Intermediate Algebra

Gregg Waterman
Oregon Institute of Technology

Contents

2 Solving Linear and Polynomial Equations

2.1 Equations and Their Solutions

> 2. (a) Determine whether a value is a solution to an equation.

An equation is simply two algebraic expressions connected by an equal sign. The equations that we will deal with for now will contain only one unknown, denoted with some letter, often x. Our objective will be to find all **solutions** to a given equation. A solution is any number that the unknown can be replaced with to give a true statement.

◇ **Example 2.1(a):** Determine whether 1 and 3 are solutions to $(x+2)^2 = 9x$.

Solution: We replace x with 1, then with 3, seeing if each makes the equation true. We will use $\stackrel{?}{=}$ instead of $=$ because we are not sure whether the two sides will in fact be equal in each case.

$$x = 1: \quad \begin{aligned} (1+2)^2 &\stackrel{?}{=} 9(1) \\ 3^2 &\stackrel{?}{=} 9 \\ 9 &= 9 \end{aligned} \qquad\qquad x = 3: \quad \begin{aligned} (3+2)^2 &\stackrel{?}{=} 9(3) \\ 5^2 &\stackrel{?}{=} 27 \\ 25 &\neq 27 \end{aligned}$$

Therefore 1 is a solution to the equation but 3 is not.

We will be spending a great deal of time trying to find all solutions to a given equation; we call the process of finding those solutions **solving the equation**. When asked to solve an equation, *the objective is to find all solutions by the most efficient way possible!* In some cases this might just involve looking at the equation:

◇ **Example 2.1(b):** Solve $x + 5 = 11$.

Solution: Here we can simply look at the equation and see that $x = 6$ is the only solution.

The following example will prove to be extremely important soon.

◇ **Example 2.1(c):** Solve $(x+5)(x-4) = 0$.

Solution: In this case, if $x = -5$ the left side becomes $(0)(-4) = 0$, so $x = -5$ is a solution to the equation. For the same reason, $x = 4$ is also a solution.

◇ **Example 2.1(d):** Solve $x^2 - 9 = 0$.

Solution: It should be clear that $x = 3$ is a solution to this equation, but note that $x = -3$ is as well!

Section 2.1 Exercises

To Solutions

1. Determine whether each of the given values is a solution to the equation.

   (a) $6x + 2 = 9x - 4$, $\quad x = -2$, $\quad x = 4$

   (b) $(x + 2)^2 = 9x$, $\quad x = 4$, $\quad x = 0$

   (c) $2(x - 6) - 7(x - 3) = 14$, $\quad x = -1$

   (d) $16x^2 = 9$, $\quad x = \frac{3}{4}$, $\quad x = -\frac{3}{4}$

   (e) $\sqrt{x + 5} = x^2 - 3x - 1$, $\quad x = 4$, $\quad x = -4$

   (f) $x^5 + 4x^2 = 3$, $\quad x = 1$, $\quad x = -1$

   (g) $\dfrac{\sqrt{x + 10}}{2} = x + 8$, $\quad x = 4$, $\quad x = -6$

   (h) $x^4 + 16 = 8x^2$, $\quad x = -2$, $\quad x = 2$

2. Consider the equation $(x + 2)(x - 7) = 0$.

   (a) Is $x = 2$ a solution to the equation?

   (b) Is $x = 7$ a solution to the equation?

   (c) Give another solution to the equation.

3. Consider the equation $x^2 = 5x$.

   (a) Is $x = 5$ a solution?

   (b) Find another solution by just looking at the equation. (**Hint:** The two easiest numbers to do arithmetic with are zero and one!)

4. Consider the equation $(x - 4)(x^2 + 2x - 15) = 0$.

   (a) Is $x = -5$ a solution to the equation?

   (b) Give a solution to the equation (other than -5 if it turned out to be a solution).

5. Evaluate each of the following.

   (a) $\left(\dfrac{3}{4}\right)^0$ $\qquad$ (b) -5^3 $\qquad$ (c) $(-2)^{-1}$ $\qquad$ (d) $\left(-\dfrac{3}{4}\right)^2$

6. If the number given is in decimal form, change it to scientific notation form. If the number given is in scientific notation form, change it to decimal form.

   (a) 0.0514 $\qquad$ (b) 8.94×10^5 $\qquad$ (c) 6.5×10^{-4} $\qquad$ (d) 7300

2.2 Solving Linear Equations

2. (b) Solve linear equations.

A linear equation is an equation in which the unknown is to the first power and is not under a root or in the bottom of a fraction. Some examples of linear equations are

$$5x + 7 = 13 \qquad \frac{w}{10} - \frac{4}{15} = \frac{w}{5} \qquad 3(x+5) - 7(x-1) = 2x - 4$$

Some equations that are *NOT* linear are

$$\sqrt{t-4} = t + 1 \qquad x^3 = 5x^2 - 6x \qquad \frac{5}{x^2-4} = \frac{2}{x-2} - \frac{7}{x+2}$$

The procedure for solving linear equations is fairly simple. The cornerstone idea is that we can add or subtract the same amount from both sides of an equation, or we can multiply or divide both sides by the same amount (other than zero). Formally we state these ideas as follows:

Properties of Equality

- For any number c, if $a = b$ then $a + c = b + c$ and $a - c = b - c$.
- For any number $c \neq 0$, if $a = b$ then $a \cdot c = b \cdot c$ and $\dfrac{a}{c} = \dfrac{b}{c}$.

Here are two simple examples of solving a linear equation - you should be familiar with these types of equations and the methods for solving them.

◇ **Example 2.2(a):** Solve $5x + 3 = 23$. Another Example

Solution: Here we can see that the solution to this equation is $x = 4$. If instead we had an equation like $5x + 3 = 20$, we would solve the equation as follows:

$$\begin{aligned} 5x + 3 &= 20 && \text{the given equation} \\ 5x &= 17 && \text{subtract } 3 \text{ from both sides} \\ x &= \tfrac{17}{5} && \text{divide both sides by } 5 \end{aligned}$$

◇ **Example 2.2(b):** Solve $3x - 4 = 7x + 2$.

