Intermediate Algebra

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Contents

11 Miscellaneous

11.1 Completing the Square

> 11. (a) Solve quadratic equations by completing the square.

Consider the equation $(x+3)^2 = 16$, which can be solved as shown below and to the left.

$$\begin{aligned} (x+3)^2 &= 16 \\ (x+3)(x+3) &= 16 \\ x^2+6x+9 &= 16 \\ x^2+6x-7 &= 0 \\ (x+7)(x-1) &= 0 \\ x &= -7,\, 1 \end{aligned}$$

$$\begin{aligned} (x+3)^2 &= 16 \\ \sqrt{(x+3)^2} &= \pm\sqrt{16} \\ x+3 &= \pm 4 \\ x &= -3 \pm 4 \end{aligned}$$

$$x = -3+4 = 1, \qquad x = -3-4 = -7$$

Note that instead of expanding the left side and solving, we could have taken the square root of both sides of the original equation and solved as shown above and to the right. Of course we need to remember that when we take the square root of both sides of an equation we have to consider both the positive and negative square roots of one side, usually the side that is just a number.

In this section you will see another method for solving quadratic equations besides factoring or the quadratic formula; it is called **completing the square**. This method will appear harder than either of those two methods to most of you, but there are two reasons it is important to learn:

- It shows us where the quadratic formula came from.
- It is useful for other mathematical applications that most of you will see in future math courses.

To demonstrate the method, let's consider the equation $x^2+6x-7=0$ that appeared at one point above. Here's how we solve this equation by completing the square:

◇ **Example 11.1(a):** Use completing the square to solve $x^2+6x-7=0$.

Solution:

x^2+6x-7	$=$	0	original equation
x^2+6x	$=$	7	add seven to both sides to get rid of the constant term on the left side
x^2+6x+9	$=$	$7+9$	add nine to both sides (you'll see later how to decide what number to add)
$(x+3)(x+3)$	$=$	16	factor the left side,
$(x+3)^2$	$=$	16	and note that both factors are the same

$$\begin{aligned} \sqrt{(x+3)^2} &= \pm\sqrt{16} && \text{take the square root of both sides} \\ x+3 &= \pm 4 && \text{and solve for } x \\ x &= -3 \pm 4 && \\ x &= 1,\ -7 && \end{aligned}$$

The key step in the above process is adding nine to $x^2 + 6x$ to create a quadratic expression that factors into two factors that are the same. That is the 'completing the square' step. Note that for any number b,

$$(x + \tfrac{1}{2}b)^2 = (x + \tfrac{1}{2}b)(x + \tfrac{1}{2}b) = x^2 + bx + (\tfrac{1}{2}b)^2$$

Thus if we have something like $x^2 + bx$, the thing that needs to be added to complete the square is $(\frac{1}{2}b)^2$. In the case we just had, $b = 6$ so we add $[\frac{1}{2}(6)]^2 = 3^2 = 9$ to both sides of the equation. Before actually completing the square, let's practice this process a few times.

◇ **Example 11.1(b):** Complete the square for $x^2 + 8x$, $x^2 - 12x$ and $x^2 + 2x$.

Solution: For $x^2 + 8x$ we take half of eight, four, and square that to get 16. When the square is completed we have $x^2 + 8x + 16$. For the next expression, note first that we don't need to take the negative into account, so we complete the square by adding $(\frac{1}{2} \cdot 12)^2 = 36$: $x^2 - 12x + 36$. Finally, for $x^2 + 2x$ we complete the square by adding $(\frac{1}{2} \cdot 2)^2 = 1$ to get $x^2 + 2x + 1$.

Now we see how completing the square works for cases where b is odd. It is really the same as the Example 11.1(a), except that some fractions are involved. We'll solve $x^2 = 3x + 10$; note that here we must start by getting x^2 and $3x$ both on the left side of the equation.

◇ **Example 11.1(c):** Use completing the square to solve $x^2 = 3x + 10$.

