Intermediate Algebra

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Contents

3 Equations Containing Rational Expressions

3.1 Rational Expressions

3. (a) Give values that an unknown is not allowed to have in a rational expression.
   (b) Simplify rational expressions.

Consider the following:

- $\frac{8}{4} = 2$ because $2 \cdot 4 = 8$
- $\frac{18}{3} = 6$ because $6 \cdot 3 = 18$
- $\frac{32}{8} = 4$ because $4 \cdot 8 = 32$
- $\frac{0}{5} = 0$ because $0 \cdot 5 = 0$

Now what about $\frac{5}{0}$? Let's suppose that it is some number x; that is, $\frac{5}{0} = x$. Like all of the above, it must then be the case that $x \cdot 0 = 5$. But there is no value of x that makes this true, so *we say that* $\frac{5}{0}$ *is undefined*.

Division and Zero

- As long as $a \neq 0$, $\frac{0}{a} = 0$.
- For any number a, $\frac{a}{0}$ is undefined.

Now consider the expression $\frac{x^2 - 16}{x^2 + x - 12}$, which is what we call a **rational expression** or **algebraic fraction**. This is really just a fancy way of saying that it is a fraction containing an unknown value x. It is possible that for some values of x the bottom of this fraction might be zero. The whole expression would then be undefined for those values of x, so x is not allowed to have those values.

◇ **Example 3.1(a):** Give all values that x is not allowed to have in the rational expression $\frac{x^2 - 16}{x^2 + x - 12}$.

Another Example

Solution: Because $\frac{x^2 - 16}{x^2 + x - 12} = \frac{x^2 - 16}{(x+4)(x-3)}$ and $(x+4)(x-3) = 0$ when $x = -4$ or $x = 3$, x is not allowed to be -4 or 3. We indicate this by writing $x \neq -4,\ 3$.

◇ **Example 3.1(b):** Give all values that x is not allowed to have in the rational expression $\dfrac{x^2+4x-5}{x^2-4x+3}$.

Another Example

$$\frac{x^2+4x-5}{x^2-4x+3} = \frac{x^2+4x-5}{(x-3)(x-1)}, \qquad \text{so} \qquad x \neq 1,\ 3$$

You should recognize that a fraction like $\dfrac{12}{18}$ can be reduced:

$$\frac{12}{18} = \frac{6 \cdot 2}{6 \cdot 3} = \frac{6}{6} \cdot \frac{2}{3} = 1 \cdot \frac{2}{3} = \frac{2}{3}$$

Although you probably don't show all the same steps that I have shown here, they show what really happens when you reduce. We can do the same thing for rational expressions:

$$\frac{x^2-16}{x^2+x-12} = \frac{(x+4)(x-4)}{(x+4)(x-3)} = \frac{x+4}{x+4} \cdot \frac{x-4}{x-3} = 1 \cdot \frac{x-4}{x-3} = \frac{x-4}{x-3}$$

◇ **Example 3.1(c):** Simplify $\dfrac{x^2+4x-5}{x^2-4x+3}$.

Another Example

Solution: Rather than showing all of the steps shown above, we will usually just show the original, the factoring and canceling, and the final result:

$$\frac{x^2+4x-5}{x^2-4x+3} = \frac{(x+5)\cancel{(x-1)}}{\cancel{(x-1)}(x-3)} = \frac{x+5}{x-3}$$

Section 3.1 Exercises To Solutions

1. Determine all values that the unknown is not allowed to have in each of the following.

   (a) $\dfrac{x^2+x-2}{x^2-4}$ (b) $\dfrac{x^2-2x-15}{x^2-6x+5}$ (c) $\dfrac{5x+25}{x^2-25}$

   (d) $\dfrac{x^2-7x+12}{x^2-9x+20}$ (e) $\dfrac{x^3+12x^2+35x}{2x^2+10x}$ (f) $\dfrac{x^2-4}{x^2-4x+4}$

2. Reduce each of the rational expressions from Exercise 1.

3. Factor each difference of squares.

   (a) x^2-25 (b) $4x^2-9$ (c) $16x^2-1$

4. Factor each expression completely. Do this by first factoring out any common factors, then factoring whatever remains after that.

