Intermediate Algebra

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Contents

1 Expressions and Exponents

1.1 Exponents

1. (a) Evaluate expressions involving positive integer exponents, without and with a calculator.

NOTE: In the past you have taken 5×7 to mean 'five times seven.' From now on we will denote multiplication of two numbers using either a dot, $5 \cdot 7$, or parentheses, $5(7)$. This is done because the symbol $\times$ is easily confused with the letter x, which we will be using to signify an unknown number.

One thing multiplication does for us is to simplify repeated addition of the same number:

$$7 + 7 + 7 + 7 + 7 = 5(7) = 35$$

As you probably know, exponents are used to simplify repeated multiplication:

$$7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 = 7^5$$

So for any natural number (the numbers 1, 2, 3, 4, ...), a^n means n a's multiplied together. This is an exponential expression, the value a is called the **base** and n is the **exponent**.

◇ **Example 1.1(a):** Compute 2^3, 3^2 and 7^1.

Solution: $2^3 = 2 \cdot 2 \cdot 2 = 8\,,\qquad 3^2 = 3 \cdot 3 = 9\,,\qquad 7^1 = 7$

◇ **Example 1.1(b):** Find $\left(\frac{2}{3}\right)^4$.

Solution: $\left(\frac{2}{3}\right)^4 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{16}{81}$. Note for future reference that $\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4}$.

◇ **Example 1.1(c):** Compute $(-3)^2$ and $(-2)^3$.

Solution: $(-3)^2 = (-3)(-3) = 9\,,\qquad (-2)^3 = (-2)(-2)(-2) = -8$

In general, a negative number to an even power is positive, and a negative number to an odd power is negative.

◇ **Example 1.1(d):** Compute -3^2.

Solution: In the expression $(-3)^2$ the exponent applies to the number -3, including the sign, so the result is as in the previous example. In the expression -3^2 the exponent applies only to the number 3, and the negative sign is recorded after computing:

$$-3^2 = -(3 \cdot 3) = -9$$

One thing we want to get clear right now is the difference between $(-3)^2$ *and* -3^2*.* Think about the previous two examples carefully.

◇ **Example 1.1(e):** Compute $2^3 \cdot 2^4$.

Solution: $$2^3 \cdot 2^4 = (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2) = 2^7$$

Summary of Exponents

- Exponents indicate repeated multiplication: $a^n = \overbrace{a \cdot a \cdot a \cdots a}^{n \text{ times}}$
- $a^1 = a$
- $(-a)^n = (-a)(-a)(-a)\cdots(-a)$ and $-a^n = -(a \cdot a \cdot a \cdots a)$

We often refer to 5^2 as 'five squared,' because its value is the area of a square that is five units on a side. Similarly, 'five cubed' refers to the quantity 5^3, the volume of a cube with edges of length five units.

Suppose that you needed to find something like 12^7. You certainly wouldn't want to do it by hand, and it is even a bit annoying to multiply seven twelves on a calculator. Fortunately, your calculator has a key which will do exponents for you.

◇ **Example 1.1(f):** Use your calculator to compute 12^7.

Solution: To compute 12^7, enter the number 12, then hit the $\frown$ key or the y^x key followed by the number 7, and finish by hitting $=$. You should get 35831808.

◇ **Example 1.1(g):** Compute $(-5)^6$ using your calculator.

Solution: To compute $(-5)^6$ you don't need to use the parentheses that your calculator has. Just enter -5, then use the $\frown$ key or the y^x key with the exponent 6. To compute -5^6 simply compute 5^6 *then* change the sign.

Section 1.1 Exercises

To Solutions

1. Simplify each of the following *without using a calculator*.

   (a) 3^3 (b) $\left(\frac{1}{2}\right)^6$ (c) 10^2 (d) -5^3

   (e) $(-5)^3$ (f) $(-5)^2$ (g) -5^2 (h) $\left(-\frac{4}{3}\right)^2$

2. Use your calculator to find the value of each of the following exponential expressions:

   (a) 4^9 (b) -3^{10} (c) $(-3)^{10}$ (d) 5^{12}

3. Try using your calculator to find the value of 23^8. You will get something that you may not recognize; we will learn about this soon.

1.2 Evaluating Numerical Expressions

1. (b) Apply order of operations to evaluate polynomial and rational expressions without a calculator.

   (c) Apply order of operations to evaluate polynomial and rational expressions with a calculator.

An **expression** is a collection of numbers that are to be combined using mathematical operations like addition, multiplication, and so on. Some examples are

$$12 - 4(3+2) \qquad \frac{3(5)-7}{4} \qquad x + 3y \qquad \frac{4r-s}{t}$$

The letters x, y, r, s and t represent numbers whose values we don't know - hence they are called **unknowns**. We will call the first two expressions **arithmetic expressions** or **numerical expressions**, meaning that all of the numbers involved are known. More often in this course we will be dealing with **algebraic expressions**, which are expressions containing unknown numbers that are represented by letters. The second two expressions are examples of these. As you may already know, $3(5)$ means three times five, and $3y$ means 3 times y. The long horizontal bar indicates that a division is to be performed.

