x+2} \\
0
\end{array}
$$

The example at the beginning of this section, of dividing $252 \div 3$, is a bit special in that 3 goes evenly into 252. When this does not happen we need to express our result in some other way. The method we will use is demonstrated at the top of the next page for $257 \div 3$. Things proceed as they did in the other example until we reach the point seen to the left there. We might be inclined to stop at that point, since three will not go into two. But in fact it does, *just not evenly*. Two divided by three is the fraction $\frac{2}{3}$, so we simply get the result that $257 \div 3 = 85\frac{2}{3}$, as shown below. For future reference, note that this could be thought of as $85 + \frac{2}{3}$.

$$
\begin{array}{r}
85 \\
3\,\overline{)\,257} \\
\underline{24} \\
17 \\
\underline{15} \\
2
\end{array}
\qquad\qquad
\begin{array}{r}
85\frac{2}{3} \\
3\,\overline{)\,257} \\
\underline{24} \\
17 \\
\underline{15} \\
2
\end{array}
$$

Divide by three and put here

If we want to divide $(3x^3 - x^2 - 13x + 7) \div (x + 2)$ we proceed as usual until we reach the point shown below and to the left. Then, just as in the numerical case, we form the fraction $\frac{5}{x+2}$ as was done in the numerical case, then add it to the expression we have obtained so far, as shown to the right below.

$$
\begin{array}{r}
3x^2 - 7x + 1 \\
x+2\,\overline{)\,3x^3 - x^2 - 13x + 7} \\
\underline{3x^3 + 6x^2}\phantom{{} - 13x + 7} \\
-7x^2 - 13x\phantom{{} + 7} \\
\underline{-7x^2 - 14x}\phantom{{} + 7} \\
x + 7 \\
\underline{x + 2} \\
5
\end{array}
\qquad
\begin{array}{r}
3x^2 - 7x + 1 + \frac{5}{x+2} \\
x+2\,\overline{)\,3x^3 - x^2 - 13x + 7}\phantom{+ \frac{5}{x+2}} \\
\underline{3x^3 + 6x^2}\phantom{{} - 13x + 7 + \frac{5}{x+2}} \\
-7x^2 - 13x\phantom{{} + 7 + \frac{5}{x+2}} \\
\underline{-7x^2 - 14x}\phantom{{} + 7 + \frac{5}{x+2}} \\
x + 7\phantom{+ \frac{5}{x+2}} \\
\underline{x + 2}\phantom{+ \frac{5}{x+2}} \\
5\phantom{+ \frac{5}{x+2}}
\end{array}
$$

◇ **Example 10.1(b):** Divide $(x^3 + 2x^2 - 38x + 11) \div (x - 5)$

Solution:

$$
\begin{array}{r}
x^2 + 7x - 3 \\
x-5\,\overline{)\,x^3 + 2x^2 - 38x + 11} \\
\underline{x^3 - 5x^2}\phantom{{} - 38x + 11} \\
7x^2 - 38x\phantom{{} + 11} \\
\underline{7x^2 - 35x}\phantom{{} + 11} \\
-3x + 11 \\
\underline{-3x + 15} \\
-4
\end{array}
\qquad
\begin{array}{r}
x^2 + 7x - 3 - \frac{4}{x-5} \\
x-5\,\overline{)\,x^3 + 2x^2 - 38x + 11}\phantom{- \frac{4}{x-5}} \\
\underline{x^3 - 5x^2}\phantom{{} - 38x + 11 - \frac{4}{x-5}} \\
7x^2 - 38x\phantom{{} + 11 - \frac{4}{x-5}} \\
\underline{7x^2 - 35x}\phantom{{} + 11 - \frac{4}{x-5}} \\
-3x + 11\phantom{- \frac{4}{x-5}} \\
\underline{-3x + 15}\phantom{- \frac{4}{x-5}} \\
-4\phantom{- \frac{4}{x-5}}
\end{array}
$$

This tells us to add $\dfrac{-4}{x-5}$, but that is the same as $-\dfrac{4}{x-5}$, so we really *subtract* $\dfrac{4}{x-5}$

Section 10.1 Exercises To Solutions

1. Perform each of the following divisions.

   (a) $(5x^3 - 2x^2 - 12x + 9) \div (x - 1)$ (b) $(4x^3 + 13x^2 - 2x - 15) \div (x + 3)$

   (c) $(x^3 - 13x + 12) \div (x + 4)$ **Hint:** Write $x^3 - 13x + 12$ as $x^3 + 0x^2 - 13x + 12$.

   (d) $(15x^3 + 16x^2 - x - 2) \div (3x - 1)$ (e) $(6x^3 + 13x^2 - x + 10) \div (2x + 5)$

2. Perform each of the following divisions.

   (a) $(x^2 + 3x + 7) \div (x + 1)$ (b) $(x^2 + 9x - 2) \div (x - 5)$

   (c) $(3x^2 + 4x - 7) \div (x - 1)$ (d) $(2x^2 - 7x + 9) \div (x - 2)$

3. Find each of the following, if possible (complex numbers *NOT* allowed). If not, write DNE for "does not exist." Use the facts that $a^{\frac{1}{n}} = \sqrt[n]{a}$ and $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$.

   (a) $8^{\frac{4}{3}}$ (b) $9^{-\frac{1}{2}}$ (c) $16^{\frac{1}{2}}$ (d) $(-27)^{\frac{1}{3}}$

4. Perform each division, giving your answers in $a + bi$ form.

   (a) $(5 + i) \div (3 - 2i)$ (b) $3i \div (5 - i)$ (c) $(1 + i) \div (6 + 2i)$

5. Solve each of the following quadratic equations, allowing complex solutions.

   (a) $x^2 - 10x + 43 = 0$ (b) $x^2 = 2x - 8$ (c) $x^2 - 8x + 28 = 0$

10.2 Adding and Subtracting Rational Expressions

10. (b) Add and subtract rational expressions.

We have seen rational expressions, also called algebraic fractions, in several contexts already. Here are some examples of rational expressions:

$$\frac{6}{x-2} \qquad \frac{x^2+5x+6}{x^2+6x+9} \qquad \frac{3x}{x^2-4}$$

We have discussed the domains of such expressions, simplifying them, and solving equations containing them. We have graphed functions of the form $f(x) = \dfrac{6}{x-2}$. In this section we will add and subtract rational expressions.

Since rational expressions are really just fractions, it might be good to start by reminding ourselves how we add and subtract fractions. *To add or subtract fractions, we have to have a common denominator.* The same holds true for rational expressions as well. Let's look at a couple examples involving fractions.

◇ **Example 10.2(a):** Subtract $\dfrac{7}{3} - \dfrac{2}{3}$

Solution: These have a common denominator, so $\dfrac{7}{3} - \dfrac{2}{3} = \dfrac{7-2}{3} = \dfrac{5}{3}$.

◇ **Example 10.2(b):** Add $\dfrac{3}{4} + \dfrac{1}{6}$

Solution: Here we need to get a common denominator. The least common denominator is 12:

$$\frac{3}{4} + \frac{1}{6} = \frac{3}{4} \cdot \frac{3}{3} + \frac{1}{6} \cdot \frac{2}{2} = \frac{9}{12} + \frac{2}{12} = \frac{11}{12}$$

Note that in the second example we multiply each fraction by some form of the number one so that the denominators become the same. We also could have multiplied the first fraction by $\frac{6}{6}$ and the second by $\frac{4}{4}$, but that does not result in the smallest possible common denominator (which we officially refer to as the **least common denominator**), and we have to reduce when we are done:

◇ **Example 10.2(c):** Add $\dfrac{3}{4} + \dfrac{1}{6}$

Solution:

$$\frac{3}{4} + \frac{1}{6} = \frac{3}{4} \cdot \frac{6}{6} + \frac{1}{6} \cdot \frac{4}{4} = \frac{18}{24} + \frac{4}{24} = \frac{22}{24} = \frac{11}{12}$$

This is no real problem, but *when working with algebraic fractions, not using the least common denominator can make exercises very difficult.*

Adding and subtracting algebraic fractions is pretty straightforward when they have common denominators. The only thing that can trip us up is forgetting to distribute a negative sign when subtracting, like in the second example on the next page.

◇ **Example 10.2(d):** Add $\dfrac{x+2}{x^2-5x+6}+\dfrac{3x}{x^2-5x+6}$

Solution: These two fractions have the same denominator, so we simply add the numerators and put the result over the common denominator:

$$\frac{x+2}{x^2-5x+6}+\frac{3x}{x^2-5x+6}=\frac{(x+2)+3x}{x^2-5x+6}=\frac{x+2+3x}{x^2-5x+6}=\frac{4x+2}{x^2-5x+6}$$

◇ **Example 10.2(e):** Subtract $\dfrac{2x+1}{x-3}-\dfrac{x+1}{x-3}$

Solution: When subtracting algebraic fractions like these, we must be very careful with how the minus sign is handled. In this case we must subtract the entire numerator of the second fraction, so the minus sign must distribute to both parts of that numerator:

$$\frac{2x+1}{x-3}-\frac{x+1}{x-3}=\frac{(2x+1)-(x+1)}{x-3}=\frac{2x+1-x-1}{x-3}=\frac{x}{x-3}$$

Note in both examples how we are careful to add or subtract the entire numerators. In the case of addition we can get away with being a little sloppy about this, but when subtracting we must be careful to distribute the subtraction throughout the numerator of the second fraction.

To understand how to add or subtract rational expressions that don't have the same denominators, let's take a careful look at the exercise of adding the rational expressions shown below and to the left. The first thing we need to do to add them is to factor the denominators, as shown to the right below.

$$\frac{3x}{x^2-9}+\frac{5}{x^2+3x} \qquad\qquad \frac{3x}{(x+3)(x-3)}+\frac{5}{x(x+3)}$$

This allows us to take a careful look at the denominators. Now we want to make the denominators the same, and we should note right off that *both denominators contain the factor* $x+3$ *already.* What we then need to do is get the other factors, $x-3$ and x in both denominators. Since the first fraction's denominator is missing the x, we multiply that fraction by $\dfrac{x}{x}$ and we multiply the second fraction by $\dfrac{x-3}{x-3}$ to get $x-3$ in its denominator:

$$\frac{3x}{(x+3)(x-3)}\cdot\frac{x}{x}+\frac{5}{x(x+3)}\cdot\frac{x-3}{x-3}$$

We then finish as follows.

$$\frac{3x^2}{x(x+3)(x-3)}+\frac{5x-15}{x(x+3)(x-3)}=\frac{3x^2+5x-15}{x(x+3)(x-3)}$$

We usually distribute and combine like terms in the numerator, but it is customary to leave the denominator in factored form. If the numerator can be factored, we should factor it and see if any factors cancel, like shown in the next example.

◇ **Example 10.2(f):** Subtract $\dfrac{x+5}{4x+12} - \dfrac{x}{x^2-9}$

Solution:

$$\begin{aligned}
\frac{x+5}{4x+12} - \frac{x}{x^2-9} &= \frac{x+5}{4(x+3)} - \frac{x}{(x+3)(x-3)} \\
&= \frac{(x+5)(x-3)}{4(x+3)(x-3)} - \frac{4x}{4(x+3)(x-3)} \\
&= \frac{x^2+2x-15-4x}{4(x+3)(x-3)} \\
&= \frac{x^2-2x-15}{4(x+3)(x-3)} \\
&= \frac{(x-5)(x+3)}{4(x+3)(x-3)} \\
&= \frac{x-5}{4(x-3)}
\end{aligned}$$

◇ **Example 10.2(g):** Add $\dfrac{3x-2}{x^2-x-2} + \dfrac{4x-3}{x^2-4}$

Solution:

$$\begin{aligned}
\frac{3x-2}{x^2-x-2} + \frac{4x-3}{x^2-4} &= \frac{3x-2}{(x+1)(x-2)} + \frac{4x-3}{(x+2)(x-2)} \\
&= \frac{(3x-2)(x+2)}{(x+1)(x-2)(x+2)} + \frac{(4x-3)(x+1)}{(x+2)(x-2)(x+1)} \\
&= \frac{3x^2+4x-4}{(x+1)(x-2)(x+2)} + \frac{4x^2+x-3}{(x+2)(x-2)(x+1)} \\
&= \frac{3x^2+4x-4+4x^2+x-3}{(x+1)(x-2)(x+2)} \\
&= \frac{7x^2+5x-7}{(x+1)(x-2)(x+2)}
\end{aligned}$$

Section 10.2 Exercises

To Solutions

1. Perform the indicated addition or subtraction and simplify your answer.

   (a) $\dfrac{7}{x+5} - \dfrac{2}{x-1}$ (b) $\dfrac{3}{x-2} + \dfrac{4}{x+2}$ (c) $\dfrac{1}{x+4} - \dfrac{5}{x-3}$

2. Graph $f(x) = \dfrac{-4}{x+2}$, indicating clearly at least four points on the graph.

3. Perform each of the following divisions.

(a) $(x^2 - 5x + 1) \div (x - 2)$

(b) $(x^3 - 7x^2 + 12x) \div (x - 2)$

4. Solve each of the following quadratic equations, allowing complex solutions.

(a) $x^2 + 6x + 10 = 0$

(b) $9x^2 + 30x + 33 = 0$

(c) $x^2 + 1 = 4x$

10.3 Simplifying Complex Fractions

10. (c) Divide rational expressions; simplify complex rational expressions.