Solution: Here we need to get the two x terms together on one side, and the numbers 4 and 2 on the other. In this case, we subtract $3x$ from both sides to avoid having a negative coefficient for x, and subtract 2 from both sides as well:

$3x - 4 = 7x + 2$	the original equation
$-4 = 4x + 2$	subtract $3x$ from both sides
$-6 = 4x$	subtract 2 from both sides
$\frac{-6}{4} = x$	divide both sides by 4
$x = -\frac{3}{2}$	reduce the resulting fraction, a negative divided by a positive is a negative

Of course we can instead begin by subtracting $7x$ from both sides and adding four to both sides:

$3x - 4 = 7x + 2$	the original equation
$-4x - 4 = 2$	subtract $7x$ from both sides
$-4x = 6$	add 4 to both sides
$x = \frac{6}{-4}$	divide both sides by -4
$x = -\frac{3}{2}$	reduce the resulting fraction

Some equations will involve a few more steps, but shouldn't be any more difficult to solve if you learn a few basic techniques and work carefully. Here is the general procedure that we use when solving linear equations:

Solving Linear Equations

(1) If the equation contains parentheses or other grouping, eliminate them by applying the distributive property.

(2) If the equation contains fractions, eliminate them by multiplying both sides of the equation by the least common denominator (or any common denominator.)

(3) Add or subtract terms from both sides to get all terms with the unknown on one side, all without the unknown on the other. Combine like terms.

(4) Divide both sides by the coefficient of the unknown.

NOTE: The **coefficient** of the unknown is the number that it is multiplied by.

We ALWAYS show our work when solving an equation by first writing the original equation, then showing each successive step below the previous one, as shown in Examples 2.2(a) and (b). **The last line should be the solutions(s), labeled as shown in the previous examples.**

◇ **Example 2.2(c):** Solve $2(x-6)-7(x-3)=14$. Another Example

Solution: In this case we begin by distributing to eliminate parentheses, then we combine like terms. At that point we will have something like the equation in Example 2.2(a).

$2(x-6)-7(x-3)$	$=$	14	the original equation
$2x-6-7x+21$	$=$	14	distribute the 2 and -7 on the left side
$-5x+15$	$=$	14	combine like terms on the left side
$-5x$	$=$	-1	subtract 15 from both sides
x	$=$	$\frac{-1}{-15}$	divide both sides by -15
x	$=$	$\frac{1}{15}$	a negative divided by a negative is a positive

◇ **Example 2.2(d):** Solve $\dfrac{2}{3}x-\dfrac{1}{5}=\dfrac{3}{10}$. Another Example Another Example

Solution: It is possible to solve this equation in the manner used in Example 2.2(a) if you are good with either fractions or your calculator. Here we will see a different method, *that has applications to more difficult equations for which your calculator won't be able to help you as much*. We begin by multiplying both sides of the equation by a number that 3, 5, and 10 all go into, preferably the smallest such number, thirty.

$\frac{2}{3}x-\frac{1}{5}$	$=$	$\frac{3}{10}$	the original equation
$30\left(\frac{2}{3}x-\frac{1}{5}\right)$	$=$	$30\left(\frac{3}{10}\right)$	multiply both sides by 30
$30\left(\frac{2}{3}x\right)-30\left(\frac{1}{5}\right)$	$=$	$30\left(\frac{3}{10}\right)$	distribute the 30 to both terms on the left side
$\overset{10}{\cancel{30}}\left(\frac{2}{\cancel{3}}x\right)-\overset{6}{\cancel{30}}\left(\frac{1}{\cancel{5}}\right)$	$=$	$\overset{3}{\cancel{30}}\left(\frac{3}{\cancel{10}}\right)$	multiply each fraction by 30, cancelling first
$20x-6$	$=$	9	result from multiplying each fraction by 30
$20x$	$=$	15	add 6 to both sides
x	$=$	$\frac{3}{4}$	divide both sides by 20 and reduce

Section 2.2 Exercises To Solutions

1. Solve each of the following equations.

(a) $4x+7=5$

(b) $2x-21=-4x+39$

(c) $\dfrac{9}{5}x-1=2x$

(d) $7(x-2)=x+2(x+3)$

(e) $8x=10-3x$

(f) $\dfrac{x}{4}+\dfrac{1}{2}=1-\dfrac{x}{8}$

(g) $2(x+4)-5(x+10)=6$

(h) $5x+1=2x+8$

2. For the following,

- evaluate numerical expressions (those not containing unknowns)
- simplify algebraic expressions (those containing unknowns)

(a) $10x - 3(2x+4)$ (b) 3^{-2} (c) $10 - 3(2-4)$

(d) $(-3)^2$ (e) $12x + 5[2x - 3(x-1)]$ (f) -3^2

3. Evaluate $-2x^2 + 5x - 3$ for $x = -3$

4. Simplify each of the following exponential expressions. Give all answers without negative exponents.

(a) $(2x^7)^4$ (b) $\frac{6z^3}{3z^4}$ (c) $3s^5(-7s)$ (d) $-\frac{36x^6y^8}{24x^3y^9}$

2.3 Adding, Subtracting and Multiplying Polynomials

2. (c) Add and subtract polynomial expressions.
   (d) Multiply polynomial expressions.

A **polynomial** is an *expression* consisting of the powers $x, x^2, x^3, \ldots$ of some unknown (in this case denoted by x), each multiplied by some number, then added or subtracted along with possibly a number. Some examples are

$$x^3 - 5x^2 + 7x - 9, \qquad 7a^4 - 3a^2 + 5, \qquad 3x + 2, \qquad 48t - 16t^2$$

There is some specific language used in discussing polynomials:

- The highest power of the unknown is called the **degree** of the polynomial. The first polynomial above is third degree, the second is fourth degree, the third polynomial is first degree, and the last one is second degree.
- Second degree polynomials are often called **quadratic polynomials**, or just "quadratics."
- Each power of the unknown, along the number that it is multiplied by and the sign before it is called a **term** of the polynomial. The terms of the first polynomial above are x^3, $-5x^2$, $7x$ and -9. The terms of the last one are $48t$ and $-16t^2$.
- The number that is added or subtracted, along with its sign, is also a term of the polynomial, called the **constant term**. The terms of the first polynomial are then x^3, $-5x^2$, $7x$, and -9, with -9 being the constant term. The constant terms of the second and third polynomials are 5 and 2, respectively.
- Each term has a degree, which is the power of the unknown in that term. For example, the term $-5x^2$ of the first polynomial is the second-degree term.
- The number (with its sign) that a power of the unknown is multiplied by is called the **coefficient** of that power of the unknown. In the first polynomial, -5 is the coefficient of x^2 and 7 is the coefficient of x.