The act of determining the single numerical value for a numerical expression is called **evaluating** the expression. *We want to be certain that everyone gets the same value when evaluating an expression*, so when evaluating most expressions we will need to refer to what is called the **order of operations**. Consider the expression $3 + 5(4)$, which contains an addition and a multiplication. If we do the addition first we get 32 and if we do the multiplication first we get 23. So, depending on which operation we do first, we can get two different values for this expression. The order of operations tell us the order in which the operations in an expression are performed; they were developed to make sure that we all interpret an expression in the same way.

Order of Operations

(1) **Parentheses:** Operations in parentheses must be performed first. If there is more than one operation within a set of parentheses, then the operations within the parentheses must follow the remaining rules.

(2) **Exponents:** Exponents are applied next - remember that they apply only to the number that they are directly "attached" to.

(3) **Multiplication and Division:** These both have equal priority. When it is not clear from other things which to do first, do them from left to right.

(4) **Addition and Subtraction:** These also have equal priority, and are done from left to right.

Evaluating Numerical Expressions

◇ **Example 1.2(a):** Evaluate the expression $3 + 5(4)$.

Solution: There are two operations, an addition and a multiplication. Since multiplication is higher on the list than addition, the multiplication is done first, giving $3+20$. We then perform the addition to get 23.

NOTE: You will usually be asked to show how an expression is evaluated, step-by-step. One way to do this is to write the original expression followed by an equal sign, followed by the expression that results when the first operation is performed, followed by another equal sign and the result after the second operation is performed. This is repeated until the final result is obtained. The process from Example 1 would then be illustrated as shown below and to the left.

$$3 + 5(4) = 3 + 20 = 23 \qquad\qquad \begin{array}{c} 3+5(4) \\ 3+20 \\ 23 \end{array}$$

Another option is to write the original expression, then write each step below the previous one, as shown above and to the right. *Note that, either way, the original expression is always given first!*

◇ **Example 1.2(b):** Evaluate the expression $(3 + 5)4$.

Solution: Compare this expression with the expression from Example 1.2(a). The same numbers and operations are involved, but parentheses have been inserted to tell us to do the addition first in this case:

$$(3 + 5)4 = 8(4) = 32$$

◇ **Example 1.2(c):** Evaluate $9 - 4 + 6$.

Solution: The expression contains both an addition and a subtraction, neither of which is necessarily to be done before the other. In this case, we simply work from left to right:

$$\begin{array}{c} 9-4+7 \\ 5+7 \\ 12 \end{array}$$

If we had intended for the addition to be done first, we would have to use parentheses to indicate that:

$$9 - (4 + 7) = 9 - 11 = -2$$

Note that the two results are different!

Sometimes we want parentheses inside parentheses; in those cases we usually use the 'square brackets' [ ] instead of ( ) for the 'outer' set. An example is the expression $5[3 - 2(7 + 1)]$. Here we do what is in the parentheses first, then the square brackets.

◇ **Example 1.2(d):** Evaluate $5[3-2(7+1)]$.

Solution:
$$\begin{gathered} 5[3-2(7+1)] \\ 5[3-2(8)] \\ 5[3-16] \\ 5[-13] \\ -65 \end{gathered}$$

NOTE: Soon we will discuss the distributive property, which you are probably already familiar with. It could be used in the previous example, but *it is simpler to just follow the order of operations when working with known numbers*, as in this last example. For other situations we will have to use the distributive property.

A fraction bar in an expression indicates division of the expression above the bar (the **numerator**) by the expression below the bar (the **denominator**). In this case the bar also acts as two sets of parentheses, one enclosing the numerator and one enclosing the denominator. In other words, something like

$$\frac{3(7)-5}{2+8} \qquad \text{means} \qquad \frac{(3(7)-5)}{(2+8)}$$

We don't usually put in the parentheses shown in the second form of the expression above, but we do need to understand that they are implied.

◇ **Example 1.2(e):** Evaluate $\dfrac{3(7)-5}{2+8}$.

$$\frac{3(7)-5}{2+8}=\frac{21-5}{10}=\frac{16}{10}=\frac{8}{5}$$

NOTE: Fractions must always be reduced when the numerator and denominator contain common factors. They *can* be left in what is called **improper form**, which means that the numerator is greater than the denominator.

There will be times that you will want to evaluate expressions using your calculator. Your calculator 'knows' the order of operations, which can be an advantage or a disadvantage. When using your calculator, you need to 'think the way it does!' Here are two examples:

◇ **Example 1.2(f):** Evaluate the expression $5[3-2(7+1)]$ using your calculator.