Any time we see a fraction like $\frac{15}{20}$ we know that we generally wish to have it in its reduced form, sometimes called "lowest terms." We usually just recognize that five will go into both the numerator and denominator, so we divide five into each to get $\frac{3}{4}$. The sequence of equalities below and to the left shows the actual steps of this process. Below and to the right we show how we could "un-reduce" a fraction if we wanted to!

$$\frac{15}{20} = \frac{3 \cdot 5}{4 \cdot 5} = \frac{3}{4} \cdot \frac{5}{5} = \frac{3}{4} \cdot 1 = \frac{3}{4} \qquad\qquad \frac{1}{3} = \frac{1}{3} \cdot 1 = \frac{1}{3} \cdot \frac{7}{7} = \frac{7}{21}$$

This "un-reducing" process is exactly what we did in the previous section to get common denominators. In this section we will sometimes do this to put things called **complex fractions** into what is considered a more proper form. A complex fraction is a fraction whose numerator and/or denominator contains fractions, like

$$\frac{1 + \frac{4}{x} - \frac{5}{x^2}}{\frac{1}{x} - \frac{6}{x^2}}$$

This is considered undesirable, so we correct it by multiplying both the top and bottom of the "larger" fraction by the smallest quantity that eliminates the "smaller" fractions. The following example shows how to do this for the fraction just given.

◇ **Example 10.3(a):** Simplify $\dfrac{1 + \frac{4}{x} - \frac{5}{x^2}}{\frac{1}{x} - \frac{6}{x^2}}$

Solution:

$$\frac{1 + \frac{4}{x} - \frac{5}{x^2}}{\frac{1}{x} - \frac{6}{x^2}} = \frac{1 + \frac{4}{x} - \frac{5}{x^2}}{\frac{1}{x} - \frac{6}{x^2}} \cdot \frac{x^2}{x^2} = \frac{1(x^2) + \frac{4}{x}(x^2) - \frac{5}{x^2}(x^2)}{\frac{1}{x}(x^2) - \frac{6}{x^2}(x^2)} = \frac{x^2 + 4x - 5}{x - 6}$$

Note that the numerator of the simplified fraction factors to $(x + 5)(x - 1)$. Because neither of those factors will cancel with the denominator, there is no need to factor the numerator.

If we were to have multiplied the top and bottom both by x we would have eliminated the fractions $\frac{4}{x}$ and $\frac{1}{x}$, but not $\frac{5}{x^2}$ and $\frac{6}{x^2}$. However, multiplying the top and bottom by x^2 will eliminate all the fractions. x^3 would as well, but we should always use the smallest power of x that will "get the job done." When finished with eliminating the complex fractions, we should always see if the result will simplify, as shown in the next example.

◇ **Example 10.3(b):** Simplify $\dfrac{1-\dfrac{1}{x}-\dfrac{12}{x^2}}{1-\dfrac{16}{x^2}}$

Solution:

$$\frac{1-\dfrac{1}{x}-\dfrac{12}{x^2}}{1-\dfrac{16}{x^2}} = \frac{1-\dfrac{1}{x}-\dfrac{12}{x^2}}{1-\dfrac{16}{x^2}} \cdot \frac{x^2}{x^2} = \frac{x^2-x-12}{x^2-16} = \frac{(x-4)(x+3)}{(x+4)(x-4)} = \frac{x+3}{x+4}$$

Recall that to divide a fraction by another fraction we simply multiply the first by the reciprocal of the second. But before multiplying we should cancel any like terms in the numerators and denominators. The next example shows this.

◇ **Example 10.3(c):** Divide $\dfrac{3}{8} \div \dfrac{9}{16}$

Solution: $$\frac{3}{8} \div \frac{9}{16} = \frac{3}{8} \cdot \frac{16}{9} = \frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{8}}} \cdot \frac{\overset{2}{\cancel{16}}}{\underset{3}{\cancel{9}}} = \frac{1}{1} \cdot \frac{2}{3} = \frac{2}{3}$$

To divide two rational expressions we do the same thing, but we need to factor the numerators and denominators of the fractions and cancel any like terms before multiplying.

◇ **Example 10.3(d):** Simplify $\dfrac{\dfrac{x+3}{x-5}}{\dfrac{x-1}{x-5}}$

Solution: $$\frac{\dfrac{x+3}{x-5}}{\dfrac{x-1}{x-5}} = \frac{x+3}{x-5} \div \frac{x-1}{x-5} = \frac{x+3}{\cancel{x-5}} \cdot \frac{\cancel{x-5}}{x-1} = \frac{x+3}{x-1}$$

The next example will be similar to this last one, except that we will be required to factor before cancelling.

◇ **Example 10.3(e):** Simplify $\dfrac{\dfrac{x-1}{x^2+2x-15}}{\dfrac{x^2+x-2}{x+5}}$

Solution:

$$\frac{\dfrac{x-1}{x^2+2x-15}}{\dfrac{x^2+x-2}{x+5}} = \frac{x-1}{x^2+2x-15} \div \frac{x^2+x-2}{x+5} = \frac{x-1}{x^2+2x-15} \cdot \frac{x+5}{x^2+x-2} =$$

$$\frac{\cancel{x-1}}{\cancel{(x+5)}(x-3)} \cdot \frac{\cancel{x+5}}{(x+2)\cancel{(x-1)}} = \frac{1}{(x-3)(x+2)}$$

Section 10.3 Exercises To Solutions

1. Simplify each of the following complex fractions.

   (a) $\dfrac{1-\dfrac{3}{x}}{1+\dfrac{5}{x}}$ (b) $\dfrac{1+\dfrac{3}{x}+\dfrac{2}{x^2}}{1-\dfrac{1}{x^2}}$

2. Simplify each of the following.

   (a) $\dfrac{\dfrac{x-2}{x^2-1}}{\dfrac{x^2+x-6}{x-1}}$ (b) $\dfrac{\dfrac{x^2-1}{x^2+6x+8}}{\dfrac{x^2+2x+1}{x^2-x-20}}$

3. Simplify each of the following.

   (a) $\dfrac{\dfrac{x+2}{x-3}}{\dfrac{x^2-4}{x+3}}$ (b) $\dfrac{1-\dfrac{2}{x}-\dfrac{15}{x^2}}{1+\dfrac{4}{x}+\dfrac{3}{x^2}}$ (c) $\dfrac{\dfrac{x+2}{x-5}}{\dfrac{x^2+5x+6}{x^2-25}}$

   (d) $\dfrac{1-\dfrac{3}{y}}{y-\dfrac{9}{y}}$ (e) $\dfrac{\dfrac{x+2}{x^2-4}}{\dfrac{x+7}{x^2+10x+21}}$ (f) $\dfrac{\dfrac{1}{x}+\dfrac{2}{x^2}}{1-\dfrac{4}{x^2}}$

4. Rationalize each denominator, reduce and/or simplify your answer when possible.

   (a) $\dfrac{10}{\sqrt{14}}$ (b) $\dfrac{6\sqrt{5}}{\sqrt{2}}$

5. Perform each of the indicated computations, giving your answers in $a + bi$ form.

(a) $(3-8i)(2+5i)$ (b) $(3-8i)-(2+5i)$ (c) $(3-8i)\div(2+5i)$

6. Simplify each square root, **allowing complex numbers**.

(a) $-\sqrt{12}$ (b) $\sqrt{-36}$ (c) $\sqrt{-50}$

7. Solve each of the following quadratic equations, allowing complex solutions.

(a) $x^2-10x+29=0$ (b) $x^2+25=8x$ (c) $x^2+4x+5=0$

8. Divide $(5x^3+8x^2-24x+8)\div(5x-2)$.

9. Perform the indicated addition or subtraction and simplify your answer.

(a) $\dfrac{8}{x-2}-\dfrac{5}{x-3}$ (b) $\dfrac{2}{x^2-5x+6}+\dfrac{3}{x^2-4}$

11 Miscellaneous

11.1 Completing the Square

> 11. (a) Solve quadratic equations by completing the square.

Consider the equation $(x+3)^2 = 16$, which can be solved as shown below and to the left.

$$\begin{aligned} (x+3)^2 &= 16 \\ (x+3)(x+3) &= 16 \\ x^2+6x+9 &= 16 \\ x^2+6x-7 &= 0 \\ (x+7)(x-1) &= 0 \\ x &= -7,\ 1 \end{aligned}$$

$$\begin{aligned} (x+3)^2 &= 16 \\ \sqrt{(x+3)^2} &= \pm\sqrt{16} \\ x+3 &= \pm 4 \\ x &= -3 \pm 4 \end{aligned}$$

$$x = -3+4 = 1, \qquad x = -3-4 = -7$$

Note that instead of expanding the left side and solving, we could have taken the square root of both sides of the original equation and solved as shown above and to the right. Of course we need to remember that when we take the square root of both sides of an equation we have to consider both the positive and negative square roots of one side, usually the side that is just a number.

In this section you will see another method for solving quadratic equations besides factoring or the quadratic formula; it is called **completing the square**. This method will appear harder than either of those two methods to most of you, but there are two reasons it is important to learn:

- It shows us where the quadratic formula came from.
- It is useful for other mathematical applications that most of you will see in future math courses.

To demonstrate the method, let's consider the equation $x^2+6x-7=0$ that appeared at one point above. Here's how we solve this equation by completing the square:

◇ **Example 11.1(a):** Use completing the square to solve $x^2+6x-7=0$.

Solution:

$x^2+6x-7 = 0$		original equation
$x^2+6x = 7$		add seven to both sides to get rid of the constant term on the left side
$x^2+6x+9 = 7+9$		add nine to both sides (you'll see later how to decide what number to add)
$(x+3)(x+3) = 16$		factor the left side,
$(x+3)^2 = 16$		and note that both factors are the same

$$\begin{aligned}
\sqrt{(x+3)^2} &= \pm\sqrt{16} && \text{take the square root of both sides} \\
x+3 &= \pm 4 && \text{and solve for } x \\
x &= -3 \pm 4 \\
x &= 1, \ -7
\end{aligned}$$

The key step in the above process is adding nine to $x^2 + 6x$ to create a quadratic expression that factors into two factors that are the same. That is the 'completing the square' step. Note that for any number b,

$$(x + \tfrac{1}{2}b)^2 = (x + \tfrac{1}{2}b)(x + \tfrac{1}{2}b) = x^2 + bx + (\tfrac{1}{2}b)^2$$

Thus if we have something like $x^2 + bx$, the thing that needs to be added to complete the square is $(\frac{1}{2}b)^2$. In the case we just had, $b = 6$ so we add $[\frac{1}{2}(6)]^2 = 3^2 = 9$ to both sides of the equation. Before actually completing the square, let's practice this process a few times.

◇ **Example 11.1(b):** Complete the square for $x^2 + 8x$, $x^2 - 12x$ and $x^2 + 2x$.

Solution: For $x^2 + 8x$ we take half of eight, four, and square that to get 16. When the square is completed we have $x^2 + 8x + 16$. For the next expression, note first that we don't need to take the negative into account, so we complete the square by adding $(\frac{1}{2} \cdot 12)^2 = 36$: $x^2 - 12x + 36$. Finally, for $x^2 + 2x$ we complete the square by adding $(\frac{1}{2} \cdot 2)^2 = 1$ to get $x^2 + 2x + 1$.

Now we see how completing the square works for cases where b is odd. It is really the same as the Example 11.1(a), except that some fractions are involved. We'll solve $x^2 = 3x + 10$; note that here we must start by getting x^2 and $3x$ both on the left side of the equation.

◇ **Example 11.1(c):** Use completing the square to solve $x^2 = 3x + 10$.

Solution:

$$(\tfrac{-3}{2})^2 = \tfrac{9}{4}$$

$$\begin{aligned}
x^2 &= 3x + 10 \\
x^2 - 3x &= 10 \\
x^2 - 3x + \tfrac{9}{4} &= 10 + \tfrac{9}{4} \\
(x - \tfrac{3}{2})(x - \tfrac{3}{2}) &+ \tfrac{40}{4} + \tfrac{9}{4} \\
(x - \tfrac{3}{2})^2 &= \tfrac{49}{4}
\end{aligned}$$

$$\begin{aligned}
\sqrt{(x - \tfrac{3}{2})^2} &= \pm\sqrt{\tfrac{49}{4}} \\
x - \tfrac{3}{2} &= \pm\tfrac{7}{2} \\
x &= \tfrac{3}{2} \pm \tfrac{7}{2} \\
x &= \tfrac{3}{2} + \tfrac{7}{2} = \tfrac{10}{2} = 5, \text{ or} \\
x &= \tfrac{3}{2} - \tfrac{7}{2} = -\tfrac{4}{2} = -2
\end{aligned}$$

Completing the square can also be used to solve quadratic equations whose solutions contain roots:

◇ **Example 11.1(d):** Use completing the square to solve $x^2 - 10x + 22 = 0$.

Solution:

$(\frac{-10}{2})^2 = 25$

$$\begin{aligned} x^2 - 10x + 22 &= 0 \\ x^2 - 10x &= -22 \\ x^2 - 10x + 25 &= 3 \\ (x-5)(x-5) &= 3 \\ (x-5)^2 &= 3 \end{aligned}$$

$$\begin{aligned} \sqrt{(x-5)^2} &= 3 \\ x - 5 &= \pm\sqrt{3} \\ x &= 5 \pm \sqrt{3} \end{aligned}$$

$$x = 5 + \sqrt{3}, \qquad x = 5 - \sqrt{3}$$

Now we see that completing the square can be used for quadratic equations that have complex number solutions:

◇ **Example 11.1(e):** Use completing the square to solve $x^2 - 4x + 13 = 0$.