From the examples we note several additional things:

- The unknown can be represented by any letter, and the letter doesn't really matter. The polynomial $s^3 - 5s^2 + 7s - 9$ is the same as $x^3 - 5x^2 + 7x - 9$; the letter used to represent the unknown makes no difference when we go to evaluate the polynomial for assorted values of the unknown.
- It is not necessary that all powers of the unknown less than the degree be represented.
- There need not be a constant term.
- The terms of a polynomial can be arranged in any order, although we usually arrange them so that the degree of each term is lower than the one before it.

One final item of importance is that a number alone can be thought of as a polynomial of degree zero, since something like $3x^0$ is really just 3 times 1, or 3.

If there is more than one term of a particular degree in a polynomial, we usually combine them using the distributive property. For example, $-4x^7 + 2x^7 = (-4+2)x^7 = -2x^7$.

◇ **Example 2.3(a):** Combine the like terms: $3x^2 - 5x + 2 - x^2 - 8x - 5$

Solution: The like terms are $3x^2$ and $-x^2$, $-5x$ and $-8x$, and 2 and -5. We combine them like this:

$$3x^2 - 5x + 2 - x^2 - 8x - 5 = 3x^2 - x^2 - 5x - 8x + 2 - 5 = 2x^2 - 13x - 3$$

We don't usually show the middle step above.

Adding or Subtracting Polynomials

To add or subtract polynomials we follow these steps:

> **Adding or Subtracting Polynomials**
>
> - Get rid of the parentheses on the first polynomial. Distribute the negative sign to all terms of the second polynomial when subtracting, simply get rid of the parentheses on the second when adding.
> - Combine like terms.

Here are some examples:

◇ **Example 2.3(b):** Add $(x^2 - 7x + 3) + (6x^2 + x - 9)$. Another Example

Solution: Because the polynomials are being added and there is no number in front of either set of parentheses to be distributed, we can just get rid of the parentheses and combine like terms:

$$(x^2 - 7x + 3) + (6x^2 + x - 9) = x^2 - 7x + 3 + 6x^2 + x - 9 = 7x^2 - 6x - 6$$

◇ **Example 2.3(c):** Subtract $(5x^2 - 3x + 1) - (3x^2 - 8x + 3)$. Another Example

Solution: In this case the subtraction has to distribute to all terms of the second polynomial:

$$\begin{aligned}(5x^2 - 3x + 1) - (3x^2 - 8x + 3) &= 5x^2 - 3x + 1 - 3x^2 - (-8x) - 3 \\ &= 5x^2 - 3x + 1 - 3x^2 + 8x - 3 \\ &= 2x^2 + 5x - 2\end{aligned}$$

We will usually not show the second step above, but instead jump to the third step or not even show any intermediate steps at all.

Note the difference in how the work is shown in the above two examples. In Example 2.3(b) we are really just simplifying the expression $(x^2 - 7x + 3) + (6x^2 + x - 9)$, and in 2.3(c) the expression $(5x^2 - 3x + 1) - (3x^2 - 8x + 3)$ is being simplified. If we have room we sometimes show our work horizontally as in Example 2.3(b). When there is not enough room to do so we usually show one step to the right, then work downward after that, like in Example 2.3(c).

Multiplying Polynomials

Recall that to simplify something like $5(2x - 7)$ we multiply the entire quantity $2x - 7$ by five, by distributing:

$$5(2x - 7) = 5(2x) + 5(-7) = 10x - 35$$

We do the very same thing with something more complicated:

$$7x^2(x^2 - 5x + 2) = (7x^2)(x^2) + (7x^2)(-5x) + (7x^2)(2) = 7x^4 - 35x^3 + 14x^2$$

The first part of this, $7x^2$ is called a **monomial**, meaning a polynomial with just one term. Notice that we simply distribute the monomial to each term of the polynomial.

Now suppose we want to simplify $(x - 3)(x^2 - 5x + 2)$. To do this we have to "double-distribute." First we distribute the $(x-3)$ factor to each term of $x^2 - 5x + 2$, then we distribute those terms into $(x - 3)$:

$$\begin{aligned} (x - 3)(x^2 - 5x + 2) &= (x - 3)x^2 + (x - 3)(-5x) + (x - 3)2 \\ &= (x)(x^2) + (-3)(x^2) + (x)(-5x) + (-3)(-5x) + (x)(2) + (-3)(2) \end{aligned}$$

At this point we are not done, but let's make an observation. Notice that we have each term of the first polynomial $x-3 = x+(-3)$ times each term of the second, $x^2-5x+2 = x^2+(-5x)+2$. This is precisely how we multiply two polynomials, we *multiply each term of the first times each term of the second and add the results*. Finishing the above gives us a final result of

$$x^3 - 3x^2 - 5x^2 + 15x + 2x - 6 = x^3 - 8x^2 + 17x - 6$$

Multiplying Polynomials

- To multiply a monomial times a polynomial, distribute the monomial to each term of the polynomial and multiply.
- To multiply two polynomials, first distribute each term of the first polynomial to each term of the second and multiply. Be careful with signs! Then combine like terms.

◇ **Example 2.3(d):** Multiply $3x^2(x + 1)$. Another Example

$$3x^2(x + 1) = 3x^2(x) + 3x^2(1) = 3x^3 + 3x^2$$

In the above calculation we would not generally show the second step, and from here on I will no longer show such intermediate steps.