Solution: Our calculators do not have brackets, but we just use parentheses instead, and we use $\times$ for multiplication. So this is entered into our calculators as

$$5\times(3-2\times(7+1))=$$

Try it, making sure you get -65 as your result! (Be sure to include the $=$ sign at the end to complete the calculation.)

◇ **Example 1.2(g):** Use your calculator to evaluate $\dfrac{3(7)-5}{2+8}$.

Solution: The key to evaluating something like this is to recall that it should be interpreted as $\dfrac{(3(7)-5)}{(2+8)}$. Thus it is entered in the calculator as

$$(3 \times 7 - 5) \div (2+8) =$$

Try it; the result should be 1.6, the decimal equivalent of $\frac{8}{5}$.

Evaluating Algebraic Expressions

As stated previously, an algebraic expression is one containing unknown numbers (often only one) that are represented by letters. On many occasions we will evaluate algebraic expressions for given values of the unknown or unknowns in the expression. Suppose that we are asked to evaluate

$$3x - 5y \qquad \text{for} \qquad x = 2,\ y = -7$$

This means to replace x with 2 and y with -7 in the algebraic expression, and evaluate the resulting numerical expression.

◇ **Example 1.2(h):** Evaluate $3x - 5y$ for $x = 2,\ y = -7$.

Solution: A good strategy for evaluating expressions like these is to replace each unknown with a set of parentheses with space between them, then fill the parentheses with the numbers and evaluate:

$$3x - 5y \quad \Rightarrow \quad 3(\quad) - 5(\quad) \quad \Rightarrow \quad 3(2) - 5(-7) = 6 + 35 = 41$$

All we would show when doing this is the sequence $3(2) - 5(-7) = 6 + 35 = 41$; the rest of what I've shown is just what we would be thinking.

◇ **Example 1.2(i):** Evaluate $5x - 2x^2$ for $x = 3$.

Solution:

$$5(3) - 2(3)^2$$
$$15 - 2(9)$$
$$15 - 18$$
$$-3$$

Notice how the order of operations were followed when evaluating this expression, in computing $(3)^2$ *before* multiplying by two.

◇ **Example 1.2(j):** Evaluate $5x - 2x^2$ for $x = -3$.

Solution: Here one must be a little bit careful because we are evaluating the expression for a *negative* value of x:

$$5(-3) - 2(-3)^2 = -15 - 2(9) = -15 - 18 = -33$$

Section 1.2 Exercises

To Solutions

1. Evaluate each of the following expressions, without using a calculator. Give any answers that are not whole numbers as fractions in reduced form.

   (a) $5 - 3(7)$ (b) $\dfrac{8+4}{2} - 1 + 5(4)$ (c) $[2(3+7) - 5](4-6)$

   (d) $\dfrac{10-7}{4} + \dfrac{4+1}{2(3)}$ (e) $3(5-1) - 5(4+1)$ (f) $(2-3)[5+2(7-1)]$

2. Evaluate each of the expressions from Exercise 1 using your calculator, *without writing down any intermediate values*. Use the parentheses ( ) on your calculator, and remember that the calculator "knows" the order of operations. If rounding is necessary, round to the hundredth's place, which is two places past the decimal.

3. Evaluate each expression for the given value or values of the unknowns.

   (a) $2l + 2w, \quad l = 13,\ w = 5$ (b) $\frac{1}{2}bh, \quad b = 7,\ h = 3$

   (c) $\dfrac{6t}{t-1}, \quad t = 5$ (d) $6x - 3(x+2), \quad x = 4$

4. Evaluate each of the following arithmetic expressions without using a calculator. Remember that exponents take precedence over all other operations except those taking place in parentheses. Note also that *an exponent applies only to the numbers (or unknown) that it is directly "attached" to.*

   (a) $5^2 - 2(7)^2$ (b) $(5+4)(-2)^3$ (c) $8 + 4(-3)^2$

   (d) $3(5)^2 - 7(5) + 14$ (e) $4(-3) - 2(-3)^2$ (f) $\dfrac{100 - 5(3)^2}{33}$

   (g) $-(-2)^2 + 5(-2)$ (h) $4(3) - 3^2$ (i) $4(-3) - (-3)^2$

5. Evaluate each of the above expressions with a calculator, without recording any intermediate values.

6. Evaluate each of the following algebraic expressions for the given value of the unknown *without using a calculator*.

   (a) $x^2 - 5x + 3, \quad x = 4$ (b) $x^2 - 5x + 3, \quad x = -4$

   (c) $(3x-5)^2 \quad x = -1$ (d) $x^2 + 3x + 4, \quad x = 5$

7. Use your calculator to evaluate each of the following:

   (a) $P(1+r)^t$ for $P = 800$, $r = 0.05$ and $t = 7$ (b) $-16t^2 + 48t + 5, \quad t = 2$

8. (a) Evaluate $5x - 3x$ and $2x$ for $x = 4$. What do you notice?