Solution:

$$\begin{aligned} x^2 - 4x + 13 &= 0 \\ x^2 - 4x &= -13 \\ x^2 - 4x + 4 &= -13 + 4 \\ (x-2)(x-2) &= -9 \\ (x-2)^2 &= -9 \end{aligned}$$

$$\begin{aligned} \sqrt{(x-2)^2} &= -9 \\ x - 2 &= \pm\sqrt{-9} \\ x &= 2 \pm 3i \end{aligned}$$

$$x = 2 + 3i, \qquad x = 2 - 3i$$

Section 11.1 Exercises

To Solutions

1. Suppose that we wish to complete the square for $x^2 - 10x$.

   (a) Take half of ten, square it, and add the result to $x^2 - 10x$. Do that - at this point you have completed the square.

   (b) Factor the result. Both factors should be the same. Note that the number in each factor is what you got when you took half of ten!

   (c) How would things change if you completed the square for $x^2 + 10x$?

2. Complete the square for each of the following, then factor the resulting quadratic. The last few will involve working with fractions, since the value of a is odd.

   (a) $x^2 + 2x$ (b) $x^2 - 12x$ (c) $x^2 - 4x$

   (d) $x^2 - 9x$ (e) $x^2 - x$ (f) $x^2 + 8x$

3. Solve each of the following by completing the square. Refer to the above examples at first, but try to get to where you don't need it any more. Check your answers by solving by factoring.

   (a) $x^2 + 4x + 3 = 0$ (b) $x^2 + 6x = 16$ (c) $x^2 = 4x + 12$

4. Solve each of the following by completing the square.

   (a) $x^2 - 2x - 4 = 0$ (b) $x^2 - 6x + 7 = 0$ (c) $x^2 - 10x + 13 = 0$

   (d) $x^2 + 2x = 48$ (e) $x^2 = 14x + 51$

5. Solve each of the following quadratic equations by completing the square, **allowing complex solutions**.

   (a) $x^2 + 13 = -6x$ (b) $x^2 + 10x + 26 = 0$ (c) $x^2 + x = 6$

6. Simplify each of the following.

   (a) $\dfrac{1 - \dfrac{1}{x}}{1 + \dfrac{4}{x} - \dfrac{5}{x^2}}$ (b) $\dfrac{\dfrac{1}{x} + \dfrac{3}{x^2}}{1 - \dfrac{6}{x} - \dfrac{7}{x^2}}$ (c) $\dfrac{\dfrac{x^2 - 4}{3}}{\dfrac{x + 2}{6}}$

11.2 Linear Inequalities

> 11. (b) Solve linear inequalities in one unknown.

Let's recall the notation $a > b$, which says that the number a is greater than the number b. Of course that is equivalent to $b < a$, b is less than a. So, for example, we could write $5 > 3$ or $3 < 5$, both mean the same thing. Note that it is not correct to say $5 > 5$, since five is not greater than itself! On the other hand, when we write $a \geq b$ it means a is greater than *or equal to* b, so we could write $5 \geq 5$ and it would be a true statement. Of course $a \geq b$ is equivalent to $b \leq a$.

In this section we will consider algebraic inequalities; that is, we will be looking at inequalities that contain an unknown value. An example would be

$$3x + 5 \leq 17$$

and our goal is to *solve* the inequality. This means to find all values of x that make this true. Let's begin by finding some values that make the inequality true, and some others that don't.

◇ **Example 11.2(a):** Find some values of x that make the inequality $3x + 5 \leq 17$ true, and some that make it false.

Solution: Let's try $x = 0$: $3(0) + 5 \stackrel{?}{\leq} 17 \quad \Rightarrow \quad 5 \leq 17$ True for $x = 0$

How about $x = 5$: $3(5) + 5 \stackrel{?}{\leq} 17 \quad \Rightarrow \quad 20 \nleq 17$ False for $x = 5$

Try $x = 1$: $3(1) + 5 \stackrel{?}{\leq} 17 \quad \Rightarrow \quad 8 \leq 17$ True for $x = 1$

And $x = 6$: $3(6) + 5 \stackrel{?}{\leq} 17 \quad \Rightarrow \quad 23 \nleq 17$ False for $x = 6$

Maybe $x = 2$: $3(2) + 5 \stackrel{?}{\leq} 17 \quad \Rightarrow \quad 11 \leq 17$ True for $x = 2$

0, 1 and 2 make the inequality true, 5 and 6 do not.

From the above it appears that somewhere between two and five is a "dividing line" between values that make the inequality true and ones that don't. The equation $3x+5 = 17$ has solution $x = 4$, so four satisfies the "or equal to" part of $\leq$. If we were to choose a value of x that was a little larger than four, like 4.1, then $3x$ will be a little larger than twelve, and $3x + 5$ will be a little larger than 17. At this point it should be clear that all the numbers less than or equal to four should make the inequality true. We usually "say" this by writing $x \leq 4$ or $(-\infty, 4]$.

It turns out that we don't have to go through this whole thought process every time we want to solve a simple inequality like this. Instead, we can just solve the inequality in almost the same way as we would solve the equation formed by replacing the $\leq$ symbol with $=$. (You'll see later why the word "almost" was used here.) The next example shows this.

⋄ **Example 11.2(b):** Solve the inequality $3x+5 \leq 17$

Solution:

$$\begin{aligned} 3x+5 &\leq 17 \\ 3x &\leq 12 \\ x &\leq 4 \end{aligned}$$

⋄ **Example 11.2(c):** Solve the inequality $8x-3(x-1) > 2x+10$

Solution:

$$\begin{aligned} 8x-3(x-1) &> 2x+10 \\ 8x-3x+3 &> 2x+10 \\ 5x+3 &> 2x+10 \\ 3x &> 7 \\ x &> \tfrac{7}{3} \end{aligned}$$

There is a pretty good way to determine whether you were probably correct in solving an inequality. The solution $x > \frac{7}{3}$ means that anything greater than $\frac{7}{3}$ should make the inequality true, and anything less than $\frac{7}{3}$ should make it false. Since $\frac{7}{3} = 2\frac{1}{3}$, three should make the inequality true and two should make it false. Let's check:

$$\begin{aligned} 8(3)-3(3-1) &\stackrel{?}{>} 2(3)+10 \\ 24-3(2) &\stackrel{?}{>} 6+10 \\ 18 &> 16 \end{aligned} \qquad\qquad \begin{aligned} 8(2)-3(2-1) &\stackrel{?}{>} 2(2)+10 \\ 16-3(1) &\stackrel{?}{>} 4+10 \\ 13 &\not> 14 \end{aligned}$$

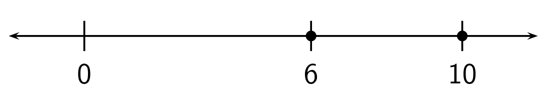
The symbol $\not>$ means "not greater than." The above computations indicate that our solution is probably correct.

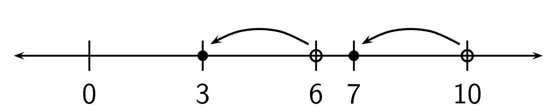
Suppose now that we use the method that we used in Examples 11.2(b) and (c) to solve the inequality $-2x+1 \leq 7$. The result is $x \leq -3$, so -4 should make the inequality true and -2 should make it false. Let's see:

$$\begin{aligned} -2(-4)+1 &\stackrel{?}{\leq} 7 \\ 8+1 &\stackrel{?}{\leq} 7 \\ 9 &\not\leq 7 \end{aligned} \qquad\qquad \begin{aligned} -2(-2)+1 &\stackrel{?}{\leq} 7 \\ 4+1 &\stackrel{?}{\leq} 7 \\ 4+1 &\leq 7 \end{aligned}$$

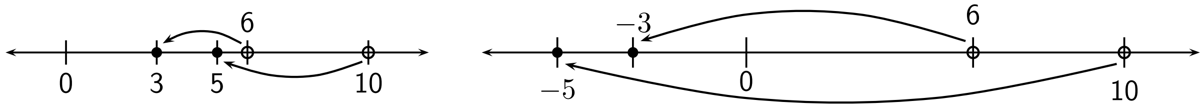
Hmmm... Something appears to be wrong here!

Consider the *true* inequality $6 < 10$; if we put dots at each of these values on the number line the larger number, ten, is farther to the right, as shown to the left below. If we subtract, say, three from both sides of the inequality we get $3 < 7$. The effect of this is to shift each number to the left by five units, as seen in the diagram below and to the right, so their relative positions don't change.





Suppose now that we divide both sides of the same inequality $6 < 10$ by two. We then get $3 < 5$, which is clearly true, and shown on the diagram below and to the left. If, on the other hand, we divide both sides by -2 we get $-3 < -5$. This is not true! The diagram below and to the right shows what is going on here; when we divide both sides by a negative we reverse the positions of the two values on the number line, so to speak.



The same can be seen to be true if we multiply both sides by a negative. This leads us to the following principle.

Solving Linear Inequalities

To solve a linear inequality we use the same procedure as for solving a linear equation, *except that we reverse the direction of the inequality whenever we multiply or divide BY a negative.*

◇ **Example 11.2(d):** Solve the inequality $3x - 5(x-1) > 12$

Solution:

$$\begin{aligned} 3x - 5(x-1) &> 12 \\ 3x - 5x + 5 &> 12 \\ -2x + 5 &> 12 \\ -2x &> 7 \\ x &< -\tfrac{7}{2} \end{aligned}$$

Section 11.2Exercises To Solutions

1. (a) Solve $4x > -12$. Then check your answer by checking a value that should be a solution and one that should not, as we have been doing in this section.

   (b) Solve $-5x > 30$ and check your answer in the same way again.

   (c) The direction of the inequality should change in only one of parts (a) and (b). Which part is it? Both exercises have negatives in them; why does the inequality change direction in one case, but not the other?

2. (a) Solve $-3x + 7 \leq 16$.

   (b) Solve the same inequality by first adding $3x$ to both sides, then subtracting 16 from both sides. This method can always be used to avoid dividing by a negative!

3. Solve each of the following inequalities.

(a) $5x + 4 \geq 18$

(b) $6x - 3 < 63$

(c) $8 - 5x \leq -2$

(d) $4x + 2 > x - 7$

(e) $5x - 2(x - 4) > 35$

(f) $5y - 2 \leq 9y + 2$

(g) $3(x - 2) + 7 < 2(x + 5)$

(h) $7 - 4(3x + 1) \geq 2x - 5$

(i) $4(x + 3) \geq x - 3(x - 2)$

(j) $-4x + 3 < -2x - 9$

(k) $8 - 5(x + 1) \leq 4$

(l) $7 - 2x \leq 13$

4. Solve each of the following quadratic equations, allowing complex solutions. **Use the quadratic formula.**

(a) $2x^2 - 6x + 5 = 0$ (b) $x^2 - 2x + 5 = 0$ (c) $x^2 + 4x + 11 = 0$

5. Divide $(x^3 + 9x^2 + 26x + 24) \div (x + 2)$

6. Perform the indicated addition or subtraction and simplify your answer.

(a) $\dfrac{5}{x^2 + 4x} - \dfrac{2}{x^2 - 16}$ (b) $\dfrac{3}{x^2 - 2x} - \dfrac{5}{x - 2}$

7. Simplify each of the following.

(a) $\dfrac{\dfrac{1}{x} - \dfrac{2}{x^2}}{1 + \dfrac{3}{x} - \dfrac{10}{x^2}}$ (b) $\dfrac{\dfrac{x^2 - 4}{x + 3}}{\dfrac{x + 2}{2x + 6}}$ (c) $\dfrac{1 - \dfrac{9}{x^2}}{1 - \dfrac{1}{x} - \dfrac{6}{x^2}}$

8. Solve each of the following by completing the square. Check your answers by solving by factoring.

(a) $x^2 + 5x = 14$ (b) $x^2 = 2x + 3$ (c) $x^2 + 5x = 6$

9. Solve each of the following by completing the square.

(a) $x^2 + 6x = 19$ (b) $x^2 + 12x - 36 = 0$ (c) $x^2 = 4x + 20$

10. Solve each of the following quadratic equations by completing the square, **allowing complex solutions**.

(a) $x^2 + 14x + 53 = 0$ (b) $x^2 = 2x - 8$

A Solutions to Exercises

A.1 Chapter 1 Solutions

Section 1.1 Solutions Back to 1.1 Exercises

1. (a) 27 (b) $\frac{1}{64}$ (c) 100 (d) -125
   (e) -125 (f) 25 (g) -25 (h) $\frac{16}{9}$
2. (a) 262,144 (b) $-59,049$ (c) 59,049 (d) 244,140,625

Section 1.2 Solutions Back to 1.2 Exercises

1. (a) -16 (b) 25 (c) -30 (d) $\frac{19}{12}$ (e) -13 (f) -17
3. (a) 36 (b) $\frac{21}{2}$ or $10\frac{1}{2}$ (c) $\frac{15}{2}$ or $7\frac{1}{2}$ (d) 6
4. (a) -73 (b) -72 (c) 44 (d) 54 (e) -30
   (f) $\frac{5}{3}$ (g) -14 (h) 3 (i) -21
6. (a) -1 (b) 39 (c) 64 (d) 44
7. (a) 1125.68 (b) 37
8. (a) 8, 8, the results are the same (b) 20, 20, the results are the same (c) 50, 50