   (a) $20x^4 - 5x^2$

   (b) $30x^3 + 21x^2 - 36x$

5. Factor each of the following by grouping.

   (a) $3x^3 + x^2 - 12x - 4$

   (b) $x^3 - 5x^2 - 9x + 45$

   (c) $2x^3 + 7x^2 - 2x - 7$

6. Solve each equation.

   (a) $4x^2 = 25$

   (b) $x^3 = 9x^2 + 22x$

   (c) $x^3 + 5x^2 - x - 5 = 0$

   (d) $\frac{1}{15}x^2 = \frac{1}{6}x + \frac{1}{10}$

   (e) $21 + 4x = x^2$

   (f) $2x + x^2 - 15 = 0$

3.2 Multiplying Rational Expressions

3. (c) Multiply rational expressions and simplify the results.

Recall that to multiply two fractions we simply multiply their numerators and denominators (tops and bottoms):

$$\frac{3}{4} \cdot \frac{2}{5} = \frac{6}{20} = \frac{3}{10}$$

It is usually more efficient to factor the numerators and denominators and cancel any common factors *BEFORE* multiplying:

◇ **Example 3.2(a):** Multiply $\frac{9}{25} \cdot \frac{10}{27}$.

$$\frac{9}{25} \cdot \frac{10}{27} = \frac{9}{5 \cdot 5} \cdot \frac{2 \cdot 5}{3 \cdot 9} = \frac{5}{5} \cdot \frac{9}{9} \cdot \frac{2}{3 \cdot 5} = 1 \cdot 1 \cdot \frac{2}{15} = \frac{2}{15}$$

The above are all the steps that occur in the process of multiplying the two fractions, but we usually won't show all of those steps. Here is what we would usually show:

$$\frac{\overset{1}{\cancel{9}}}{\underset{5}{\cancel{25}}} \cdot \frac{\overset{2}{\cancel{10}}}{\underset{3}{\cancel{27}}} = \frac{1 \cdot 2}{5 \cdot 3} = \frac{2}{15}$$

We use the same process to multiply rational expressions:

◇ **Example 3.2(b):** Multiply $\frac{x+1}{x^2-4} \cdot \frac{x+2}{3x+3}$

Solution: We first factor the numerators and denominators of both fractions, and see that there are common factors of $x+1$ and $x+2$ that can be cancelled:

$$\frac{x+1}{x^2-4} \cdot \frac{x+2}{3x+3} = \frac{\cancel{x+1}}{\cancel{(x+2)}(x-2)} \cdot \frac{\cancel{x+2}}{3\cancel{(x+1)}} = \frac{1}{3(x-2)}$$

Notice that there is "nothing left" on top after common factors are canceled, but we must put a one there to keep the remaining factors in the bottom, where they belong. Notice also that we do not usually multiply factors back together at the end, after cancelling.

◇ **Example 3.2(c):** Multiply $\frac{y-1}{y^2-y-6} \cdot \frac{y^2+5y+6}{y^2-1}$ Another Example

$$\frac{y-1}{y^2-y-6} \cdot \frac{y^2+5y+6}{y^2-1} = \frac{\cancel{y-1}}{(y-3)\cancel{(y+2)}} \cdot \frac{(y+3)\cancel{(y+2)}}{(y+1)\cancel{(y-1)}} = \frac{y+3}{(y-3)(y+1)}$$

When we want to multiply a polynomial times a rational expression, we simply give the polynomial a denominator of one:

◇ **Example 3.2(d):** Multiply $(4x^2-9)\cdot\dfrac{x+3}{2x+3}$

$$\begin{aligned}(4x^2-9)\cdot\frac{x+3}{2x+3} &= \frac{4x^2-9}{1}\cdot\frac{x+3}{2x+3}\\ &= \frac{\cancel{(2x+3)}(2x-3)}{1}\cdot\frac{x+3}{\cancel{2x+3}}\\ &= (2x-3)(x+3) \quad \text{or} \quad 2x^2+3x-9\end{aligned}$$

Soon we will see situations where we need to multiply a polynomial times a sum or difference of rational expressions. To do this we simply distribute the polynomial to each of the rational expressions, then multiply as in Example 3.2(d). Be careful to distribute negative signs!