   (b) Evaluate the same two expressions for $x = 10$.

   (c) Evaluate $5(x+7)$ and $5x + 35$ for $x = 3$ and $x = -7$.

1.3 Simplifying Linear Algebraic Expressions

1. (d) Apply order of operations and the distributive property to simplify linear algebraic expressions.

In the previous section we learned how to evaluate numerical expressions. In this section we'll be working with algebraic expressions, and our objective with such expressions is usually to 'simplify' them. When doing Exercise 8 from the previous section you should have noticed that the expressions $5x - 3x$ and $2x$ give the same results when evaluating for $x = 4$. We might think this is just an accident, but the same thing happens when evaluating for $x = 10$. When two algebraic expressions give the same result when evaluated for any value of the unknown (the same value in *BOTH* expressions) we say the expressions are equal. In this case we write $5x - 3x = 2x$. This should make sense intuitively; if we have five of some number and we remove (subtract) three of the same number, then two of the unknown number should remain.

Similarly, $5(x + 7) = 5x + 35$. This can be understood as saying that we can add two numbers and multiply the result by five, but we will get the same result if we multiply each number individually by five first, *then* add those two results. You probably know that this is a result of the distributive property, which we'll discuss soon.

A large part of this course will be devoted to simplifying expressions, which is the process of taking an expression and finding an equivalent expression that is somehow 'simpler.' We will have two main tools for doing this: 'combining like terms' and the distributive property. (Combining like terms is really the distributive property, as we'll see.) An example of combining like terms is that $5x - 3x$, as you saw before, is equivalent to $2x$. We can think that x is some unknown amount, then see that we start with five of that amount and remove three *of the same amount*, resulting in two of the unknown amount left. We write $5x - 3x = 2x$. Similarly, $13t + 5t = 18t$. Note that $2l + 2w$ cannot be simplified because l and w are likely (but not necessarily) different amounts. Or, $2l$ and $2w$ are not "like terms," so they can't be combined.

◇ **Example 1.3(a):** Combine like terms when possible: $8x + 3x$, $8x + 3y$, $8x^2 + 3x$

Solution: Like the other examples just given, $8x + 3x = 11x$. We cannot simplify $8x + 3y$ because x and y are likely different values, and we can also not simplify $8x^2 + 3x$ because $x^2 \neq x$ (unless $x = 1$ or $x = 0$).

In a later section we'll see examples similar to the last one above, in which we can combine some like terms.

We mentioned previously the distributive property:

The Distributive Property

For three numbers a, b and c, any of which might be unknowns,

$$a(b + c) = ab + ac \qquad \text{and} \qquad a(b - c) = ab - ac$$

◇ **Example 1.3(b):** Simplify $8(x-2)$.

Solution: Applying the subtraction version of the distributive property, we have

$$8(x-2) = 8(x) - 8(2) = 8x - 16$$

We usually leave out the middle step above - I just put it in to make it completely clear what is happening.

◇ **Example 1.3(c):** Simplify $3(x+4)+7x$.

Solution: In this case we must apply the **distributive property** before combining like terms:

$$3(x+4)+7x = 3x+12+7x = 10x+12$$

Whenever you are uncertain about whether you correctly simplified an expression, it is a good idea to check your work by evaluating both the original expression and the final result for some number value(s). *It is best not to use zero or one when doing this!* Let's test the original expression and the result for $x=2$:

$$3(2+4)+7(2) = 3(6)+14 = 18+14 = 32 \qquad \text{and} \qquad 10(2)+12 = 20+12 = 32$$

Note that we do not distribute to evaluate $3(2+4)$, but apply order of operations instead. Since we got the same result from evaluating both the original expression and the simplified expression for $x=2$, it is likely that that we simplified the expression correctly.

◇ **Example 1.3(d):** Simplify $2x-5(x+1)$.

Solution: It is *VERY* important to note that when we see something like this we must distribute not only the number five, but the minus sign as well:

$$2x-5(x+1) = 2x-5(x)-5(1) = 2x-5x-5 = -3x-5$$

We usually don't show the second step; I put it in to try to emphasize what is happening.

◇ **Example 1.3(e):** Simplify $3-4(3x-2)$.

Solution: Don't forget that subtracting a negative amounts to adding a positive:

$$\begin{gathered} 3-4(3x-2) \\ 3-4(3x)-4(-2) \\ 3-12x-(-8) \\ 3-12x+8 \\ 11-12x \end{gathered}$$

Again, we won't usually show the second step.

◇ **Example 1.3(f):** Simplify $17x + 4[15 - 3(x + 7)]$.