Section 1.3 Solutions Back to 1.3 Exercises

1. (a) 4 (b) $-2x + 10$ (c) 4
2. (a) $x - 7$ (b) $-x + 26$ (c) $9t - 4$ (d) $-2x + 2$
3. (a) $-27x + 14$ (b) $30x + 12$ (c) $-8x + 20$
   (d) $3x - 9$ (e) $8x + 21$ (f) $9x + 15$
4. (a) -42 (b) $-28x - 10$ (c) $2x - 19$
   (d) -8 (e) -25 (f) $6x - 2$

Section 1.4 Solutions Back to 1.4 Exercises

1. (a) $12x^8$ (b) $8x^3y^{15}$ (c) $9y^{10}$ (d) $16x^{12}y^4$
2. (a) x^7 (b) y^{14} (c) $s^{10}t^{15}$ (d) $\dfrac{r^5}{s^3}$ (e) $-\dfrac{3y^3}{2x}$
   (f) $3z^{10}$ (g) y^3 (h) $\dfrac{8y^3}{x^6}$ (i) $\dfrac{1}{y^3}$
3. (a) 9 (b) -9 (c) $\frac{125}{8}$ (d) 75 (e) -12 (f) 4 (g) $-\frac{7}{4}$

4. (a) 38 (b) 1

5. (a) $2x-15$ (b) $-6x+32$ (c) $7x-8$

Section 1.5 Solutions Back to 1.5 Exercises

1. (a) $\frac{1}{25}$ (b) $\frac{1}{16}$ (c) $\frac{1}{16}$ (d) $-\frac{1}{9}$
2. (a) $\frac{1}{3}$ (b) $\frac{16}{9}$ (c) 1 (d) $\frac{1}{25}$
   (e) $\frac{3}{4}$ (f) $\frac{1000}{27}$ (g) 16 (h) -32
3. (a) 3.44×10^5 (b) 1.567×10^{14}
4. (a) 98500 (b) 73.28
5. (a) 4.3×10^3 (b) 0.00167 (c) 4.3×10^{-4} (d) 1.507×10^5
   (e) 3.69×10^{-5} (f) 2650 (g) 31,420,000 (h) 6.2×10^{-3}
6. (a) s^2t^3 (b) $\dfrac{9}{2x^{13}}$ (c) $\dfrac{b^3}{2}$ (d) $\dfrac{3b^{10}}{2a^3c^3}$ (e) $10x^5$ (f) $-125z^{13}$
7. (a) -9 (b) -20 (c) $\frac{35}{2}$ (or $17\frac{1}{2}$) (d) 6 (e) $\frac{12}{7}$ (f) -39
8. Same as for Exercise 7.
9. (a) 12 (b) $\frac{1}{5}$

A.2 Chapter 2 Solutions

Section 2.1 Solutions Back to 2.1 Exercises

1. (a) -2 is not a solution, 4 is not
   (b) 4 is a solution, 0 is not
   (c) -1 is a solution
   (d) Both $\frac{3}{4}$ and $-\frac{3}{4}$ are solutions
   (e) 4 is not a solution, -4 is not
   (f) 1 is not a solution, -1 is
   (g) 4 is not a solution, -6 is
   (h) Both -2 and 2 are solutions
2. (a) 2 is not a solution (b) 7 is a solution
   (c) Another solution is $x=-2$
3. (a) 5 is a solution (b) 0 is a solution
4. (a) Yes, -5 is a solution. (b) 4 is also a solution
5. (a) 1 (b) -125 (c) $-\frac{1}{2}$ (d) $\frac{9}{16}$
6. (a) 5.14×10^{-2} (b) $894,000$ (c) 0.00065 (d) 7.3×10^3

Section 2.2 Solutions Back to 2.2 Exercises

1. (a) $-\frac{1}{2}$ (b) 10 (c) -5 (d) 5 (e) $\frac{10}{11}$
   (f) $\frac{4}{3}$ (g) -16 (h) $\frac{7}{3}$

2. (a) $4x - 12$ (b) $\frac{1}{9}$ (c) 16 (d) 9 (e) $7x + 15$ (f) -9

3. -36

4. (a) $16x^{28}$ (b) $\dfrac{2}{z}$ (c) $-21s^6$ (d) $-\dfrac{3x^3}{2y}$

Section 2.3 Solutions Back to 2.3 Exercises

1. (a) $9x^2 - 3x - 4$ (b) $-3x^2 + 10x + 2$ (c) $x^3 - 3x^2 - 9x - 1$
   (d) $4x^3 - 11x^2 + 7x - 3$

2. (a) $2x^2 + 2x + 4$ (b) $-x^2 + 4x + 10$ (c) $a^2 + 5a + 4$

3. (a) $x^2 + 5x - 14$ (b) $5x^3 + 7x^2 - 2x$ (c) $2x^3 + 19x^2 - 58x - 5$
   (d) $x^2 + 6x + 9$ (e) $3x^2 + 17x + 20$ (f) $3x^2 + 14x - 5$

4. (a) -1 (b) $-\frac{6}{5}$ (c) -18

Section 2.4 Solutions Back to 2.4 Exercises

1. $4x^2(2x^2 - x + 4)$

2. (a) $(x-5)(x+2)$ (b) $(x-4)(x-3)$ (c) $(x+3)(x-5)$
   (d) $(x+9)(x+1)$ (e) $(x-4)(x-7)$ (f) $(x-2)(x+6)$

3. (a) $(7x-4)(x+1)$ (b) $(3x-2)(2x+1)$ (c) $2x+1)(3x+5)$
   (d) $(3x+2)(x-7)$ (e) $(10x-3)(x+3)$ (f) $(5x-2)(3x-3)$

4. (a) $(x+3)(x-3)$ (b) $(5x+1)(5x-1)$ (c) $(3x+7)(3x-7)$

5. (a) $x(x+4)(x+1)$ (b) $3x(x+7)(x-2)$ (c) $x^2(x+5)(x+1)$

6. (a) $(x+2)(x^2-3)$ (b) $(2x+3)(x+1)(x-1)$ (c) $(x+2)(x+5)(x-5)$

7. (a) $4x^2(x^2+x+1)$ (b) $(5x-1)(2x-3)$ (c) $(3x-5)(x+1)$
   (d) $(9x+2)(x+1)$ (e) $(2x^2+5)(3x-2)$ (f) $(4x+7)(4x-7)$
   (g) $(3-x)(1+8x)$ (h) Can't be factored (i) $12x(x^2+2x+4)$
   (j) $(3x+1)(2x-7)$ (k) $(2x-1)(x-7)$ (l) $(x+2)(x-2)(x+3)$
   (m) $(2y-3)(5y+5)$ (n) $(1+5x)(1-5x)$ (o) Can't be factored
   (p) $(2x+3)(3x-5)$ (q) $(7x+4)(x+1)$ (r) $(2-x)(4+3x)$

8. (a) 36 (b) -12

9. (a) $\dfrac{8}{27}$ (b) $\frac{1}{25}$ (c) 1 (d) $-\frac{1}{25}$

10. (a) $-4x^2 - 13x + 5$ (b) $-12x - 7$

11. (a) $6x^3 - 29x^2 + 43x - 20$ (b) $9x^2 + 30x + 25$ (c) $3x^3 - 21x^2 - 15x$
(d) $6x^3 + 7x^2 - 5$ (e) $x^2 - 2x + 1$ (f) $5x^3 - 29x^2 - 7x + 6$

Section 2.5 Solutions Back to 2.5 Exercises

1. (a) $x = -2, 6$ (b) $x = 2, 4$ (c) $x = -4, 6$
2. (a) $x = 0, -5, 2$ (b) $x = -1, -7$ (c) $x = 0, -2, 3$
3. (a) $x = 12, 1$ (b) $x = 3, 5$ (c) $x = 4, -4$
(d) $x = 0, -\frac{1}{5}$ (e) $x = -\frac{1}{3}, 7$ (f) $x = -1, -4, 4$
(g) $a = -3, -5$ (h) $x = \frac{3}{2}, -5$ (i) $x = 0, -1, -2$
(j) $x = -1, -6$ (k) $x = 0, 2$ (l) $x = -\frac{1}{2}, 3, -3$
4. (a) $56x^{19}$ (b) $4y^7$ (c) $\dfrac{25s^8t^{16}}{16}$ (d) $\dfrac{u^3}{2}$
5. (a) 237 (b) 4.9×10^{-3} (c) 0.16 (d) 5.3×10^4
6. (a) $x = -5$ (b) $x = \frac{1}{4}$ (c) $x = -17$

A.3 Chapter 3 Solutions

Section 3.1 Solutions Back to 3.1 Exercises

1. (a) $x \neq 2, -2$ (b) $x \neq 1, 5$ (c) $x \neq -5, 5$
(d) $x \neq 4, 5$ (e) $x \neq 0, -5$ (f) $x \neq 2$
2. (a) $\dfrac{x-1}{x-2}$ (b) $\dfrac{x+3}{x-1}$ (c) $\dfrac{5}{x-5}$
(d) $\dfrac{x-3}{x-5}$ (e) $\dfrac{x+7}{2}$ (f) $\dfrac{x+2}{x-2}$
3. (a) $(x+5)(x-5)$ (b) $(2x+3)(2x-3)$ (c) $(4x+1)(4x-1)$
4. (a) $5x^2(2x+1)(2x-1)$ (b) $3x(5x-4)(2x+3)$
5. (a) $(3x+1)(x+2)(x-2)$ (b) $(x-5)(x+3)(x-3)$ (c) $(2x+7)(x+1)(x-1)$
6. (a) $x = \frac{5}{2}, -\frac{5}{2}$ (b) $x = 0, 11, -2$ (c) $x = -5, -1, 1$
(d) $x = -\frac{1}{2}, 3$ (e) $x = -3, 7$ (f) $x = 3, -5$

Section 3.2 Solutions Back to 3.2 Exercises

1. (a) $\dfrac{1}{x-3}$ (b) $\dfrac{x+3}{x-2}$ (c) $x+5$ (d) $\dfrac{x+3}{x-2}$
2. (a) $15-2x$ (b) $-8x-4$ (c) $3x^2+22x-35$ (d) $-3x+15$
3. (a) $\frac{1}{6}$ (b) $\dfrac{25}{16}$ (c) 1 (d) $\dfrac{4}{9}$
4. (a) $4x^2-4x+1$ (b) $3x^2-3x+9$
   (c) $x^3+x^2-18x+10$ (d) $-x^2+5x+3$
5. (a) $x \neq -2, 2$ (b) $x \neq 5, -2$ (c) $x \neq -1, -2$
6. (a) $\dfrac{x-5}{x-2}$ (b) $\dfrac{1}{x-5}$ (c) $\dfrac{x-4}{x+2}$

Section 3.3 Solutions Back to 3.3 Exercises

1. (a) $x \neq 0, \quad x=4$ (b) $x \neq 3, -5, \quad x=-\frac{11}{7}$
   (c) $x \neq -7, \quad x=2$ (d) $x \neq 3, 0, \quad x=-2, -3$
   (e) $x \neq 1, -1, \quad x=3$ (f) $x \neq 2, -4, \quad x=3$
2. (a) $x=0, \frac{4}{3}$ (b) $x=\frac{4}{3}$ (c) $x=-\frac{1}{2}, -3, 3$ (d) $x=2, -2$
   (e) $t=\frac{3}{2}$ (f) $x=\frac{1}{2}, -3$ (g) $x=-2, \frac{5}{2}$ (h) $a=\frac{5}{4}$

A.4 Chapter 4 Solutions

Section 4.1 Solutions Back to 4.1 Exercises

1. (a) 5 (b) 3 (c) 2 (d) DNE (e) 1
   (f) -7 (g) -3 (h) -3 (i) $\frac{4}{5}$ (j) DNE
2. (a) $3\sqrt{5}$ (b) $2\sqrt{2}$ (c) $\sqrt{15}$ (d) $6\sqrt{2}$ (e) $7\sqrt{2}$
   (f) $2\sqrt{3}$ (g) DNE (h) $5\sqrt{3}$
3. (a) $x \neq 3$ (b) $x \neq -2, -3$ (c) $x \neq -1, -3$
4. (a) $x+1$ (b) $\dfrac{x-3}{x+2}$ (c) $\dfrac{2x-8}{x+1}$
5. (a) $3x$ (b) $\dfrac{x^2+9x+20}{x-3}$
6. (a) $x \neq 0, \quad x=-1, -6$ (b) $x \neq 3, -3, \quad x=5$

Section 4.2 Solutions

Back to 4.2 Exercises

1. (a) $34 - 6\sqrt{5}$ (b) 13 (c) $15 + 5\sqrt{2} - 6\sqrt{3} - 2\sqrt{6}$
   (d) $8 - 4\sqrt{3} + 2\sqrt{7} - \sqrt{21}$
2. (a) $3\sqrt{2}$ (b) 5 (c) $\sqrt{35}$
   (d) $2\sqrt[3]{10}$ (e) $30\sqrt{2}$ (f) $10\sqrt{3}$
   (g) $20 + 4\sqrt{7}$ (h) $6 - 3\sqrt{3} + 2\sqrt{5} - \sqrt{15}$
   (i) $5 - \sqrt{2} - 5\sqrt{10} + 2\sqrt{5}$
3. (a) $x \neq 3, -3,\quad x = -5, 6$ (b) $x \neq 0, 1,\quad x = 5$
4. (a) -1 (b) 1 (c) 0 (d) 10 (e) $\frac{8}{5}$
5. (a) $4\sqrt{2}$ (b) $2\sqrt{6}$ (c) $3\sqrt{2}$ (d) $5\sqrt{2}$