Solution:

$$(\tfrac{-3}{2})^2 = \tfrac{9}{4}$$

$$\begin{aligned} x^2 &= 3x + 10 \\ x^2 - 3x &= 10 \\ x^2 - 3x + \tfrac{9}{4} &= 10 + \tfrac{9}{4} \\ (x - \tfrac{3}{2})(x - \tfrac{3}{2}) &+ \tfrac{40}{4} + \tfrac{9}{4} \\ (x - \tfrac{3}{2})^2 &= \tfrac{49}{4} \end{aligned}$$

$$\begin{aligned} \sqrt{(x - \tfrac{3}{2})^2} &= \pm\sqrt{\tfrac{49}{4}} \\ x - \tfrac{3}{2} &= \pm\tfrac{7}{2} \\ x &= \tfrac{3}{2} \pm \tfrac{7}{2} \\ x &= \tfrac{3}{2} + \tfrac{7}{2} = \tfrac{10}{2} = 5, \text{ or} \\ x &= \tfrac{3}{2} - \tfrac{7}{2} = -\tfrac{4}{2} = -2 \end{aligned}$$

Completing the square can also be used to solve quadratic equations whose solutions contain roots:

◇ **Example 11.1(d):** Use completing the square to solve $x^2 - 10x + 22 = 0$.

Solution:

$$(\tfrac{-10}{2})^2 = 25$$

$$\begin{aligned} x^2 - 10x + 22 &= 0 \\ x^2 - 10x &= -22 \\ x^2 - 10x + 25 &= 3 \\ (x-5)(x-5) &= 3 \\ (x-5)^2 &= 3 \end{aligned}$$

$$\begin{aligned} \sqrt{(x-5)^2} &= 3 \\ x - 5 &= \pm\sqrt{3} \\ x &= 5 \pm \sqrt{3} \end{aligned}$$

$$x = 5 + \sqrt{3}, \qquad x = 5 - \sqrt{3}$$

Now we see that completing the square can be used for quadratic equations that have complex number solutions:

◇ **Example 11.1(e):** Use completing the square to solve $x^2 - 4x + 13 = 0$.

Solution:

$$\begin{aligned} x^2 - 4x + 13 &= 0 \\ x^2 - 4x &= -13 \\ x^2 - 4x + 4 &= -13 + 4 \\ (x-2)(x-2) &= -9 \\ (x-2)^2 &= -9 \end{aligned}$$

$$\begin{aligned} \sqrt{(x-2)^2} &= -9 \\ x - 2 &= \pm\sqrt{-9} \\ x &= 2 \pm 3i \end{aligned}$$

$$x = 2 + 3i, \qquad x = 2 - 3i$$

Section 11.1 Exercises

To Solutions

1. Suppose that we wish to complete the square for $x^2 - 10x$.

   (a) Take half of ten, square it, and add the result to $x^2 - 10x$. Do that - at this point you have completed the square.

   (b) Factor the result. Both factors should be the same. Note that the number in each factor is what you got when you took half of ten!

   (c) How would things change if you completed the square for $x^2 + 10x$?

2. Complete the square for each of the following, then factor the resulting quadratic. The last few will involve working with fractions, since the value of a is odd.

   (a) $x^2 + 2x$ (b) $x^2 - 12x$ (c) $x^2 - 4x$

   (d) $x^2 - 9x$ (e) $x^2 - x$ (f) $x^2 + 8x$

3. Solve each of the following by completing the square. Refer to the above examples at first, but try to get to where you don't need it any more. Check your answers by solving by factoring.

   (a) $x^2 + 4x + 3 = 0$ (b) $x^2 + 6x = 16$ (c) $x^2 = 4x + 12$

4. Solve each of the following by completing the square.

   (a) $x^2 - 2x - 4 = 0$ (b) $x^2 - 6x + 7 = 0$ (c) $x^2 - 10x + 13 = 0$

   (d) $x^2 + 2x = 48$ (e) $x^2 = 14x + 51$

5. Solve each of the following quadratic equations by completing the square, **allowing complex solutions**.

   (a) $x^2 + 13 = -6x$ (b) $x^2 + 10x + 26 = 0$ (c) $x^2 + x = 6$

6. Simplify each of the following.

   (a) $\dfrac{1 - \dfrac{1}{x}}{1 + \dfrac{4}{x} - \dfrac{5}{x^2}}$ (b) $\dfrac{\dfrac{1}{x} + \dfrac{3}{x^2}}{1 - \dfrac{6}{x} - \dfrac{7}{x^2}}$ (c) $\dfrac{\dfrac{x^2 - 4}{3}}{\dfrac{x + 2}{6}}$

11.2 Linear Inequalities

11. (b) Solve linear inequalities in one unknown.