Solution: When there are parentheses inside brackets, it is usually easiest to distribute to eliminate the parentheses first, combine like terms within the brackets, then distribute to eliminate the brackets:

$$17x + 4[15 - 3(x + 7)] = 17x + 4[15 - 3x - 21] =$$

$$17x + 4[-3x - 6] = 17x - 12x - 24 = 5x - 24$$

Simplifying Linear Expressions

To simplify a linear expression, take the following steps.

- Apply the distributive property to eliminate *the innermost parentheses*, taking care to distribute negatives.
- Combine like terms.
- Repeat the above two steps for any parentheses or brackets that remain.

Section 1.3 Exercises

To Solutions

1. Consider the expression $3x - 5(x - 2)$.

   (a) Evaluate the expression for $x = 3$. *Use order of operations, not the distributive property.*

   (b) Simplify the expression; for this you will need the distributive property.

   (c) Evaluate your answer to (b) for $x = 3$. Your answer should be the same as you got for (a).

2. Simplify each of the following.

   (a) $3(2x - 4) - 5(x - 1)$

   (b) $2x + 3(7 - x) + 5$

   (c) $5t + 4(t - 1)$

   (d) $6 - (2x + 4)$

3. Simplify each expression. Refer to Example 1.3(f) if you are not sure what to do.

   (a) $x - 7[3x - (2 - x)]$

   (b) $-3[2x - 4(3x + 1)]$

   (c) $4[7 - 2(x + 1)]$

   (d) $(x - 7) + [3x - (x + 2)]$

   (e) $x - 7[x - (2x + 3)]$

   (f) $4x - 5[3(2x - 1) - 7x]$

4. Simplify the expressions with unknowns in them, evaluate the numerical expressions. When evaluating the numerical expressions, *DO NOT* use the distributive property; instead, just apply the order of operations.

(a) $(2+5)[3-(7+2)]$

(b) $2x+5[x-(7x+2)]$

(c) $3(2x-5)-4(x+1)$

(d) $2+5(3-7+2)$

(e) $3(2-5)-4(3+1)$

(f) $-2(2x+1)+5x(3-1)$

1.4 Algebraic Expressions With Exponents

1. (e) Simplify algebraic expressions with positive integer exponents.

In this section we will simplify algebraic expressions containing exponents. We will see that this can be done by one of two ways. A person can either work each such problem "from scratch" by simply applying the definition of an exponent, or one can use instead some "rules" that we will now derive. Note the following:

$$x^3 \cdot x^2 = (x \cdot x \cdot x) \cdot (x \cdot x) = x \cdot x \cdot x \cdot x \cdot x = x^5 = x^{3+2}$$

and

$$(x^3)^2 = (x \cdot x \cdot x)^2 = (x \cdot x \cdot x) \cdot (x \cdot x \cdot x) = x^6 = x^{3 \cdot 2}$$

These indicate that the following rules hold for exponents: $x^m x^n = x^{m+n}$ and $(x^m)^n = x^{mn}$. *It is easy to confuse these two rules, but if you think back to the two examples given you can always sort out which is which.* We can also see that

$$(xy)^3 = (xy)(xy)(xy) = (x \cdot x \cdot x)(y \cdot y \cdot y) = x^3y^3$$

This demonstrates the rule that $(xy)^m = x^m y^m$.

◇ **Example 1.4(a):** Simplify $(5x)^2$.

Solution: By the third rule above, $(5x)^2 = 5^2x^2 = 25x^2$. We can also "barehand" it:

$$(5x)^2 = (5x)(5x) = 5 \cdot 5 \cdot x \cdot x = 25x^2$$

◇ **Example 1.4(b):** Simplify $(5x^7)^2$.

Solution: Combining the third rule with the second, $(5x^7)^2 = 5^2(x^7)^2 = 25x^{7 \cdot 2} = 25x^{14}$. We can also use the definition of an exponent and the first rule above:

$$(5x^7)^2 = (5x^7)(5x^7) = 5 \cdot 5 \cdot x^7 \cdot x^7 = 25x^{7+7} = 25x^{14}$$

◇ **Example 1.4(c):** Simplify $(4x^5)(2x^7)$.

Solution: $$(4x^5)(2x^7) = 4 \cdot 2 \cdot x^5 \cdot x^7 = 8x^{12}$$

◇ **Example 1.4(d):** Simplify $(4x^5)(2x^7)^3$.

Solution: $$(4x^5)(2x^7)^3 = 4x^5 \cdot 2^3(x^7)^3 = 4x^5 \cdot 8x^{21} = 32x^{26}$$

In Example 1.1(b) we saw that

$$\left(\frac{2}{3}\right)^4 = \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} = \frac{16}{81}$$

This indicates that $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$. We can also see that

$$\frac{x^8}{x^3} = \frac{x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot}{x\cdot x\cdot x} = \frac{x}{x}\cdot\frac{x}{x}\cdot\frac{x}{x}\cdot\frac{x\cdot x\cdot x\cdot x\cdot x}{1} = 1\cdot 1\cdot 1\cdot x^5 = x^5 = x^{8-3}$$

so $\frac{x^m}{x^n} = x^{m-n}$, at least when $m > n$.