Section 4.3 Solutions

Back to 4.3 Exercises

1. (a) $x = -\frac{3}{2},\ 5$ (b) $x = -\frac{1}{4},\ -3$
2. (a) $x = 1 + \sqrt{5},\ 1 - \sqrt{5}$ (b) $x = 3 + \sqrt{2},\ 3 - \sqrt{2}$
   (c) $x = -5 + 2\sqrt{3},\ -5 - 2\sqrt{3}$ (d) $x = \frac{1}{2} + \frac{\sqrt{3}}{2},\ \frac{1}{2} - \frac{\sqrt{3}}{2}$
   (e) $x = \frac{2}{3} + \frac{\sqrt{5}}{3},\ \frac{2}{3} - \frac{\sqrt{5}}{3}$ (f) $x = -\frac{1}{5} + \frac{3\sqrt{7}}{5},\ -\frac{1}{5} - \frac{3\sqrt{7}}{5}$
3. (a) $x = -5, -3, 3$ (b) $x = 4, -\frac{3}{2}$ (c) $x = \frac{1}{3}, -\frac{1}{3}$ (d) $x = 0, 2, 5$
4. (a) $x \neq 1, -4,\quad x = -8$ (b) $x \neq 4, -4,$ no solution
5. (a) -4 (b) $12 + 2\sqrt{2} - 18\sqrt{5} - 3\sqrt{10}$

Section 4.4 Solutions

Back to 4.4 Exercises

1. (a) $x = 2$ (b) $x = -2, 3$ (c) $x = -\frac{3}{2}$
2. (a) $x = 4$ (b) $x = -2, -5$
3. (a) $x = -\frac{3}{4}$ (b) $x = \frac{4}{5}$ (c) $x = 1$ $x = -4$ is not a solution
   (d) $x = -3$ (e) no solution ($x = \frac{2}{3}$ doesn't check) (f) $x = 2, 3$
4. (a) $\dfrac{x+3}{x+7}$ (b) $x^2 + 7x + 10$ (c) $-2x + 17$ (d) $27 + x^2$
5. (a) $x = -4 + \sqrt{3},\ -4 - \sqrt{3}$ (b) $x = -3 + \sqrt{2},\ -3 - \sqrt{2}$

A.5 Chapter 5 Solutions

Section 5.1 Solutions Back to 5.1 Exercises

1. 6 inches 2. 88.2 square inches 3. 5 inches

4. 2.6 inches 5. (a) \$1.10 (b) \$21.05

6. \$2870.50 7. (a) \$1416 (b) 22.2 years

8. (a) 1, 5 sec (b) 3 sec (c) 0.62, 5.38 sec (d) never (e) 6.04 seconds

9. \$4 or \$8 10. 12.9 inches

11. (a) DNE (b) DNE (c) -4 (d) 10 (e) 6

12. (a) $-3-5\sqrt{5}$ (b) $37+20\sqrt{3}$

13. (a) $7-7\sqrt{5}+2\sqrt{3}-2\sqrt{15}$ (b) $-2-5\sqrt{7}$ (c) $7-4\sqrt{3}$

14. (a) $x=3$ (b) no solution ($x=-4$ doesn't check)
    (c) $x=-1$ $x=-5$ doesn't check (d) $x=-5$

Section 5.2 Solutions Back to 5.2 Exercises

1. (a) 22.9 years (b) $r=\dfrac{A-P}{Pt}$ 2. $R=\dfrac{PV}{nT}$ 3. $b=\dfrac{ax+8}{x}$

4. $l=\dfrac{P-2w}{2}$ 5. $r=\dfrac{C}{2\pi}$

6. (a) $y=-\frac{3}{4}x-2$ (b) $y=-\frac{5}{2}x-5$ (c) $y=\frac{3}{2}x+\frac{5}{2}$
   (d) $y=\frac{3}{4}x-2$ (e) $y=-\frac{3}{2}x+\frac{5}{2}$

7. (a) $x=-\frac{10}{3}$ (b) $x=\dfrac{-4}{a-b}$

8. (a) $x=\dfrac{d-b}{a-c}$ (b) $x=\dfrac{8.99}{2.065}=4.35$ (c) $x=\dfrac{c}{a+b}$ (d) $x=\dfrac{-10}{a-c}$

9. $P=\dfrac{A}{1+rt}$

10. (a) $\sqrt{17}$ (b) $\frac{2\sqrt{5}}{3}$ (c) $-6\sqrt{2}$

11. (a) $x=3+2\sqrt{7},\ 3-2\sqrt{7}$ (b) $x=1+3\sqrt{2},\ 1-3\sqrt{2}$

12. (a) $\frac{1}{2},\frac{5}{2}$ seconds (b) 3 seconds (c) 33.2 feet (d) 1, 2 seconds
    (e) $\frac{3}{2}$ seconds (f) never

Section 5.3 Solutions Back to 5.3 Exercises

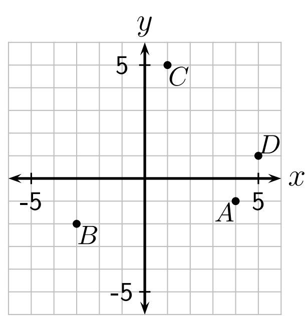
1. **Equation:** $2w + 2(2w - 3.2) = 57.1$ **Answer:** $w = 10.6$ inches, $l = 18$ inches
2. **Equation:** $0.055x = 10.42$ **Answer:** $x = \$189.95$
3. **Equation:** $x^2 + (2x)^2 = 15^2$ **Answer:** $x = 3\sqrt{5} = 6.7$ feet (one leg), other leg is 13.4 feet or $6\sqrt{5}$ feet
4. **Equation:** $x + 0.02x = 7.37$ **Answer:** $x = \$7.22$
5. **Equation:** $\frac{1}{2}x + (\frac{1}{2}x + 1.4) + x = 42$ **Answer:** $x = 20.3$ (longest side) shortest side is 10.15 and middle side is 11.55
6. (a) $x = \frac{9}{2}$ (b) $x = 8$ $x = -1$ doesn't check (c) $x = \frac{1}{2}$
7. (a) $x = \dfrac{-8}{a-b}$ (b) $x = \dfrac{bc-7}{a-b}$

A.6 Chapter 6 Solutions

Section 6.1 Solutions Back to 6.1 Exercises

1. Any of $(-4,-20), (-3,-17), (-2,-14), (-1,-11), (0,-8), (1,-5), (2,-2), (3,1), (4,4), \ldots$
2. Any of $(0,5), (1,4), (-1,4), (2,1), (-2,1), (3,-4), (-3,-4), \ldots$ The same value of y results from x and $-x$
3. (a) $A(4,2),\ B(-3,1),\ C(-3,-2),\ D(1,-5)$

3. (b)



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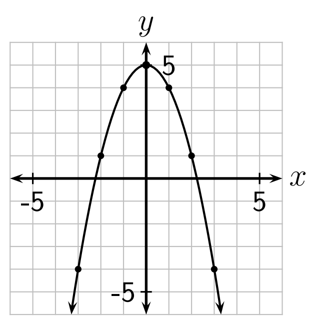

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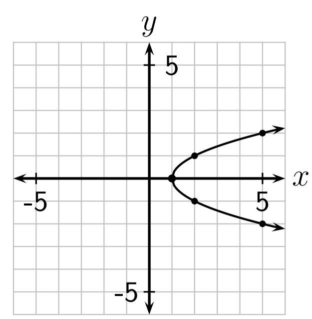
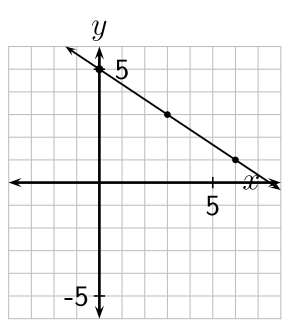
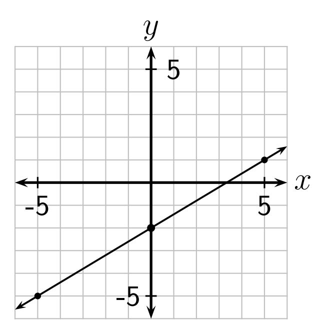


6. (a) $y = -\frac{2}{3}x + 5$ (b) multiples of three (c) $(-6,9), (-3,7), (0,5), (3,3), (6,1), (9,-1)$

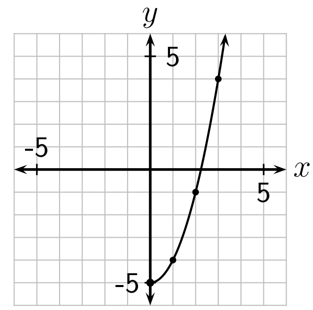
6. (d)



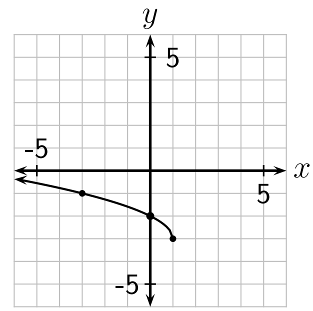
7.



8. (a) $(0,-5)$, $(1,-4)$, $(2,-1)$, $(3,4)$, $(4,11)$, ...
   (b) The smallest value that can be used for y is -5. The solution in that case is $(0,-5)$.
   (c) $(\sqrt{13},7)$
   (d) $(3.61,7)$
   (e) See graph to the right.



9. (a) $y=\sqrt{1-x}-3$
   (b) The *largest* value of x that can be used is $x=1$, giving the solution $(1,-3)$.
   (c) See graph to the right.



10. (a) -1 (b) $159-30\sqrt{6}$ (c) -65

11. **Equation:** $x+0.40x=59.95$ **Answer:** $x=\$42.82$

12. **Equation:** $x^2+8^2=(x+2)^2$ **Answer:** 8, 15, 17

Section 6.2 Solutions

Back to 6.2 Exercises

1. x-intercept: -3 y-intercept: 5

2. x-intercepts: $-1,3$ y-intercept: -3

3. x-intercept: $\frac{9}{2}$ or $4\frac{1}{2}$ y-intercept: 3

4. (a) $(0,0)$, $(5,0)$ (b) $(0,0)$

5. (a) -2 (b) 5 (c) 1

6. (a) x-intercept: $\frac{2}{3}$ y-intercept: -1
   (b) x-intercept: 0 y-intercepts: 0,2
   (c) x-intercept: -4 y-intercept: 2
   (d) x-intercept: -3 y-intercept: 1
   (e) x-intercept: 10 y-intercept: 6
   (f) x-intercepts: $3,-3$ y-intercepts: $4,-4$

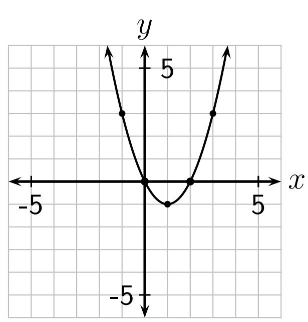
7. (a) $x\neq 3$, $x=4$ (b) $x\neq -2,1$, $x=2$

8. (a) $x=-2+3\sqrt{5},\ -2-3\sqrt{5}$ (b) $x=\frac{3}{5}+\frac{\sqrt{2}}{5},\ \frac{3}{5}-\frac{\sqrt{2}}{5}$

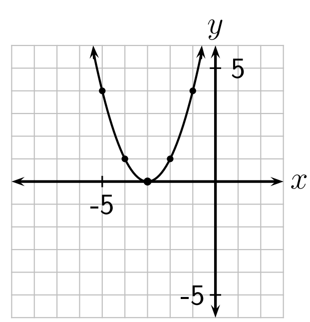
9. (a) $y=\frac{2}{3}x-2$ (b) $y=-\frac{3}{5}x-2$

10. \$6.50

11. (a)



(b)



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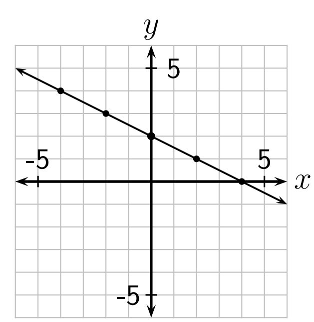


Section 6.3 Solutions Back to 6.3 Exercises

1. $\frac{3}{1}$ or 3   2. $\frac{2}{3}$ 3. (a) $\frac{1}{2}$ (b) 0   4. (a) $-\frac{3}{4}$ (b) $\frac{4}{3}$

5. (a) -2 (b) $\frac{2}{5}$ (c) 0 (d) undefined   (e) 3   (f) -2

6. (a) $-\frac{5}{2}$ (b) 0 (c) $\frac{3}{4}$ (d) undefined   (e) $\frac{4}{3}$ (f) $-\frac{1}{3}$

7. 3   8. $\frac{1}{2}$

9. **Equation:** $2l + 2(\frac{1}{2}l + 3) = 39$ **Answer:** $l = 11, w = \frac{1}{2}(11) + 3 = 8.5$

10. **Equation:** $x + 0.06x = 10.55$ **Answer:** $x = \$9.95$

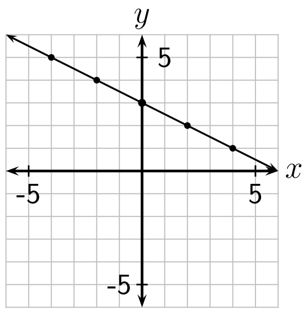
11. (a) x-intercepts: 0,2 y-intercept: 0   (b) x-intercept: -3 y-intercept: 9