◇ **Example 2.3(e):** Multiply $(x+5)(x-3)$. Another Example

$$(x+5)(x-3) = x^2 - 3x + 5x - 15 = x^2 + 2x - 15$$

◇ **Example 2.3(f):** Multiply $(2x-4)(x+5)$. Another Example

$$(2x-4)(x+5) = 2x^2 + 10x - 4x - 20 = 2x^2 + 6x - 20$$

◇ **Example 2.3(g):** Multiply $(3x-4)(3x+4)$.

$$(3x-4)(3x+4) = 9x^2 + 12x - 12x - 16 = 9x^2 - 16$$

This last example illustrates something that will soon be of importance to us. Look at the results of the last three examples, together:

$$(x+5)(x-3) = x^2+2x-15, \qquad (2x-4)(x+5) = 2x^2+6x-20, \qquad (3x-4)(3x+4) = 9x^2-16$$

Notice that in the first two cases there was a middle term containing an x, whereas that term "disappeared" in Example 2.3(g). The result, $9x^2 - 16$, is something that we call a **difference of squares**.

◇ **Example 2.3(h):** Multiply $(x-2)(x^2-7x+4)$. Another Example

$$(x-2)(x^2-7x+4) = x^3 - 7x^2 + 4x - 2x^2 + 14x - 8 = x^3 - 9x^2 + 18x - 8$$

Section 2.3 Exercises

To Solutions

1. Combine the like terms in each expression.

   (a) $7x^2 - 3x + 2x^2 - 4$

   (b) $-5x^2 + 3x - 2 + 2x^2 + 7x + 4$

   (c) $x^3 - 4x^2 + x^2 - 3x - 6x - 1$

   (d) $4x^3 + 5x - 11x^2 + 2x - 3$

2. Add or subtract the polynomials, as indicated.

   (a) $(2x^2 - 5x + 7) + (7x - 3)$

   (b) $(5x + 7) - (x^2 + x - 3)$

   (c) $(-2a^2 + 4a - 5) - (-3a^2 - a - 9)$

3. For each of the following, multiply the polynomials.

(a) $(x+7)(x-2)$

(b) $x(5x^2+7x-2)$

(c) $(2x-5)(x^2+12x+1)$

(d) $(x+3)^2$

(e) $(x+4)(3x+5)$

(f) $(3x-1)(x+5)$

4. Solve each of the following equations.

(a) $3x+5=2$

(b) $4x-3(5x+2)=4(x+3)$

(c) $\frac{2}{3}x-1=\frac{3}{4}x+\frac{1}{2}$

2.4 Factoring Polynomials

> 2. (e) Factor quadratic trinomials and differences of squares. Factor polynomial expressions (quadratic, higher degree) by factoring out common factors, or grouping.

In the last section you saw how to multiply two polynomials together. We will have need to do that on occasion, but more often we will want to take a single polynomial and break it down into factors that could then be multiplied back together to get the original polynomial. One of the main reasons for developing this skill is to be able to solve equations containing polynomials, like the equation $x^2 + 2x = 15$.

◇ **Example 2.4(a):** Solve $x^2 + 2x = 15$.

Solution: We subtract 15 from both sides to get $x^2 + 2x - 15 = 0$, then factor the left side to get $(x + 5)(x - 3) = 0$. Here we see that if x was -5 we'd have

$$(-5 + 5)(-5 - 3) = (0)(-8) = 0,$$

so -5 is a solution to $x^2 + 2x = 15$. We can check this in the original equation:

$$(-5)^2 + 2(-5) = 25 - 10 = 15.$$

By the same reasoning, $x = 3$ is also a solution.

The simplest type of factoring is factoring out a common factor. Although it is very easy to do, students often overlook it later, so make note of it now! The next example shows this type of factoring.

◇ **Example 2.4(b):** Factor $2x^3 - 10x$.

Solution: Here we see that both terms have factors of both 2 and x, so we "remove" them:

$$2x^3 - 10x = (2x)x^2 - (2x)5 = 2x(x^2 - 5)$$

Factoring Quadratic Trinomials

Perhaps the most common type of factoring that you will do throughout all of your math courses is factoring expressions like

$$x^2 + 4x + 3 \qquad \text{and} \qquad 2x^2 - 5x + 3$$

which we call **quadratic trinomials**. The word 'quadratic' refers to the fact that the highest power of x is two, and 'trinomial' means the expressions have three terms. To give us some insight into how to factor the first of these, we see that

$$(x+a)(x+b) = x^2 + bx + ax + ab = x^2 + (a+b)x + ab \tag{1}$$

The x^2 term comes from multiplying the two x's at the fronts of $x+a$ and $x+b$. The constant term, ab, comes from multiplying the numbers a and b. Lastly, the middle term comes from the sum of ax and bx. This process is what many of us think of as FOIL: first, outside, inside, last. The next example shows how we use these ideas by first thinking about F and L, first and last, then O and I, outside and inside..

◇ **Example 2.4(c):** Factor $x^2 + 4x + 3$.

Solution: To get the x^2 first term of x^2+4x+3 we must have factors like $(x \quad)(x \quad)$. Because the constant term 3 only has factors 1 and 3, the factors of x^2+4x+3 must look like $(x \quad 1)(x \quad 3)$. Next we see that the outside and inside products of what we have so far are $3x$ and $1x$. If these were both positive their sum would be $+4x$, the middle term of the trinomial. Let's check what we have concluded:

$$(x+1)(x+3) = x^2 + 3x + x + 3 = x^2 + 4x + 3,$$

the desired result. Thus $x^2 + 4x + 3$ factors to $(x+1)(x+3)$.

In (1) on the previous page we see that the coefficient (number in front) of x is the sum of a and b. Thus when factoring $x^2 + 4x + 3$ we might realize that a and b can't both be negative or we would get a negative coefficient of x. We would then go directly to the correct factoring right away.

◇ **Example 2.4(d):** Factor $x^2 - 9x - 10$.