Rules of Exponents

- $x^m x^n = x^{m+n}$
- $(x^m)^n = x^{mn}$
- $(xy)^m = x^m y^m$
- $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$
- $\frac{x^m}{x^n} = x^{m-n}$ if $m > n$

At this point we should consider the last rule to only be valid when $m > n$. When this is not the case we get a computation like this:

$$\frac{x^3}{x^8} = \frac{x\cdot x\cdot x}{x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot} = \frac{x}{x}\cdot\frac{x}{x}\cdot\frac{x}{x}\cdot\frac{1}{x\cdot x\cdot x\cdot x\cdot x} = 1\cdot 1\cdot 1\cdot\frac{1}{x^5} = \frac{1}{x^5}$$

We can now use the rules for exponents and the principle just shown to simplify fractions containing exponents.

◇ **Example 1.4(e):** Simplify $\frac{15x^2}{10x^7}$.

Solution: Here we can separate the given fraction into the product of two fractions, reduce each, and multiply the result back together:

$$\frac{15x^2}{10x^7} = \frac{15}{10}\cdot\frac{x^2}{x^7} = \frac{3}{2}\cdot\frac{1}{x^5} = \frac{3}{2x^5}$$

◇ **Example 1.4(f):** Simplify $\frac{(4x^2)^3}{8x^{10}}$.

Solution: In this case we must first apply the third power to the numerator (as prescribed by the order of operations), then we can reduce the fraction in the same way as in the previous example.

$$\frac{(4x^2)^3}{8x^{10}} = \frac{4^3(x^2)^3}{8x^{10}} = \frac{64x^6}{8x^{10}} = \frac{8}{x^4}$$

◇ **Example 1.4(g):** Simplify $\left(\dfrac{21x^9}{35x^2}\right)^3$.

Solution: For this type of exercise it is generally simplest to reduce the fraction inside the parentheses first, as done below, then apply the fourth rule.

$$\left(\frac{21x^9}{35x^2}\right)^3 = \left(\frac{3x^7}{5}\right)^3 = \frac{(3x^7)^3}{5^3} = \frac{27x^{21}}{125}$$

Section 1.4 Exercises To Solutions

1. Simplify each of the following.

   (a) $(4x)(3x^7)$ (b) $(2xy^5)^3$ (c) $(3y^5)^2$ (d) $(2x^3y)^4$

2. Simplify each of the following exponential expressions. Give all answers without negative exponents (which in theory you are not supposed to know about at this point).

   (a) $x^2 \cdot x^5$ (b) $(y^7)^2$ (c) $(s^2t^3)^5$ (d) $\dfrac{r^7s}{r^2s^4}$

   (e) $\dfrac{-24x^3y^5}{16x^4y^2}$ (f) $3(z^2)^5$ (g) $\dfrac{x^3y^6}{x^3y^3}$ (h) $\left(\dfrac{8x^2y}{4x^4}\right)^3$

3. Evaluate each *without using a calculator*.

   (a) $(-3)^2$ (b) -3^2 (c) $\left(\dfrac{5}{2}\right)^3$ (d) $(5-2)(4+1)^2$

   (e) $(7-3)[5-2(3+1)]$ (f) $7-3[5-2(3-1)]$ (g) $\dfrac{8+3(3-1)}{-2(1+3)}$

4. Evaluate each expression for the given value of the unknown.

   (a) a^2-5a+2 for $a=-4$ (b) $\dfrac{3x-x^2}{x-3}$ for $x=-1$

5. Simplify each:

   (a) $8x-3(2x+5)$ (b) $8-3[2x-(4+4)]$ (c) $2(5x-1)-3(x+2)$

1.5 Negative Exponents and Scientific Notation

1. (f) Evaluate and simplify numerical expressions involving integer (including zero) exponents without use of a calculator.
   (g) Change numbers from decimal form to scientific notation and vice versa without using a calculator.

Negative Exponents

In the last section we saw that $\frac{x^8}{x^3} = x^5$ and $\frac{x^3}{x^8} = \frac{1}{x^5}$. In the first case the result can be obtained using the rule $\frac{x^m}{x^n} = x^{m-n}$. If we were to apply the same rule to the second situation we would get $\frac{x^3}{x^8} = x^{3-8} = x^{-5}$, which would make sense if x^{-5} was equal to $\frac{1}{x^5}$. So to make things work out we define negative exponents by $x^{-n} = \frac{1}{x^n}$, and we can then remove the condition that $m > n$ for the fifth rule of exponents.

◇ **Example 1.5(a):** Simplify 5^{-3}.