Let's recall the notation $a > b$, which says that the number a is greater than the number b. Of course that is equivalent to $b < a$, b is less than a. So, for example, we could write $5 > 3$ or $3 < 5$, both mean the same thing. Note that it is not correct to say $5 > 5$, since five is not greater than itself! On the other hand, when we write $a \geq b$ it means a is greater than *or equal to* b, so we could write $5 \geq 5$ and it would be a true statement. Of course $a \geq b$ is equivalent to $b \leq a$.

In this section we will consider algebraic inequalities; that is, we will be looking at inequalities that contain an unknown value. An example would be

$$3x + 5 \leq 17$$

and our goal is to *solve* the inequality. This means to find all values of x that make this true. Let's begin by finding some values that make the inequality true, and some others that don't.

◇ **Example 11.2(a):** Find some values of x that make the inequality $3x + 5 \leq 17$ true, and some that make it false.

Solution: Let's try $x = 0$: $3(0) + 5 \stackrel{?}{\leq} 17 \quad \Rightarrow \quad 5 \leq 17$ True for $x = 0$

How about $x = 5$: $3(5) + 5 \stackrel{?}{\leq} 17 \quad \Rightarrow \quad 20 \nleq 17$ False for $x = 5$

Try $x = 1$: $3(1) + 5 \stackrel{?}{\leq} 17 \quad \Rightarrow \quad 8 \leq 17$ True for $x = 1$

And $x = 6$: $3(6) + 5 \stackrel{?}{\leq} 17 \quad \Rightarrow \quad 23 \nleq 17$ False for $x = 6$

Maybe $x = 2$: $3(2) + 5 \stackrel{?}{\leq} 17 \quad \Rightarrow \quad 11 \leq 17$ True for $x = 2$

0, 1 and 2 make the inequality true, 5 and 6 do not.

From the above it appears that somewhere between two and five is a "dividing line" between values that make the inequality true and ones that don't. The equation $3x+5 = 17$ has solution $x = 4$, so four satisfies the "or equal to" part of $\leq$. If we were to choose a value of x that was a little larger than four, like 4.1, then $3x$ will be a little larger than twelve, and $3x + 5$ will be a little larger than 17. At this point it should be clear that all the numbers less than or equal to four should make the inequality true. We usually "say" this by writing $x \leq 4$ or $(-\infty, 4]$.

It turns out that we don't have to go through this whole thought process every time we want to solve a simple inequality like this. Instead, we can just solve the inequality in almost the same way as we would solve the equation formed by replacing the $\leq$ symbol with $=$. (You'll see later why the word "almost" was used here.) The next example shows this.

◇ **Example 11.2(b):** Solve the inequality $3x+5 \leq 17$

Solution:

$$\begin{aligned} 3x+5 &\leq 17 \\ 3x &\leq 12 \\ x &\leq 4 \end{aligned}$$

◇ **Example 11.2(c):** Solve the inequality $8x-3(x-1) > 2x+10$

Solution:

$$\begin{aligned} 8x-3(x-1) &> 2x+10 \\ 8x-3x+3 &> 2x+10 \\ 5x+3 &> 2x+10 \\ 3x &> 7 \\ x &> \tfrac{7}{3} \end{aligned}$$

There is a pretty good way to determine whether you were probably correct in solving an inequality. The solution $x > \frac{7}{3}$ means that anything greater than $\frac{7}{3}$ should make the inequality true, and anything less than $\frac{7}{3}$ should make it false. Since $\frac{7}{3} = 2\frac{1}{3}$, three should make the inequality true and two should make it false. Let's check:

$$\begin{aligned} 8(3)-3(3-1) &\overset{?}{>} 2(3)+10 \\ 24-3(2) &\overset{?}{>} 6+10 \\ 18 &> 16 \end{aligned} \qquad\qquad \begin{aligned} 8(2)-3(2-1) &\overset{?}{>} 2(2)+10 \\ 16-3(1) &\overset{?}{>} 4+10 \\ 13 &\ngtr 14 \end{aligned}$$

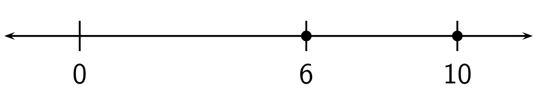
The symbol $\ngtr$ means "not greater than." The above computations indicate that our solution is probably correct.