Solution: Again we must have factors like $(x \quad)(x \quad)$ and, because $(2)(5) = 10$ the factors of $x^2 - 9x - 10$ might be $(x \quad 2)(x \quad 5)$. In this case the outside and inside products are $5x$ and $2x$. The sum of these can't be nine, regardless of the sign of each, so the combination of 2 and 5 to get 10 is not what we want. Choosing 1 and 10 instead, we have $(x \quad 1)(x \quad 10)$, and the outside and inside products are $10x$ and $1x$. If the $10x$ was negative and the $1x$ positive, their sum would be $-9x$, which is what we want. Checking, we see that

$$(x+1)(x-10) = x^2 - 10x + x - 10 = x^2 - 9x - 10$$

This is the desired result, so $x^2 - 9x - 10$ factors to $(x+1)(x-10)$.

In both of the above examples, the value of a in the quadratic trinomials $ax^2 + bx + c$ was one, the easiest situation to deal with. When a is not one, the process of factoring becomes yet a little more complicated. There are various strategies available for factoring. If you already have a method you use to *efficiently and correctly* factor, by all means continue using it. However, if your method is slow or not reliable for giving correct factorings, you may wish to try the method I'm about to describe. The method involves some amount of trial and error, which is unavoidable without reducing the process to a huge set of rules. With a bit of practice and some 'number sense' you will be able to reduce the number of trials needed to get a correct factoring. At the top of the next page are the steps for this method; you might wish to not read them, and instead refer to them as you read through Example 2.4(e).

Factoring $ax^2 + bx + c$

(1) Choose two numbers m and n for which $mn = a$ and set up factors $(mx \quad)(nx \quad)$. Do not change these until you have exhausted all possibilities for p and q (see below)!

(2) Choose two numbers p and q for which $pq = |c|$, the absolute value of c. In other words, don't be concerned yet with the sign of c. Set up the factors $(mx \quad p)(nx \quad q)$.

(3) Find the outer and inner products mqx and pnx from the 'FOIL' process. Add them and subtract them to see if you can get the bx term from the original trinomial $ax^2 + bx + c$. If you can, go on to step (7).

(4) If you were not successful your first time through step (7), switch the order of p and q to get the factors $(mx \quad q)(nx \quad p)$ and try steps (3), and perhaps (7) again. If it still doesn't work, go on to step (5).

(5) Go back to step (2) and choose a new p and q. Repeat step (3). If you have exhausted all possibilities for p and q, go to the next step.

(6) Go back to step (1) and choose a different m and n, then repeat the other steps.

(7) Once you have found factors $(mx \quad q)(nx \quad p)$ for which $pq = |c|$ and mqx and pnx add or subtract to give bx, determine the signs within each factor that give the correct bx and the correct sign for c. If this doesn't work and you haven't tried switching p and q yet, go to step (4). If you have already been through (4), go back to step (2) now.

◇ **Example 2.4(e):** Factor $2x^2 - 5x + 3$.

1) The factors $(2x \quad)(x \quad)$ will give us the $2x^2$ term of $2x^2 - 5x + 3$.

2) We now find two factors of the 3 from $2x^2 - 5x + 3$; 1 and 3 are the only choices. This means the product looks like $(2x \quad 1)(x \quad 3)$ or $(2x \quad 3)(x \quad 1)$.

3) In the first case, multiplying the 'outside' and 'inside' gives $6x$ and x, which can be subtracted to get $5x$. The factors $(2x \quad 1)(x \quad 3)$ are then candidates for the factorization. To get the $5x$ to be negative we would have to have $(2x+1)(x-3)$, but if we 'FOIL these out we get $2x^2 - 5x - 3$ rather than $2x^2 - 5x + 3$.

4) In the first case, multiplying the 'outside' and 'inside' gives $6x$ and x, which can be subtracted to get $5x$.

5) Since $(2x \quad 1)(x \quad 3)$ didn't work, we try $(2x \quad 3)(x \quad 1)$. In this case, to get the $5x$ to be negative we would have to have $(2x-3)(x-1)$, and when we 'FOIL these out we get $2x^2 - 5x + 3$, the desired result. $2x^2 - 5x + 3$ then factors to $(2x-3)(x-1)$.

Factoring Differences of Squares

◇ **Example 2.4(f):** Factor $4x^2 - 25$.

Solution: Although we won't usually do this, let's think of this as $4x^2 + 0x - 25$ and apply the same method as used in the previous example. Although the factors could look like $(4x \quad)(x \quad)$, let's try $(2x \quad)(2x \quad)$ first. To get the constant term of 25, we could have factors that look like $(2x \quad 5)(2x \quad 5)$, and to get the 25 to be negative, one sign must be positive and the other negative. We see that

$$(2x+5)(2x-5) = 4x^2 - 10x + 10x - 25 = 4x^2 - 25,$$

so $4x^2 - 25$ factors to $(2x+5)(2x-5)$.

The quadratic expression in the previous example is called a **difference of squares**. The following is a description of how to factor a difference of squares.

Factoring a Difference of Squares $m^2x^2 - n^2$

The difference of squares $m^2x^2 - n^2$ factors to $(mx+n)(mx-n)$.

◇ **Example 2.4(g):** Factor $9x^4 - x^2$.

Solution: We see that there is a common factor of x^2, and after that we have a difference of squares:

$$9x^4 - x^2 = x^2(9x^2 - 1) = x^2(3x+1)(3x-1)$$

Factoring by Grouping

Factoring by grouping is a method that applies to some third degree polynomials with four terms, like

$$x^3 - x^2 + 7x - 7 \qquad \text{and} \qquad x^3 + 2x^2 - 4x - 8.$$

Basically, the method goes like this:

(1) Factor a common factor out of the first two terms, and a different common factor out of the third and fourth terms.

(2) The result of (1) is something of the form $something \times stuff \pm something\ else \times stuff$, where both pieces of '$stuff$' are the same. This expression only has two terms, with the common factor of '$stuff$' in both terms, so it can be factored out to get $(stuff) \times (something \pm something\ else)$.

It is quite likely that the above description makes no sense at all! Let's see what it means by looking at a couple of examples.