Solution:
$$5^{-3} = \frac{1}{5^3} = \frac{1}{125}$$

◇ **Example 1.5(b):** Simplify $(-3)^{-4}$

Solution:
$$(-3)^{-4} = \frac{1}{(-3)^4} = \frac{1}{81}$$

One must be careful not to confuse the sign of the exponent with the sign of the number that it is applied to. *A negative exponent does not cause a number to be negative!*

Now we note two things. First, we know that $\frac{x^n}{x^n} = 1$, but if we again apply the rule $\frac{x^m}{x^n} = x^{m-n}$ we get $\frac{x^n}{x^n} = x^{n-n} = x^0$. Therefore it must be the case that $x^0 = 1$ regardless of what x is (well, unless x is zero). Second, remembering that to divide by a fraction we really multiply by its reciprocal,

$$\left(\frac{3}{5}\right)^{-2} = \frac{1}{(\frac{3}{5})^2} = \frac{1}{\frac{9}{25}} = 1 \div \frac{9}{25} = 1 \cdot \frac{25}{9} = \left(\frac{5}{3}\right)^2$$

Here we see that $\left(\frac{3}{5}\right)^{-2}$ is equivalent to $\left(\frac{5}{3}\right)^2$, giving us our final rule for negative and zero exponents.

Rules of Negative and Zero Exponents

- $x^{-n} = \dfrac{1}{x^n}$
- $\left(\dfrac{x}{y}\right)^{-n} = \left(\dfrac{y}{x}\right)^{n}$
- $x^0 = 1$ for all values of x except $x = 0$. 0^0 is undefined.

Scientific Notation

Suppose that we are considering the number of molecules in some amount of a substance, and we decide that there are 2,340,000,000 molecules. There might actually be 2,341,257,956 molecules, but we usually can't determine the value this exactly, so perhaps 2,340,000,000 is the best we can do. Now this number is two billion three hundred and forty million, which is the same as 2.34 billion. A billion happens to be 10^9, so

$$2,340,000,000 = 2.34 \times 10^9$$

The second form of this number is what is called the **scientific notation** form of the number. Scientific notation is used to give very large or small numbers. The scientific notation form of a number consists of a decimal number with just one digit to the left of the decimal point, times a power of ten. Notice that to change 2,340,000,000 to 2.34 we must move the decimal point 9 places to the left; this is where we get the power of ten when changing a number from decimal form to scientific notation form.

◇ **Example 1.5(c):** Change 7100 to scientific notation.

Solution: We move the decimal point three places to the left to get 7.1. We must then multiply by 10^3 so that the actual value of the number will be 7100, rather than 7.1. The final result is then 7.1×10^3.

◇ **Example 1.5(d):** In addition to changing a decimal to scientific notation, we will at times want to change the other direction; change 4.09×10^6 to decimal form.

Solution: The decimal point must be moved six places to the right because of the exponent of six. Therefore the decimal equivalent to 4.09×10^6 is $4,090,000$.

A small number like 0.00000056 is equal to 5.6 times 0.0000001, but $0.0000001 = \frac{1}{10000000} = 10^{-7}$. Therefore we can write

$$0.00000056 = 5.6 \times 10^{-7}$$

The number seven indicates how many places the decimal must move, but since the number is small, the exponent must be negative. Similarly, to change a number like 8.45×10^{-3} to decimal form, we know that the number is small because of the negative exponent, and we know that we must move the decimal to the left to get a small number. Thus

$$8.45 \times 10^{-3} = 0.00845$$

Section 1.5 Exercises

To Solutions

1. Evaluate each of the following.

   (a) 5^{-2} (b) 2^{-4} (c) $(-2)^{-4}$ (d) -3^{-2}

2. Evaluate each of the following.

   (a) 3^{-1} (b) $\left(\frac{3}{4}\right)^{-2}$ (c) $(5)^0$ (d) $(-5)^{-2}$

   (e) $\left(\frac{4}{3}\right)^{-1}$ (f) $\left(\frac{3}{10}\right)^{-3}$ (g) $(-4)^2$ (h) $\left(-\frac{1}{2}\right)^{-5}$

3. Change each of the following large numbers into scientific notation.

   (a) 344,000 (b) 156,700,000,000,000

4. Change each of the following into decimal form.

   (a) 9.85×10^4 (b) 7.328×10^1

5. If the number given is in decimal form, change it to scientific notation form. If the number given is in scientific notation form, change it to decimal form.

   (a) 4300 (b) 1.67×10^{-3} (c) 0.00043 (d) $150,700$

   (e) 0.0000369 (f) 2.65×10^3 (g) 3.142×10^7 (h) 0.0062

6. Simplify each of the following exponential expressions. Give all answers without negative exponents (which in theory you are not supposed to know about at this point).