(c) x-intercept: 4 y-intercept: 2

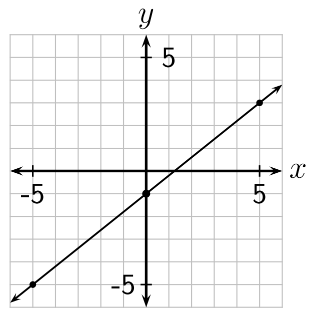
Section 6.4 Solutions Back to 6.4 Exercises

1. (a) $y = 4x - 4$ (b) $y = -\frac{3}{2}x + 4$ (c) $y = -\frac{2}{5}x + 3$

2. (a)



(b)



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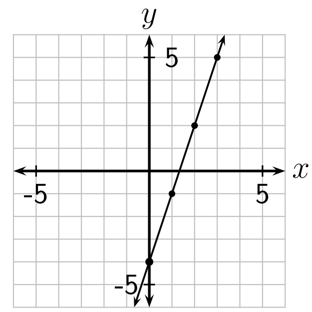
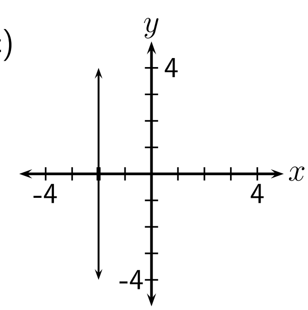
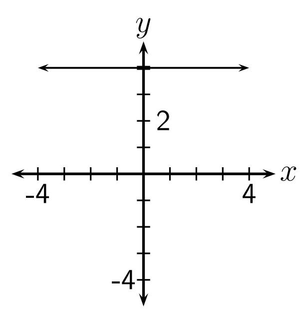


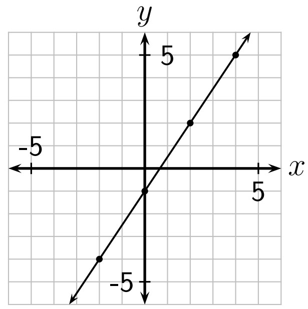
3. (a) $x = 4$ (b) $y = -1$ (c)

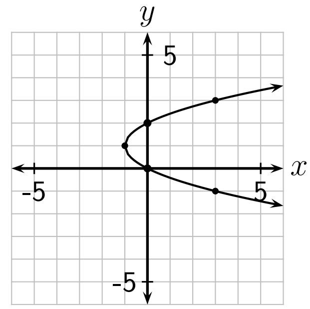


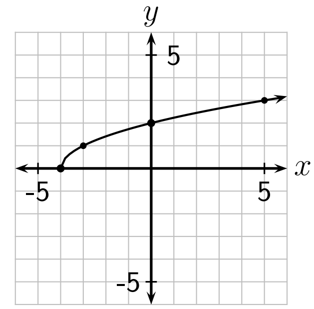
(d)



4. (a) $y = \frac{4}{3}x + 2$ (b) $y = -\frac{2}{3}x + 7$ (c) $y = 2$

5. (a) 

(b) 

(c) 

6. (a) $m = 4$ (b) $m = \frac{2}{3}$ (c) $m = -\frac{2}{5}$
   (d) $m = -\frac{1}{2}$ (e) $m = \frac{4}{5}$ (f) $m = -\frac{1}{3}$

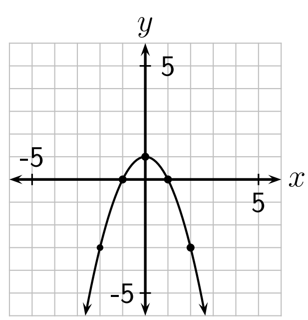
Section 6.5 Solutions Back to 6.5 Exercises

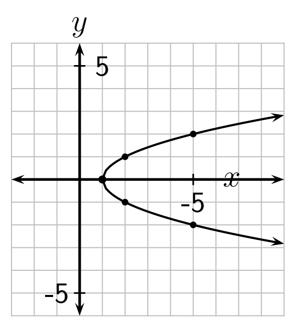
1. (a) The weight of a lizard that is 3 cm long is -18 grams. This is not reasonable because a weight cannot be negative. The problem is that the equation is only valid for lizards between 12 and 30 cm in length.
   (b) The w-intercept is -84, which would be the weight of a lizard with a length of zero inches. It is not meaningful for the same reason as given in (a).
   (c) The slope is 22 grams per centimeter. This says that for each centimeter of length gained by a lizard, the weight gain will be 22 grams.

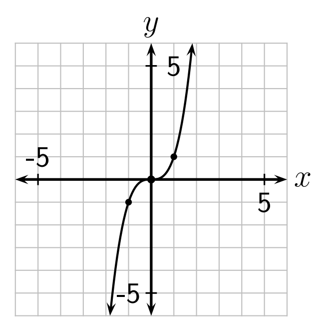
2. (a) \$2300, \$3800 (b) $P = 0.03S + 800$
   (c) the slope of the line is 0.03, the commission rate (as a decimal percent). (It has no units because it is dollars per dollar, which is one.)
   (d) The P-intercept of the line is \$800, the monthly base salary.

3. The y-intercept is \$30 and represents the daily "flat rate" charge for renting a car. The slope is \$0.24 per mile, the mileage fee for renting the car.

4. (a) $y = \frac{5}{3}x + 3$ (b) $y = \frac{5}{2}x + 10$

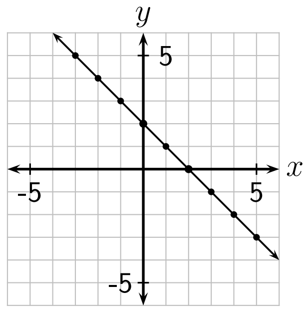
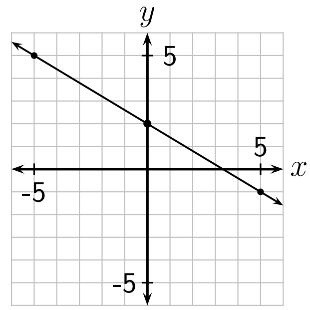
5. (a) 

(b) 

(c) 

6. (a) x-intercept: 5 y-intercept: -4 (b) x-intercepts: $5, -5$ y-intercepts: $5, -5$
   (c) x-intercept: 1 y-intercept: 1

7. (a) $y = \frac{3}{4}x - 1$ (b) $y = 2$ (c) $y = -\frac{4}{3}x + 2$

8. (a)  (c) 

9. (a) $y = 3x + 10$ (b) $x = 1$ (c) $y = 4x - 1$

A.7 Chapter 7 Solutions

Section 7.1 Solutions Back to 7.1 Exercises

1. (a) $(1, -1)$ (b) $(-1, 4)$ (c) $(-2, \frac{1}{3})$
2. (a) $(-15, -7)$ (b) $(3, 4)$ (c) $(0, -9)$
3. $(\frac{29}{22}, -\frac{5}{11})$
4. $(8, 6)$
6. (a) $(11, 7)$ (b) $(7, 2)$ (c) $(1, -1)$
7. **Equation:** $w(2w + 3) = 44$ **Answer:** $w = 4$ ($-\frac{11}{2}$ is another solution, but w cannot be negative)
8. **Equation:** $x + 0.065x = 41.37$ **Answer:** $x = \$38.85$
9. Give the slope of each line whose equation is given.

   (a) $m = -\frac{4}{7}$ (b) undefined (c) $m = -\frac{3}{5}$ (d) $m = 0$

10. (a) The F-intercept is 32°, which is the Fahrenheit temperature when the Celsius temperature is 0 degrees.

    (b) The slope of the line is $\frac{9}{5}$ degrees Fahrenheit per degree Celsius; it tells us that each degree Celsius is equivalent to $\frac{9}{5}$ degrees Fahrenheit.

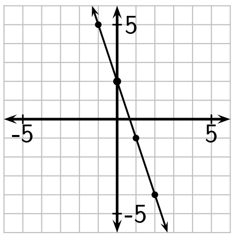
Section 7.2 Solutions Back to 7.2 Exercises

1. (a) no solution (b) $(2, -3)$ (c) no solution
2. (a) infinitely many solutions (b) infinitely many solutions (c) $(-1, 3)$
3. (a) $(4, 1)$ (b) no solution (c) infinitely many solutions (d) $(0, -3)$
4. (a) x-intercepts: $1, -1$ y-intercept: 1 (b) x-intercept: 1 y-intercepts: none

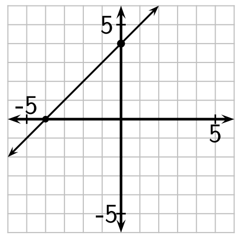
   (c) x-intercept: 0 y-intercept: 0

5. (a) $m = \frac{4}{3}$ (b) $m = 2$

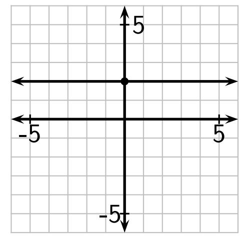
6. (a)



(b)



(c)



7. (a) $y = -\frac{1}{2}x + 4$ (b) $x = -3$ (c) $3x - 5y = 10$

8. (a) $y = 2x + 5$ (b) $y = -\frac{1}{11}x - \frac{28}{11}$ (c) $y = \frac{5}{2}x$

9. (a) The C-intercept is \$5000, which represents the costs when no Widgets are produced.
   (b) The slope is \$7 per Widget, and represents the increase in costs for each additional Widget produced. 1,491 widgets cost \$15,437 to produce or $\approx \$10.35$ per widget.

A.8 Chapter 8 Solutions

Section 8.1 Solutions Back to 8.1 Exercises

1. (a) $f(5) = 10$ (b) $f(-1) = 4$ (c) $f(-3) = 18$ (d) $f(0) = 0$

2. $f(\frac{1}{2}) = -\frac{5}{4}$

3. (a) $g(2) = 1$ (b) $g(-6) = 3$ (c) $g(-5) = 2\sqrt{2}$ (d) $g(-2) = 2.24$

4. (a) (a) $h(-4) = 3$ (b) $h(2) = 4.58$ (c) $h(7)$ doesn't exist.
   (d) We must make sure that $25 - x^2 \geq 0$. This holds for all values of x between -5 and 5, including those two.

5. (a) $f(s) = s^2 - 3s$ (b) $f(s+1) = s^2 - s - 2$ (c) $x = -2, 5$

6. (a) $h(a-3) = \frac{2}{3}a - 7$ (b) $h(a+2) = \frac{2}{3}a - \frac{11}{3}$

7. $x = \frac{21}{2}$

8. $g(s+1) = \sqrt{2-s}$

9. (a) $h(a-2) = \sqrt{21 + 4a - a^2}$ (b) $x = \sqrt{21}, -\sqrt{21}$

10. (a) $(13, -10)$ (b) $(3, 7)$ (c) $(3, 4)$

Section 8.2 Solutions Back to 8.2 Exercises

1. (a) $g[f(1)] = g[(1)^2 - 2(1)] = g[1-2] = g[-1] = -1 - 3 = -4$
   (b) $g[f(-3)] = g[(-3)^2 - 2(-3)] = g[9+6] = g[15] = 15 - 3 = 12$

2. (a) $f[g(1)] = f[1-3] = f[-2] = (-2)^2 - 2(-2) = 4 + 4 = 8$

   (b) $f[g(-3)] = f[-3-3] = f[-6] = (-6)^2 - 2(-6) = 36 + 12 = 48$

3. $(f \circ g)(-3) = (-3)^2 - 8(-3) + 15 = 9 + 24 + 15 = 48$

4. (a) $(g \circ f)(x) = g[f(x)] = g[x^2 - 2x] = x^2 - 2x - 3$

   (b) $(g \circ f)(4) = 4^2 - 2(4) - 3 = 16 - 8 - 3 = 5$

5. (a) $f[g(2)] = 10$ (b) $(g \circ f)(2) = 3$ (c) $(g \circ f)(x) = x^2 + 3x - 7$

   (d) $(f \circ g)(x) = x^2 - 11x + 28$

6. (a) $y = 3$ (b) $x = 4$ (c) $y = -\frac{5}{6}x - 2$

7. (a) $(2, 2)$ (b) infinitely many solutions (c) no solution

8. (a) $(4, 3)$ (b) no solution 9. $x = \frac{19}{2}$ 10. $x = 8$ ($x = 1$ doesn't check)

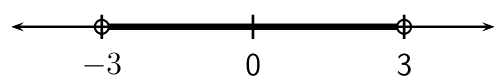
11. (a) $f(-3) = 24$ (b) $x = -5, 6$ (c) $x = \frac{1}{2} + \frac{\sqrt{3}}{2},\ x = \frac{1}{2} - \frac{\sqrt{3}}{2}$

Section 8.3 Solutions Back to 8.3 Exercises

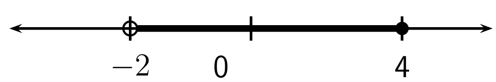
1. (a) $x < 10$ or $10 > x$

   (b) $t \geq 0$ or $0 \leq t$

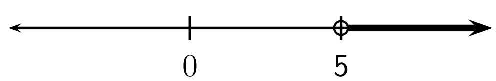
   (c) $-3 < x < 3$ or $3 > x > -3$



   (d) $-2 < x \leq 4$



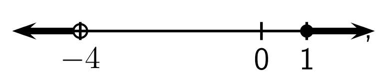
2. (a) $5 < x$ or $x > 5$,



, $(5, \infty)$

   (b) All numbers less than or equal to 3, $x \leq 3$,

   (c) All numbers less than -4 or greater than or equal to 1, $(-\infty, -4) \cup [1, \infty)$



   (d) All numbers greater than 2 and less than or equal to $5\frac{1}{2}$, $2 < x \leq 5\frac{1}{2}$, $(2, 5\frac{1}{2}]$

(e) All numbers less than -7 or greater than or equal to 1, $x < -7$ or $1 \leq x$,

(f) All numbers less than 17, 

, $(-\infty, 17)$

(g) $-10 < x \leq -3$, 

, $(-10, -3]$

3. (a) x-intercept: 5 $\quad$ y-intercept: -4 $\quad$ (b) x-intercepts: $5, -5$ $\quad$ y-intercepts: $5, -5$

   (c) x-intercept: 1 $\quad$ y-intercept: 1

4. $m = -\frac{2}{5}$

5. (a) $y = \frac{1}{4}x - 5$ $\quad$ (b) $y = -\frac{3}{4}x + \frac{1}{2}$ $\quad$ (c) $y = 4$

6. (a) $(5, 0)$ $\quad$ (b) infinitely many solutions $\quad$ (c) no solution

7. $(h \circ g)(x) = h[2x - 3] = (2x - 3)^2 - 5(2x - 3) + 2 = 4x^2 - 22x + 26$

8. (a) $f[g(-1)] = 2$ $\quad$ (b) $(g \circ f)(4) = -25$ $\quad$ (c) $(g \circ f)(x) = 7 - 2x^2$

   (d) $(f \circ g)(x) = -4x^2 - 4x + 2$

Section 8.4 Solutions $\quad$ Back to 8.4 Exercises

1. (a) All real numbers except -4. $\quad$ (b) All real numbers.