◇ **Example 2.4(h):** Factor $x^3 - x^2 + 7x - 7$.

Solution: First we factor x^2 out of the first two terms and 7 out of the third and fourth terms to get $x^2(x-1) + 7(x-1)$. This expression can be thought of as having just two terms, $x^2(x-1)$ and $7(x-1)$. Both of them have a factor of $x-1$, which can then be factored out to get $(x-1)(x^2+7)$.

Sometimes when factoring by grouping we can go one step beyond what we were able to do in this last example. The next example illustrates this.

◇ **Example 2.4(i):** Factor $x^3 + 5x^2 - 4x - 20$ completely.

Solution: Factoring x^2 out of the first two terms and -4 out of the last two gives $x^2(x+5) - 4(x+5)$. As in the last example, we then factor $(x+5)$ out of the two terms to get $(x+5)(x^2-4)$. Finally, we factor the difference of squares $x^2 - 4$ to get $(x+5)(x+2)(x-2)$.

Let's summarize the general method for factoring polynomials:

Factoring Polynomials

- Factor out common factors (of both the the numbers and the variable).
- If the polynomial has four terms, try factoring by grouping.
- Factor any quadratic factors.
- Check to see if any of your factors can be factored further.
- (Optional) Check your factoring by multiplying the factors back together.

Section 2.4 Exercises

To Solutions

1. Factor the largest possible common factor out of $8x^4 - 4x^3 + 16x^2$.

2. Factor each quadratic expression.

   (a) $x^2 - 3x - 10$ (b) $x^2 - 7x + 12$ (c) $x^2 - 2x - 15$
   (d) $x^2 + 10x + 9$ (e) $x^2 - 11x + 28$ (f) $x^2 + 4x - 12$

3. Factor each quadratic expression.

   (a) $7x^2 + 3x - 4$ (b) $6x^2 - x - 2$ (c) $6x^2 + 13x + 5$
   (d) $3x^2 - 19x - 14$ (e) $10x^2 + 27x - 9$ (f) $15x^2 - 21x + 6$

4. Factor each difference of squares.

   (a) $x^2 - 9$ (b) $25x^2 - 1$ (c) $9x^2 - 49$

5. Factor each expression completely. Do this by first factoring out any common factors, then factoring whatever remains after that.

   (a) $x^3 + 5x^2 + 4x$ (b) $3x^3 + 15x^2 - 42x$ (c) $x^4 + 6x^3 + 5x^2$

6. Factor each of the following by grouping.

   (a) $x^3 + 2x^2 - 3x - 6$ (b) $2x^3 + 3x^2 - 2x - 3$ (c) $x^3 + 2x^2 - 25x - 50$

7. Factor each of the following **completely**. Some may not be possible.

   (a) $4x^4 + 4x^3 + 4x^2$ (b) $10x^2 - 17x + 3$ (c) $3x^2 - 4x - 5$
   (d) $9x^2 + 11x + 2$ (e) $6x^3 - 4x^2 + 15x - 10$ (f) $16x^2 - 49$
   (g) $3 + 23x - 8x^2$ (h) $4x^2 + 9$ (i) $12x^3 + 24x^2 + 48x$
   (j) $6x^2 - 19x - 7$ (k) $2x^2 - 15x + 7$ (l) $x^3 + 3x^2 - 4x - 12$
   (m) $10y^2 - 5y - 15$ (n) $1 - 25x^2$ (o) $x^2 + 3x + 1$
   (p) $6x^2 - x - 15$ (q) $7x^2 + 11x + 4$ (r) $8 + 2x - 3x^2$

8. Evaluate each of the following algebraic expressions for the given value of the unknown *without using a calculator*.

   (a) $x^2 + 2x + 1, \qquad x = 5$ (b) $4x - x^2, \qquad x = -2$

9. Evaluate each of the following.

   (a) $\left(\frac{3}{2}\right)^{-3}$ (b) 5^{-2} (c) $(-7)^0$ (d) -5^{-2}

10. Add or subtract the polynomials, as indicated.

    (a) $(5x^2 - 8x + 3) - (9x^2 + 5x - 2)$ (b) $(x^2 - 4) - (x^2 + 12x + 3)$

11. For each of the following, multiply the polynomials.

    (a) $(2x - 5)(3x^2 - 7x + 4)$ (b) $(3x + 5)^2$ (c) $3x(x^2 - 7x - 5)$
    (d) $(2x - 1)(3x + 5)$ (e) $(x - 1)^2$ (f) $(x - 6)(5x^2 + x - 1)$

2.5 Solving Polynomial Equations

2. (f) Solve polynomial equations.

Look back at Example 2.4(a), where the equation $x^2 + 2x = 15$ was solved. This equation is a **polynomial equation**, which is simply an equation in which the expressions on both sides of the equal sign are polynomials. (Remember that a single number, even zero, can be thought of as a polynomial.) The highest power of the unknown that appears in the equation is called the **degree** of the equation.

To solve the equation, all terms were put on one side and the left side was factored to get $(x+5)(x-3) = 0$. The critical idea is that the factors $x+5$ and $x-3$ are being multiplied to get zero, and the only way that can happen is if one of them is zero. The only way that $x+5$ can be zero is if $x = -5$, and the only way that $x-3$ can be zero is if $x = 3$. A basic, but very important idea has been used here:

Zero Factor Property

If the product of two or more factors is zero, then at least one of the factors must be zero. Symbolically, if $A \cdot B = 0$, then either $A = 0$ or $B = 0$ (or in some cases, both).

◇ **Example 2.5(a):** Solve $2x^2 + 3x = 20$. Another Example

Solution: We begin by getting zero alone on one side and factoring:

$$\begin{aligned} 2x^2 + 3x &= 20 \\ 2x^2 + 3x - 20 &= 0 \\ (2x-5)(x+4) &= 0 \end{aligned}$$

From this we can easily see that $x = -4$ is a solution, because it will make the factor $x+4$ be zero, and the entire left side of the equation will be zero as a result. It is not as readily apparent what value of x makes $2x-5$ zero, so we set that factor to zero and solve to find out:

$$\begin{aligned} 2x - 5 &= 0 \\ 2x &= 5 \\ x &= \tfrac{5}{2} \end{aligned}$$

The equation $2x^2 + 3x = 20$ therefore has two solutions, $x = -4$ and $x = \frac{5}{2}$.