   (a) s^2t^3 (b) $\dfrac{(6x)^2}{(2x^5)^3}$ (c) $\dfrac{2a^2b^3}{4a^2}$

   (d) $\dfrac{-48ab^{10}}{-32a^4c^3}$ (e) $(2x^3)(5x^2)$ (f) $(-5z^2)^3z^7$

7. Evaluate each of the following expressions, without using a calculator. Give any answers that are not whole numbers as fractions in reduced form.

   (a) $-(-5)^2 - 3(-5) + 1$ (b) $(3-1) - 4(5+1) + 2$ (c) $4(7) - \dfrac{42}{9-5}$

   (d) $-2^2 + 5(2)$ (e) $\dfrac{4(4+2)}{19-5}$ (f) $-5^2 - 3(5) + 1$

8. Evaluate each of the expressions in Exercise 7 with a calculator, *without computing and writing down any intermediate results.*

9. Evaluate each of the following algebraic expressions for the given value of the unknown *without using a calculator*.

   (a) $x^2 - 4x \quad x = -2$ (b) $\dfrac{1}{9-x^2}, \quad x = 2$

A Solutions to Exercises

A.1 Chapter 1 Solutions

Section 1.1 Solutions Back to 1.1 Exercises

1. (a) 27 (b) $\frac{1}{64}$ (c) 100 (d) -125
   (e) -125 (f) 25 (g) -25 (h) $\frac{16}{9}$
2. (a) 262,144 (b) $-59,049$ (c) 59,049 (d) 244,140,625

Section 1.2 Solutions Back to 1.2 Exercises

1. (a) -16 (b) 25 (c) -30 (d) $\frac{19}{12}$ (e) -13 (f) -17
3. (a) 36 (b) $\frac{21}{2}$ or $10\frac{1}{2}$ (c) $\frac{15}{2}$ or $7\frac{1}{2}$ (d) 6
4. (a) -73 (b) -72 (c) 44 (d) 54 (e) -30
   (f) $\frac{5}{3}$ (g) -14 (h) 3 (i) -21
6. (a) -1 (b) 39 (c) 64 (d) 44
7. (a) 1125.68 (b) 37
8. (a) 8, 8, the results are the same (b) 20, 20, the results are the same (c) 50, 50

Section 1.3 Solutions Back to 1.3 Exercises

1. (a) 4 (b) $-2x + 10$ (c) 4
2. (a) $x - 7$ (b) $-x + 26$ (c) $9t - 4$ (d) $-2x + 2$
3. (a) $-27x + 14$ (b) $30x + 12$ (c) $-8x + 20$
   (d) $3x - 9$ (e) $8x + 21$ (f) $9x + 15$
4. (a) -42 (b) $-28x - 10$ (c) $2x - 19$
   (d) -8 (e) -25 (f) $6x - 2$

Section 1.4 Solutions Back to 1.4 Exercises

1. (a) $12x^8$ (b) $8x^3y^{15}$ (c) $9y^{10}$ (d) $16x^{12}y^4$
2. (a) x^7 (b) y^{14} (c) $s^{10}t^{15}$ (d) $\dfrac{r^5}{s^3}$ (e) $-\dfrac{3y^3}{2x}$
   (f) $3z^{10}$ (g) y^3 (h) $\dfrac{8y^3}{x^6}$ (i) $\dfrac{1}{y^3}$
3. (a) 9 (b) -9 (c) $\frac{125}{8}$ (d) 75 (e) -12 (f) 4 (g) $-\frac{7}{4}$

4. (a) 38 (b) 1

5. (a) $2x - 15$ (b) $-6x + 32$ (c) $7x - 8$

Section 1.5 Solutions

Back to 1.5 Exercises

1. (a) $\frac{1}{25}$ (b) $\frac{1}{16}$ (c) $\frac{1}{16}$ (d) $-\frac{1}{9}$
2. (a) $\frac{1}{3}$ (b) $\frac{16}{9}$ (c) 1 (d) $\frac{1}{25}$
   (e) $\frac{3}{4}$ (f) $\frac{1000}{27}$ (g) 16 (h) -32
3. (a) 3.44×10^5 (b) 1.567×10^{14}
4. (a) 98500 (b) 73.28
5. (a) 4.3×10^3 (b) 0.00167 (c) 4.3×10^{-4} (d) 1.507×10^5
   (e) 3.69×10^{-5} (f) 2650 (g) 31,420,000 (h) 6.2×10^{-3}
6. (a) s^2t^3 (b) $\dfrac{9}{2x^{13}}$ (c) $\dfrac{b^3}{2}$ (d) $\dfrac{3b^{10}}{2a^3c^3}$ (e) $10x^5$ (f) $-125z^{13}$
7. (a) -9 (b) -20 (c) $\frac{35}{2}$ (or $17\frac{1}{2}$) (d) 6 (e) $\frac{12}{7}$ (f) -39
8. Same as for Exercise 7.
9. (a) 12 (b) $\frac{1}{5}$