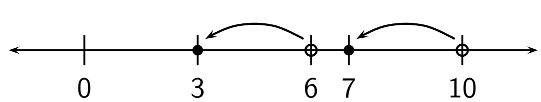
Suppose now that we use the method that we used in Examples 11.2(b) and (c) to solve the inequality $-2x+1 \leq 7$. The result is $x \leq -3$, so -4 should make the inequality true and -2 should make it false. Let's see:

$$\begin{aligned} -2(-4)+1 &\overset{?}{\leq} 7 \\ 8+1 &\overset{?}{\leq} 7 \\ 9 &\nleq 7 \end{aligned} \qquad\qquad \begin{aligned} -2(-2)+1 &\overset{?}{\leq} 7 \\ 4+1 &\overset{?}{\leq} 7 \\ 4+1 &\leq 7 \end{aligned}$$

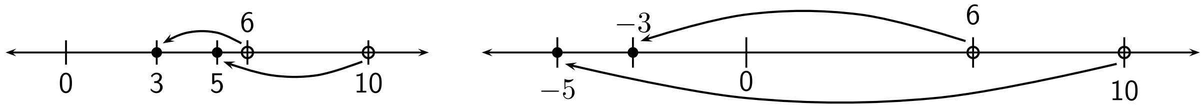
Hmmm... Something appears to be wrong here!

Consider the *true* inequality $6 < 10$; if we put dots at each of these values on the number line the larger number, ten, is farther to the right, as shown to the left below. If we subtract, say, three from both sides of the inequality we get $3 < 7$. The effect of this is to shift each number to the left by five units, as seen in the diagram below and to the right, so their relative positions don't change.





Suppose now that we divide both sides of the same inequality $6 < 10$ by two. We then get $3 < 5$, which is clearly true, and shown on the diagram below and to the left. If, on the other hand, we divide both sides by -2 we get $-3 < -5$. This is not true! The diagram below and to the right shows what is going on here; when we divide both sides by a negative we reverse the positions of the two values on the number line, so to speak.



The same can be seen to be true if we multiply both sides by a negative. This leads us to the following principle.

> **Solving Linear Inequalities**
>
> To solve a linear inequality we use the same procedure as for solving a linear equation, *except that we reverse the direction of the inequality whenever we multiply or divide BY a negative.*

◇ **Example 11.2(d):** Solve the inequality $3x - 5(x-1) > 12$

Solution:

$$\begin{aligned} 3x - 5(x-1) &> 12 \\ 3x - 5x + 5 &> 12 \\ -2x + 5 &> 12 \\ -2x &> 7 \\ x &< -\tfrac{7}{2} \end{aligned}$$

Section 11.2Exercises To Solutions

1. (a) Solve $4x > -12$. Then check your answer by checking a value that should be a solution and one that should not, as we have been doing in this section.

   (b) Solve $-5x > 30$ and check your answer in the same way again.

   (c) The direction of the inequality should change in only one of parts (a) and (b). Which part is it? Both exercises have negatives in them; why does the inequality change direction in one case, but not the other?

2. (a) Solve $-3x + 7 \leq 16$.

   (b) Solve the same inequality by first adding $3x$ to both sides, then subtracting 16 from both sides. This method can always be used to avoid dividing by a negative!

3. Solve each of the following inequalities.

(a) $5x + 4 \geq 18$

(b) $6x - 3 < 63$

(c) $8 - 5x \leq -2$

(d) $4x + 2 > x - 7$

(e) $5x - 2(x - 4) > 35$

(f) $5y - 2 \leq 9y + 2$

(g) $3(x - 2) + 7 < 2(x + 5)$

(h) $7 - 4(3x + 1) \geq 2x - 5$

(i) $4(x + 3) \geq x - 3(x - 2)$

(j) $-4x + 3 < -2x - 9$

(k) $8 - 5(x + 1) \leq 4$

(l) $7 - 2x \leq 13$

4. Solve each of the following quadratic equations, allowing complex solutions. **Use the quadratic formula.**

(a) $2x^2 - 6x + 5 = 0$ (b) $x^2 - 2x + 5 = 0$ (c) $x^2 + 4x + 11 = 0$

5. Divide $(x^3 + 9x^2 + 26x + 24) \div (x + 2)$

6. Perform the indicated addition or subtraction and simplify your answer.

(a) $\dfrac{5}{x^2 + 4x} - \dfrac{2}{x^2 - 16}$ (b) $\dfrac{3}{x^2 - 2x} - \dfrac{5}{x - 2}$