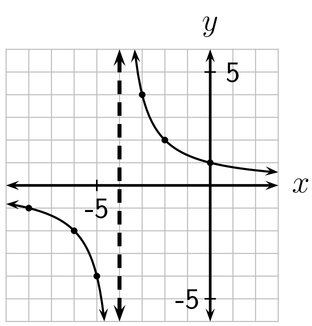
   (c) All real numbers except 2 and 3. $\quad$ (d) $x \geq -5$ OR $[-5, \infty)$

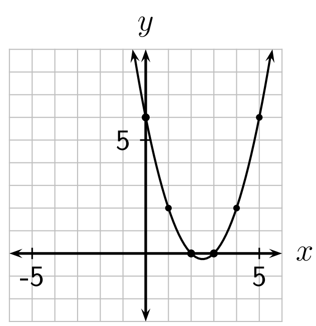
   (e) $x > 3$ OR $(3, \infty)$ $\quad$ (f) All real numbers except 4 and -4.

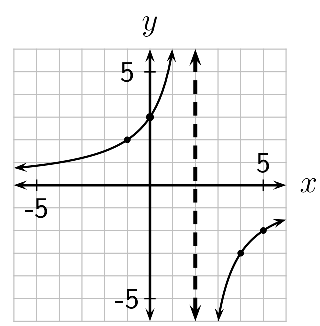
2. (a) $x = -2, 4$ $\quad$ (b) $x = 0, 2$ $\quad$ (c) $h(-3) = 12$

   (d) $x = 1 + \sqrt{6}, 1 - \sqrt{6}$, $x = 3.45, \ -1.45$

Section 8.5 Solutions $\quad$ Back to 8.5 Exercises

1. (a) 

(b) 

(c) 

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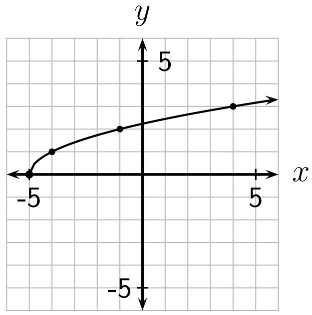
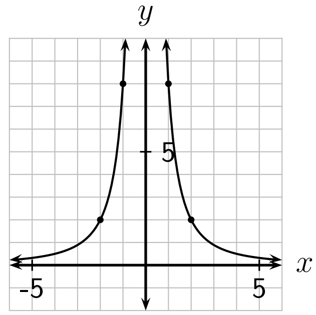
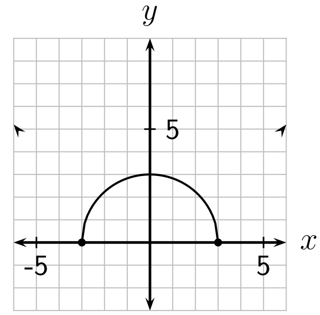


(e)


(f)


2. $(f \circ h)(x) = 3x^2 - 6x - 10$ $\qquad$ $(h \circ f)(x) = 9x^2 - 12x$

3. (a) $x \leq -3$ or $x \geq 3$ OR $(-\infty, -3] \cup [3, \infty)$ $\qquad$ (b) $-3 \leq x \leq 3$ OR $[-3, 3]$

   (c) $-3 < x < 3$ OR $(-3, 3)$

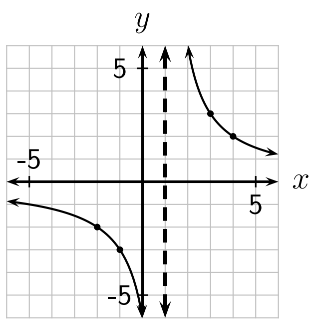
Section 8.6 Solutions $\qquad$ Back to 8.6 Exercises

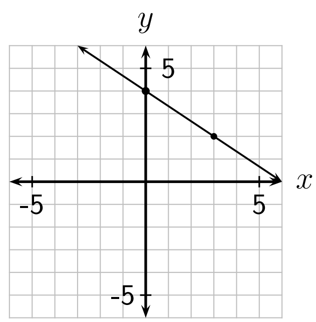
2. (a) downward, $(-1, -1)$ $\qquad$ (b) upward, $(3, -16)$

   (c) upward, $(-2, 3)$ $\qquad$ (d) downward, $(\frac{3}{2}, \frac{49}{4})$

3. (a) $(-2, 1)$ $\qquad$ (b) $(3, 4)$ $\qquad$ (c) $(3, 2)$

4. The systems in 4. (b) and (c) can easily be solved using the substitution method. See the solutions above.

5. (a)


(b)


A.9 Chapter 9 Solutions

Section 9.1 Solutions $\qquad$ Back to 9.1 Exercises

1. (a) 2 $\qquad$ (b) 10 $\qquad$ (c) -2 $\qquad$ (d) DNE

2. (a) -2 $\qquad$ (b) 9 $\qquad$ (c) DNE $\qquad$ (d) 2

3. (a) All real numbers except 7. $\qquad$ (b) All real numbers.

   (c) All real numbers except 3.

4. (a) downward, $(-1, -2)$ $\qquad$ (b) upward, $(-3, 9)$

Section 9.2 Solutions

Back to 9.2 Exercises

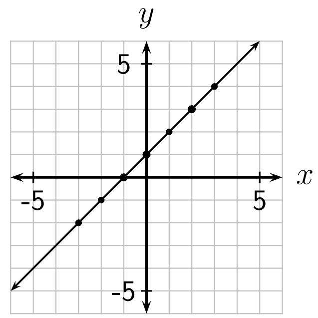
1. (a) $\frac{5\sqrt{6}}{6}$ (b) $\frac{2\sqrt{15}}{3}$ (c) $-\frac{1+\sqrt{5}}{2}$ (d) $4\sqrt{3}$

   (e) $\sqrt{3}$ (f) $\frac{3+7\sqrt{3}}{46}$

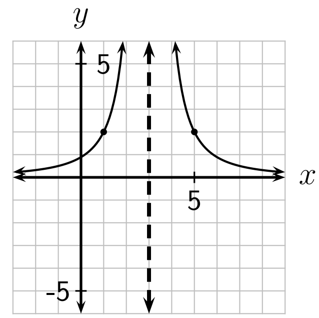
2. (a) $f[g(3)] = 3$ (b) $(f \circ g)(x) = 3 + 3x - x^2$ (c) $(f \circ g)(3) = 3$

   (d) $(g \circ f)(x) = x^2 - 3x$

3. (a)



   (b)



4. (a) $\frac{1}{32}$ (b) 27 (c) -125 (d) -1

Section 9.3 Solutions

Back to 9.3 Exercises

1. (a) $4 + 18i$ (b) $5 + 2i$ (c) $2i$ (d) $-3 + 10i$

2. (a) -15 (b) 14 (c) $18i$ (d) $15 - 6i$ (e) $2 + 8i$

   (f) $9 + 37i$ (g) $a^2 + b^2$ (h) $33 + 21i$ (i) $(c^2 - d^2) + 2cdi$

3. (a) $\frac{6}{29} + \frac{15}{29}i$ (b) $\frac{1}{15} + \frac{3}{15}i$ (c) $\frac{16}{13} + \frac{11}{13}i$ (d) $\frac{c^2 - d^2}{c^2 + d^2} + \frac{2cd}{c^2 + d^2}i$

4. (a) $5i$ (b) $2i\sqrt{6}$ (c) $-2\sqrt{2}$ (d) $10i$ (e) $5i\sqrt{3}$ (f) -8

5. (a) downward, $(5, 4)$ (b) upward, $(-1, -5)$

6. (a) $\frac{3\sqrt{5}}{5}$ (b) $-9 - 4\sqrt{5}$

Section 9.4 Solutions

Back to 9.4 Exercises

1. (a) $x = -3 + 2i,\ x = -3 - 2i$ (b) $x = -5 + i,\ x = -5 - i$

   (c) $x = 2,\ x = -3$ (d) $x = 2i,\ x = -2i$

   (e) $x = \frac{1}{2} + \frac{3}{2}i,\ x = \frac{1}{2} - \frac{3}{2}i$ (f) $x = -7 + 2i,\ x = -7 - 2i$

2. The domain of f is $(-\infty, 3]$ or $x \leq 3$, and the domain of g is $x \neq -2, 5$

3. (a) $\frac{5\sqrt{6}}{2}$ (b) $19+7\sqrt{3}$ 4. (a) $27+10i$ (b) 2

5. (a) $-5+12i$ (b) 25 (c) $2+23i$ 6. (a) $3i$ (b) $3i\sqrt{2}$ (c) -4

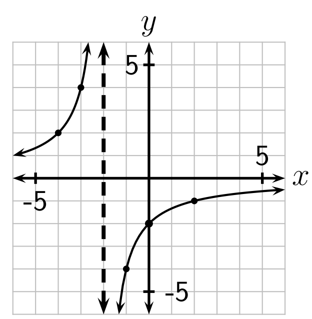
A.10 Chapter 10 Solutions

Section 10.1 Solutions Back to 10.1 Exercises

1. (a) $5x^2+3x-9$ (b) $4x^2+x-5$ (c) x^2-4x+3
   (c) $5x^2+7x+2$ (d) $3x^2-x+2$
2. (a) $x+2+\frac{5}{x+1}$ (b) $x+14+\frac{68}{x-5}$
   (c) $3x+7$ (d) $2x-3+\frac{3}{x-2}$
3. (a) 16 (b) $\frac{1}{3}$ (c) 4 (d) -3
4. (a) $1+i$ (b) $\frac{3}{26}+\frac{15}{26}i$ (c) $\frac{1}{5}+\frac{1}{10}i$
5. (a) $x=5+3i\sqrt{2},\ x=5-3i\sqrt{2}$ (b) $x=1+i\sqrt{7},\ x=1-i\sqrt{7}$
   (c) $x=4+2i\sqrt{3},\ x=4-2i\sqrt{3}$

Section 10.2 Solutions Back to 10.2 Exercises

1. (a) $\frac{5x-17}{(x+5)(x-1)}$ (b) $\frac{7x-2}{(x+2)(x-2)}$ (c) $\frac{-4x-23}{(x+4)(x-3)}$
2. See to the right



3. (a) $x-3-\frac{5}{x-2}$ (b) $x^2-5x+2+\frac{4}{x-2}$
4. (a) $x=-3+i,\ \ -3-i$
   (b) $x=-\frac{5}{3}+\frac{2\sqrt{2}}{3}i,\ x=-\frac{5}{3}+\frac{2\sqrt{2}}{3}i$
   (c) $x=2+\sqrt{3},\ x=2-\sqrt{3}$

Section 10.3 Solutions Back to 10.3 Exercises

1. (a) $\frac{x-3}{x+5}$ (b) $\frac{x+2}{x-1}$
2. (a) $\frac{1}{(x+1)(x+3)}$ (b) $\frac{(x-1)(x-5)}{(x+1)(x+2)}$
3. (a) $\frac{x+3}{(x-3)(x-2)}$ (b) $\frac{x-5}{x+1}$ (c) $\frac{x+5}{x+3}$
   (d) $\frac{1}{y+3}$ (e) $\frac{x+3}{x-2}$ (f) $\frac{1}{x-2}$

4. (a) $\frac{5\sqrt{14}}{7}$ (c) $3\sqrt{10}$

5. (a) $46 - i$ (b) $1 - 13i$ (c) $-\frac{34}{29} - \frac{31}{29}i$

6. (a) $-2\sqrt{3}$ (b) $6i$ (c) $5i\sqrt{2}$

7. (a) $x = 5 + 2i,\ x = 5 - 2i$ (b) $x = 4 + 3i,\ x = 4 - 3i$ (c) $x = -2 + i,\ x = -2 - i$

8. $x^2 + 2x - 4$

9. (a) $\dfrac{3x - 14}{(x-2)(x-3)}$ (b) $\dfrac{5x - 5}{(x-3)(x+2)(x-2)}$

A.11 Chapter 11 Solutions

Section 11.1 Solutions Back to 11.1 Exercises

1. (a) $x^2 - 10x + 25$ (b) $(x-5)(x-5) = (x-5)^2$
   (c) We'd still add 25, but it would factor to $(x+5)(x+5)$.