◇ **Example 2.5(b):** Solve $4x^3 - 8x^2 - 12x = 0$.

Solution: First we factor the left side, beginning with factoring a common factor of $4x$ out of each term:

$$\begin{aligned} 4x^3 - 8x^2 - 12x &= 0 \\ 4x(x^2 - 2x - 3) &= 0 \\ 4x(x-3)(x+1) &= 0 \end{aligned}$$

In this case there are three factors, $4x$, $x-3$ and $x+1$. The x values 0, 3 and -1 make them zero, so those are the solutions to the equation.

◇ **Example 2.5(c):** Solve $4x^2 - 8x - 12 = 0$.

Solution: This equation is very similar to the one in the previous example, but this one is second degree and the only common factor is 4, rather than $4x$.

$$\begin{aligned} 4x^2 - 8x - 12 &= 0 \\ 4(x^2 - 2x - 3) &= 0 \\ 4(x-3)(x+1) &= 0 \end{aligned}$$

The solutions in this case are just $x = 3$ and $x = -1$, because the factor 4 can't be zero, regardless of what value x has.

Note that Examples 2.5(a) and (c) were both second degree (quadratic) equations, and each had two solutions. The equation in Example 2.5(b) is third degree, and has three solutions. We might guess that the number of solutions is the the degree of the equation. This is close to correct, but what is really true is the following:

> **Solutions of Polynomial Equations**
>
> The number of solutions to a polynomial equation is *at most* the degree of the equation. (For example, a third degree polynomial has three *or fewer* solutions.)

◇ **Example 2.5(d):** Solve $x^2 - 6x + 9 = 0$.

$$\begin{aligned} x^2 - 6x + 9 &= 0 \\ (x-3)(x-3) &= 0 \end{aligned}$$

Solution: $x = 3$ is the only solution. This shows that a second degree polynomial need not have two solutions.

◇ **Example 2.5(e):** Solve $4x^2 = 25$.

$$\begin{aligned} 4x^2 &= 25 \\ 4x^2 - 25 &= 0 \\ (2x+5)(2x-5) &= 0 \end{aligned}$$

$$\begin{array}{rclcrcl} 2x+5 &=& 0 & \text{or} & 2x-5 &=& 0 \\ 2x &=& -5 & \text{or} & 2x &=& 5 \\ x &=& -\frac{5}{2} & \text{or} & x &=& \frac{5}{2} \end{array}$$

◇ **Example 2.5(f):** Solve $3x^3 - x^2 - 12x + 4 = 0$.

Solution: Because this is a third degree polynomial equation, we expect that we might have as many as three solutions. We also notice that the left side is in the correct form to apply factoring by grouping.

$$\begin{aligned} 3x^3 - x^2 - 12x + 4 &= 0 \\ x^2(3x-1) - 4(3x-1) &= 0 \\ (3x-1)(x^2-4) &= 0 \\ (3x-1)(x+2)(x-2) &= 0 \end{aligned}$$

Here we can see that two solutions are 2 and -2, and setting $3x-1$ equal to zero and solving gives a third solution of $x = \frac{1}{3}$.

Solving Polynomial Equations

- Note how many solutions you expect to find.
- Eliminate parentheses (except in special cases like the one shown in the example above) by distributing or multiplying. Eliminate fractions by multiplying both sides by the least common denominator.
- Get zero on one side of the equation, all other terms on the other side.
- Factor the non-zero side of the equation completely.
- Find the value that makes each factor zero.
  - Do this by inspection if possible.
  - When you can't determine the value that makes the factor zero by inspection, set the factor equal to zero and solve.

All of the values that make factors equal to zero are solutions to the original equation.

Section 2.5 Exercises To Solutions

1. For each of the following, first eliminate fractions by multiplying both sides by a number that eliminates all the fractions. Then get zero on one side and factor.

   (a) $\frac{1}{6}x^2 = \frac{2}{3}x + 2$ (b) $\frac{1}{3}x^2 + \frac{8}{3} = 2x$ (c) $\frac{1}{4}x^2 = \frac{1}{2}x + 6$

2. For each of the following you will need to factor out a common factor (number, unknonwn, or both), then factor again.

   (a) $x^3 + 3x^2 = 10x$ (b) $4x^2 + 32x + 28 = 0$ (c) $5x^3 = 5x^2 + 30x$

3. Solve each equation. Remember to get all terms on one side, and be sure to look for common factors!

   (a) $x^2 - 13x + 12 = 0$ (b) $x^2 + 15 = 8x$ (c) $x^2 - 16 = 0$

   (d) $5x^2 + x = 0$ (e) $3x^2 = 20x + 7$ (f) $16x + 16 = x^3 + x^2$

   (g) $3x^2 + 24x + 45 = 0$ (h) $\frac{2}{3}x^2 + \frac{7}{3}x = 5$ (i) $x^3 + 3x^2 + 2x = 0$

   (j) $x^2 + 7x + 6 = 0$ (k) $8x^2 = 16x$ (l) $2x^3 + x^2 = 18x + 9$

4. Simplify each of the following exponential expressions. Give all answers without negative exponents (which in theory you are not supposed to know about at this point).

   (a) $(2x^5)^3(7x^4)$ (b) $4y^3y^4$ (c) $\left(\frac{15s^5t^9}{12st}\right)^2$ (d) $\frac{3u^3v^2}{6v^2}$

5. If the number given is in decimal form, change it to scientific notation form. If the number given is in scientific notation form, change it to decimal form.