7. Simplify each of the following.

(a) $\dfrac{\dfrac{1}{x} - \dfrac{2}{x^2}}{1 + \dfrac{3}{x} - \dfrac{10}{x^2}}$ (b) $\dfrac{\dfrac{x^2 - 4}{x + 3}}{\dfrac{x + 2}{2x + 6}}$ (c) $\dfrac{1 - \dfrac{9}{x^2}}{1 - \dfrac{1}{x} - \dfrac{6}{x^2}}$

8. Solve each of the following by completing the square. Check your answers by solving by factoring.

(a) $x^2 + 5x = 14$ (b) $x^2 = 2x + 3$ (c) $x^2 + 5x = 6$

9. Solve each of the following by completing the square.

(a) $x^2 + 6x = 19$ (b) $x^2 + 12x - 36 = 0$ (c) $x^2 = 4x + 20$

10. Solve each of the following quadratic equations by completing the square, **allowing complex solutions**.

(a) $x^2 + 14x + 53 = 0$ (b) $x^2 = 2x - 8$

A Solutions to Exercises

A.11 Chapter 11 Solutions

Section 11.1 Solutions Back to 11.1 Exercises

1. (a) $x^2 - 10x + 25$ (b) $(x-5)(x-5) = (x-5)^2$
   (c) We'd still add 25, but it would factor to $(x+5)(x+5)$.

2. (a) $x^2 + 2x + 1 = (x+1)^2$ (b) $x^2 - 12x + 36 = (x-6)^2$
   (c) $x^2 - 4x + 4 = (x-2)^2$ (d) $x^2 - 9x + \frac{81}{4} = (x - \frac{9}{2})^2$
   (e) $x^2 - x + \frac{1}{4} = (x - \frac{1}{2})^2$ (f) $x^2 + 8x + 16 = (x+4)^2$

4. (a) $x = 1 + \sqrt{5},\ 1 - \sqrt{5}$ (b) $x = 3 + \sqrt{2},\ 3 - \sqrt{2}$
   (c) $x = 5 + 2\sqrt{3},\ 5 - 2\sqrt{3}$ (d) $x = 6,\ -8$ (e) $x = 17,\ -3$

5. (a) $x = -3 + 2i,\ -3 - 2i$ (b) $x = -5 + i,\ -5 - i$ (c) $x = 2,\ -3$

6. (a) $\dfrac{x}{x+5}$ (b) $\dfrac{x+3}{(x-7)(x+1)}$ (c) $2x - 4$

Section 11.2 Solutions Back to 11.2 Exercises

1. (a) $x > -3$ (b) $x < -6$
   (c) The direction changes in (b). The direction of the inequality changes in (b) because we divide by a negative. In (a) we divide by a positive, so the inequality doesn't change direction.

2. (a) $x \geq -3$ (b) $-3 < x$

3. (a) $x \geq -3$ (b) $-3 < x$ (c) $x \geq 2$ (d) $x \geq -3$
   (e) $x > 9$ (f) $y \geq -1$ (g) $x < 9$ (h) $x \leq \frac{4}{7}$
   (i) $x \geq -1$ (j) $x > 2$ (k) $x \geq -\frac{1}{5}$ (l) $x \geq -3$

4. (a) $x = \frac{3}{2} + \frac{1}{2}i,\ x = \frac{3}{2} - \frac{1}{2}i$ (b) $x = 1 + 2i,\ x = 1 - 2i$
   (c) $x = -2 + i\sqrt{7},\ x = -2 - i\sqrt{7}$

5. $x^2 + 7x + 12$

6. (a) $\dfrac{3x - 20}{x(x+4)(x-4)}$ (b) $\dfrac{3 - 5x}{x(x-2)}$

7. (a) $\dfrac{1}{x+5}$ (b) $2x - 8$ (c) $\dfrac{x+3}{x+2}$

9. (a) $x = -3 + 3\sqrt{3},\ -3 - 3\sqrt{3}$ (b) $x = -6 + 6\sqrt{2},\ -6 - 6\sqrt{2}$
   (c) $x = 2 + 2\sqrt{3},\ 2 - 2\sqrt{3}$

10. (a) $x = -7 + 2i,\ -7 - 2i$ (b) $x = 1 + i\sqrt{7},\ 1 - i\sqrt{7}$