2. (a) $x^2 + 2x + 1 = (x+1)^2$ (b) $x^2 - 12x + 36 = (x-6)^2$
   (c) $x^2 - 4x + 4 = (x-2)^2$ (d) $x^2 - 9x + \frac{81}{4} = (x - \frac{9}{2})^2$
   (e) $x^2 - x + \frac{1}{4} = (x - \frac{1}{2})^2$ (f) $x^2 + 8x + 16 = (x+4)^2$

4. (a) $x = 1 + \sqrt{5},\ 1 - \sqrt{5}$ (b) $x = 3 + \sqrt{2},\ 3 - \sqrt{2}$
   (c) $x = 5 + 2\sqrt{3},\ 5 - 2\sqrt{3}$ (d) $x = 6,\ -8$ (e) $x = 17,\ -3$

5. (a) $x = -3 + 2i,\ -3 - 2i$ (b) $x = -5 + i,\ -5 - i$ (c) $x = 2,\ -3$

6. (a) $\dfrac{x}{x+5}$ (b) $\dfrac{x+3}{(x-7)(x+1)}$ (c) $2x - 4$

Section 11.2 Solutions Back to 11.2 Exercises

1. (a) $x > -3$ (b) $x < -6$
   (c) The direction changes in (b). The direction of the inequality changes in (b) because we divide by a negative. In (a) we divide by a positive, so the inequality doesn't change direction.

2. (a) $x \geq -3$ (b) $-3 < x$

3. (a) $x \geq -3$ (b) $-3 < x$ (c) $x \geq 2$ (d) $x \geq -3$

(e) $x > 9$ (f) $y \geq -1$ (g) $x < 9$ (h) $x \leq \frac{4}{7}$

(i) $x \geq -1$ (j) $x > 2$ (k) $x \geq -\frac{1}{5}$ (l) $x \geq -3$

4. (a) $x = \frac{3}{2} + \frac{1}{2}i,\ x = \frac{3}{2} - \frac{1}{2}i$ (b) $x = 1 + 2i,\ x = 1 - 2i$

(c) $x = -2 + i\sqrt{7},\ x = -2 - i\sqrt{7}$

5. $x^2 + 7x + 12$

6. (a) $\dfrac{3x - 20}{x(x+4)(x-4)}$ (b) $\dfrac{3 - 5x}{x(x-2)}$

7. (a) $\dfrac{1}{x+5}$ (b) $2x - 8$ (c) $\dfrac{x+3}{x+2}$

9. (a) $x = -3 + 3\sqrt{3},\ -3 - 3\sqrt{3}$ (b) $x = -6 + 6\sqrt{2},\ -6 - 6\sqrt{2}$

(c) $x = 2 + 2\sqrt{3},\ 2 - 2\sqrt{3}$

10. (a) $x = -7 + 2i,\ -7 - 2i$ (b) $x = 1 + i\sqrt{7},\ 1 - i\sqrt{7}$

B Exercises by Performance Criterion

1. (a) Evaluate expressions involving positive integer exponents without and with a calculator.

   1.1: 1 a b c d e f g h **1.4:** 3 a b c **2.2:** 2 d f

   (b) Apply order of operations to evaluate polynomial and rational expressions without a calculator.

   1.2: 1 a b c d e f 3 a b c d 4 a b c d e f g h i 6 a b c d
   1.3: 4 a d e **1.4:** 3 d e f g 4 a b **1.5:** 7 a b c d e f, 9 a b
   2.2: 2 c 3 **2.4:** 8 a b

   (c) Apply order of operations to evaluate polynomial and rational expressions with a calculator.

   1.2: 2 5 a b c d e f g h i 7 a b **1.5:** 8 a b c d e f

   (d) Apply order of operations and the distributive property to simplify linear algebraic expressions.

   1.3: 1 2a b c d 3 a b c d e f 4 b c f **1.4:** 5 a b c **2.2:** 2 a e

   (e) Simplify algebraic expressions with positive integer exponents.

   1.4: 1 a b c d 2 a b c d e f g h **1.5:** 6 a b c d e f
   2.2: 4 a b c d **2.5:** 4 a b c d

   (f) Evaluate and simplify numerical expressions involving integer (including zero) exponents without use of a calculator.

   1.5: 1 a b c d 2 a b c d e f g h **2.1:** 5 a b c d **2.2:** 2 b
   2.4: 9 a b c d **3.2:** 3 a b c d

   (g) Change numbers from decimal form to scientific notation and vice versa without using a calculator.

   1.5: 3 a b 4 a b 5 a b c d e f g h **2.1:** 6 a b c d **2.5:** 5 a b c d

2. (a) Determine whether a value is a solution to an equation.

   2.1: 1 a b c d e f g h 2 3 4

   (b) Solve linear equations.

   2.2: 1 a b c d e f g h **2.3:** 4 a b c **2.5:** 6 a b c **3.3:** 2 b e h

   (c) Add and subtract polynomial expressions.

   2.3: 1 a b c d 2 a b c **2.4:** 10 a b **3.2:** 4 b d

   (d) Multiply polynomial expressions.

   2.3: 3 a b c d e f **2.4:** 11 a b c d e f **3.2:** 4 a c

(e) Factor quadratic trinomials and differences of squares. Factor polynomial expressions (quadratic higher degree) by factoring out common factors or grouping.

2.4: 1 2 a b c d e f 3 a b c d e f 4 a b c 5 a b c 6 a b c
2.4: 7 a b c d e f g h i j k l m n o p q r **3.1:** 3 a b c 4 a b 5 a b c

(f) Solve polynomial equations.

2.5: 1 a b c 2 a b c 3 a b c d e f g h i **3.1:** 6 a b c d e f
3.3: 2 a c d f g **4.3:** 3 a b c d

3. (a) Give values that an unknown is not allowed to have in a rational expression.

3.1: 1 a b c d e f **3.2:** 5 a b c **4.1:** 3 a b c

(b) Simplify rational expressions.

3.1: 2 a b c d e f **3.2:** 6 a b c **4.1:** 4 a b c

(c) Multiply rational expressions and simplify the results.

3.2: 1 a b c d 2 a b c d **4.1:** 5 a b **4.4:** 4 a b c d

(d) Solve rational equations.

3.3: 1 a b c d e f **4.1:** 6 a b **4.2:** 3 a b **4.3:** 4 a b
6.2: 7 a b

4. (a) Find (without a calculator) real roots of numbers when they exist.

4.1: 1 a b c d e f g h i j **4.2:** 4 a b c d e **5.1:** 11 a b c d e

(b) Simplify a square root.

4.1: 2 a b c d e f g h **4.2:** 5 a b c d **5.2:** 10 a b c

(c) Add subtract and multiply expressions containing roots.

4.2: 1 a b c d 2 a b c d e f g h i **4.3:** 5 a b
5.1: 12 a b 13 a b c **6.1:** 10 a b c

(d) Solve quadratic equations using the quadratic formula.

4.3: 1 a b 2 a b c d e f **4.4:** 5 a b **5.2:** 11 a b **6.2:** 8 a b

(e) Solve equations containing roots.

4.4: 1 a b c 2 a b 3 a b c d e f **5.1:** 14 a b c d **5.3:** 6 a b c

5. (a) Use formulas to solve applied problems.

5.1: 1 2 3 4 5 6 7 8 9 10
5.2: 12 **6.2:** 10

(b) Solve formulas for given unknowns.

5.2: 1 2 3 4 5 6 a b c d e 7 8 a b c d 9
5.3: 7 a b **6.2:** 9 a b **6.5:** 4 a b

(c) Create equations whose solutions are answers to applied problems.

5.3: 1 2 3 4 5 **6.1:** 11 12 **6.3:** 9 10 **7.1:** 7 8

6. (a) Graph the solution set of an equation in two unknowns.

   6.1: 1 2 3 4 5 6 7 8 9
   6.2: 11 a b c **6.4:** 5 a b c **6.5:** 5 a b c

   (b) Find x- and y-intercepts of an equation in two variables.

   6.2: 1 2 3 4 5 6 a b c d e f **6.3:** 11 a b c
   6.5: 5 a b c **7.2:** 4 a b c **8.3:** 3 a b c

   (c) Find the slope of a line including vertical and horizontal.

   6.3: 1 2 3 4 5 a b c d e f 6 a b c d e f **6.4:** 6 c e
   7.1: 9 a b c d

   (d) Know and apply the relationship between slopes of parallel and perpendicular lines.

   6.3: 7 8 **6.4:** 5 a b d f **7.2:** 5 a b **8.3:** 4

   (e) Graph a line given its equation; give the equation of a line having a given graph.

   6.4: 1 a b c 2 a b c 3 a b c d **6.5:** 7 a b c 8 a b c
   7.2: 6 a b c 7 a b c **8.2:** 6 a b c

   (f) Determine the equation of a line through two points.

   6.4: 4 a b c **6.5:** 9 a b c **7.2:** 8 a b c **8.3:** 5 a b c

   (g) Use a given linear model of a "real" situation and a given value of either of the two linearly related quantities to find the value of the other.

   6.5: 2 a **7.2:** 9 c

   (h) Given a linear model of a "real" situation interpret the values of the slope and intercept.

   6.5: 1 2 b c d 3 **7.1:** 10 **7.2:** 6 a b

7. (a) Solve a system of two linear equations by addition.

   7.1: 1 a b c 3 4 5 a b 6 a b c **7.2:** 1 a b c
   8.1: 11 a b c **8.6:** 3

   (b) Solve a system of two linear equations by substitution.

   7.1: 2 a b c **7.2:** 2 a b c **8.1:** 10 a b c **8.6:** 4

   (c) Recognize when a system of two linear equations has no solution or infinitely many solutions.

   7.2: 1 a b c 2 a b c 3 a b c d **8.2:** 7 a b c **8.3:** 6 a b c

   (d) Solve a system of two linear equations by graphing.

   7.2: 3 a b c d **8.2:** 8 a b

8. (a) Evaluate a function for a given numerical or algebraic value.

   8.1: 1 a b c d 2 3 a b c d 4 a b c d 5 a b 6 8 9 a
   8.2: 11 a **8.4:** 2 c

(b) Find all numerical ("input") values for which a function takes a certain ("output") value.

8.1: 5 c 7 9 b **8.2:** 9 10 11 b c **8.4:** 2 a b d

(c) Evaluate the composition of two functions for a given "input" value; determine a simplified composition function for two given functions.

8.2: 1 a b 2 a b 3 4 5 a b c d **8.3:** 7 8 a b c d
8.5: 2 **9.2:** 2 a b c d

(d) Describe sets of numbers using inequalities or interval notation.

8.3: 1 a b c d 2 a b c d e f g

(e) Determine the domains of rational and radical functions.

8.4: 1 a b c d e f **8.5:** 3 a b c **9.1:** 3 a b c **9.4:** 2

(f) Graph quadratic polynomial and simple root and rational functions.

8.5: 1 a b c d e f **8.6:** 5 **9.2:** 3 a b **10.2:** 2

(g) Determine whether the graph of a quadratic function is a parabola opening up or down and find the coordinates of the vertex of the parabola.

8.6: 1 2 **9.1:** 4 a b **9.3:** 5 a b

9. (a) Evaluate numerical expressions containing fractional exponents.

9.1: 1 a b c d 2 a b c d **9.2:** 4 a b c d **10.1:** 3 a b c d

(b) Rationalize denominators of rational expressions with roots in their denominators.

9.2: 1 a b c d e f **9.3:** 6 a b **9.4:** 3 a b **10.3:** 4 a b

(c) Add subtract and multiply complex numbers.

9.3: 1 a b c d 2 a b c d e f g h i **9.4:** 4 a b 5 a b c **10.3:** 5 a b

(d) Divide complex numbers numbers.

9.3: 3 a b c d **10.1:** 4 a b c **10.3:** 5 c

(e) Simplify square roots of negative numbers.

9.3: 4 a b c d e f **9.4:** 5 a b c **10.3:** 6 a b c

(f) Solve quadratic equations with complex solutions.

9.4: 1 a b c d e f **10.1:** 5 a b c **10.2:** 4 a b c
10.3: 7 a b c **11.2:** 7 a b c

10. (a) Divide polynomial expressions.

10.1: 1 a b c d e f 2 a b c d **10.2:** 3 a b **10.3:** 8 **11.2:** 8

(b) Add and subtract rational expressions.

10.2: 1 a b c **10.3:** 9 a b **11.2:** 10 a b

(c) Divide rational expressions; simplify complex rational expressions.

10.3: 1 a b   2a b   3 a b c d e f   **11.1:** 6 a b c   **11.2:** 9 a b c

11. (a) Solve quadratic equations by completing the square.

    11.1: 1 a b c   2 a b c d e f   3 a b c   4 a b c d e   5 a b c
    11.2: 4 a b c   5 a b c   6 a b

    (b) Solve linear inequalities in one unknown.

    11.2: 1   2   3 a b c d e f g h i j k l