   (a) 2.37×10^2 (b) 0.0049 (c) 1.6×10^{-1} (d) $53,000$

6. Solve each of the following equations.

   (a) $\frac{3x}{4} - \frac{5}{12} = \frac{5x}{6}$ (b) $11x + 9 = 3x + 11$ (c) $4(x + 1) - 7(x + 5) = 20$

A Solutions to Exercises

A.2 Chapter 2 Solutions

Section 2.1 Solutions Back to 2.1 Exercises

1. (a) -2 is not a solution, 4 is not
   (b) 4 is a solution, 0 is not
   (c) -1 is a solution
   (d) Both $\frac{3}{4}$ and $-\frac{3}{4}$ are solutions
   (e) 4 is not a solution, -4 is not
   (f) 1 is not a solution, -1 is
   (g) 4 is not a solution, -6 is
   (h) Both -2 and 2 are solutions
2. (a) 2 is not a solution
   (b) 7 is a solution
   (c) Another solution is $x = -2$
3. (a) 5 is a solution
   (b) 0 is a solution
4. (a) Yes, -5 is a solution.
   (b) 4 is also a solution
5. (a) 1 (b) -125 (c) $-\frac{1}{2}$ (d) $\frac{9}{16}$
6. (a) 5.14×10^{-2} (b) $894,000$ (c) 0.00065 (d) 7.3×10^3

Section 2.2 Solutions Back to 2.2 Exercises

1. (a) $-\frac{1}{2}$ (b) 10 (c) -5 (d) 5 (e) $\frac{10}{11}$
   (f) $\frac{4}{3}$ (g) -16 (h) $\frac{7}{3}$
2. (a) $4x - 12$ (b) $\frac{1}{9}$ (c) 16 (d) 9 (e) $7x + 15$ (f) -9
3. -36
4. (a) $16x^{28}$ (b) $\dfrac{2}{z}$ (c) $-21s^6$ (d) $-\dfrac{3x^3}{2y}$

Section 2.3 Solutions Back to 2.3 Exercises

1. (a) $9x^2 - 3x - 4$ (b) $-3x^2 + 10x + 2$ (c) $x^3 - 3x^2 - 9x - 1$
   (d) $4x^3 - 11x^2 + 7x - 3$
2. (a) $2x^2 + 2x + 4$ (b) $-x^2 + 4x + 10$ (c) $a^2 + 5a + 4$
3. (a) $x^2 + 5x - 14$ (b) $5x^3 + 7x^2 - 2x$ (c) $2x^3 + 19x^2 - 58x - 5$
   (d) $x^2 + 6x + 9$ (e) $3x^2 + 17x + 20$ (f) $3x^2 + 14x - 5$
4. (a) -1 (b) $-\frac{6}{5}$ (c) -18

Section 2.4 Solutions Back to 2.4 Exercises

1. $4x^2(2x^2 - x + 4)$

2. (a) $(x-5)(x+2)$ (b) $(x-4)(x-3)$ (c) $(x+3)(x-5)$
   (d) $(x+9)(x+1)$ (e) $(x-4)(x-7)$ (f) $(x-2)(x+6)$

3. (a) $(7x-4)(x+1)$ (b) $(3x-2)(2x+1)$ (c) $2x+1)(3x+5)$
   (d) $(3x+2)(x-7)$ (e) $(10x-3)(x+3)$ (f) $(5x-2)(3x-3)$

4. (a) $(x+3)(x-3)$ (b) $(5x+1)(5x-1)$ (c) $(3x+7)(3x-7)$

5. (a) $x(x+4)(x+1)$ (b) $3x(x+7)(x-2)$ (c) $x^2(x+5)(x+1)$

6. (a) $(x+2)(x^2-3)$ (b) $(2x+3)(x+1)(x-1)$ (c) $(x+2)(x+5)(x-5)$

7. (a) $4x^2(x^2+x+1)$ (b) $(5x-1)(2x-3)$ (c) $(3x-5)(x+1)$
   (d) $(9x+2)(x+1)$ (e) $(2x^2+5)(3x-2)$ (f) $(4x+7)(4x-7)$
   (g) $(3-x)(1+8x)$ (h) Can't be factored (i) $12x(x^2+2x+4)$
   (j) $(3x+1)(2x-7)$ (k) $(2x-1)(x-7)$ (l) $(x+2)(x-2)(x+3)$
   (m) $(2y-3)(5y+5)$ (n) $(1+5x)(1-5x)$ (o) Can't be factored
   (p) $(2x+3)(3x-5)$ (q) $(7x+4)(x+1)$ (r) $(2-x)(4+3x)$

8. (a) 36 (b) -12

9. (a) $\dfrac{8}{27}$ (b) $\frac{1}{25}$ (c) 1 (d) $-\frac{1}{25}$

10. (a) $-4x^2 - 13x + 5$ (b) $-12x - 7$

11. (a) $6x^3 - 29x^2 + 43x - 20$ (b) $9x^2 + 30x + 25$ (c) $3x^3 - 21x^2 - 15x$
    (d) $6x^3 + 7x^2 - 5$ (e) $x^2 - 2x + 1$ (f) $5x^3 - 29x^2 - 7x + 6$

Section 2.5 Solutions Back to 2.5 Exercises

1. (a) $x = -2, 6$ (b) $x = 2, 4$ (c) $x = -4, 6$

2. (a) $x = 0, -5, 2$ (b) $x = -1, -7$ (c) $x = 0, -2, 3$

3. (a) $x = 12, 1$ (b) $x = 3, 5$ (c) $x = 4, -4$
   (d) $x = 0, -\frac{1}{5}$ (e) $x = -\frac{1}{3}, 7$ (f) $x = -1, -4, 4$
   (g) $a = -3, -5$ (h) $x = \frac{3}{2}, -5$ (i) $x = 0, -1, -2$
   (j) $x = -1, -6$ (k) $x = 0, 2$ (l) $x = -\frac{1}{2}, 3, -3$

4. (a) $56x^{19}$ (b) $4y^7$ (c) $\dfrac{25s^8t^{16}}{16}$ (d) $\dfrac{u^3}{2}$

5. (a) 237 (b) 4.9×10^{-3} (c) 0.16 (d) 5.3×10^{4}

6. (a) $x = -5$ (b) $x = \frac{1}{4}$ (c) $x = -17$