◇ **Example 3.2(e):** Multiply $(x+2)(x-1)\cdot\left(\dfrac{4x}{x^2+x-2}-\dfrac{3}{x+2}\right)$.

Solution: Here we begin by putting $(x+2)(x-1)$ over one and distributing to both parts of the expression $\dfrac{4x}{x^2+x-2}-\dfrac{3}{x+2}$.

$$\begin{aligned}&(x+2)(x-1)\left(\frac{4x}{x^2+x-2}-\frac{3}{x+2}\right)\\ &\quad= \frac{(x+2)\cancel{(x-1)}}{1}\cdot\frac{4x}{(x+2)\cancel{(x-1)}}-\frac{\cancel{(x+2)}(x-1)}{1}\cdot\frac{3}{\cancel{x+2}}\\ &\quad= 4x-3(x-1)=4x-3x+3=x+3\end{aligned}$$

Section 3.2 Exercises

To Solutions

1. Multiply each.

(a) $\dfrac{x+1}{x-4}\cdot\dfrac{x-4}{x^2-2x-3}$

(b) $\dfrac{x^2+x-2}{x^2-4}\cdot\dfrac{x+3}{x-1}$

(c) $\dfrac{x^2+2x-15}{x^2-9}\cdot(x+3)$

(d) $\dfrac{x+1}{x^2-4}\cdot\dfrac{x^2+5x+6}{x+1}$

2. Multiply each.

(a) $6x \cdot \left(\frac{5}{2x} - \frac{1}{3} \right)$

(b) $x(x+1) \cdot \left(\frac{x-3}{x+1} - \frac{x+4}{x} \right)$

(c) $(x+5)(x-5) \cdot \left(\frac{3x}{x-5} + \frac{7}{x+5} \right)$

(d) $(x-3)(x+3) \cdot \left(\frac{2x}{x^2-9} - \frac{5}{x+3} \right)$

3. Evaluate each of the following.

(a) 6^{-1}

(b) $\left(\frac{4}{5} \right)^{-2}$

(c) 6^0

(d) $\left(-\frac{2}{3} \right)^2$

4. Perform the indicated operation on the polynomial(s).

(a) $(2x-1)^2$

(b) $(2x^2+5)+(x^3-3x+4)$

(c) $(x+5)(x^2-4x+2)$

(d) $(x+5)-(x^2-4x+2)$

5. Determine all values that the unknown is not allowed to have in each of the following.

(a) $\frac{x^2-3x-10}{x^2-4}$

(b) $\frac{x+2}{x^2-3x-10}$

(c) $\frac{x^2-3x-4}{x^2+3x+2}$

6. Reduce each of the rational expressions from Exercise 5.

3.3 Solving Rational Equations

3. (d) Solve rational equations.

A rational equation is an equation containing rational expressions. The procedure for solving rational equations is as follows:

Solving Rational Equations

- Factor the denominators of all rational expressions, and determine what values the unknown is not allowed to have.
- Multiply both sides of the equation by *JUST ENOUGH* of the factors of the denominators to "kill off" all the denominators. Be sure to distribute carefully and take all signs properly into account.
- Solve the resulting linear or polynomial equation.
- Eliminate any of the solutions you obtained that are also values that the unknown is not allowed to have.

◇ **Example 3.3(a):** Solve $1 - \frac{1}{x} = \frac{12}{x^2}$

$x \neq 0$

$$\begin{aligned} \frac{x^2}{1}\left(1 - \frac{1}{x}\right) &= \left(\frac{12}{x^2}\right)\frac{x^2}{1} \\ x^2 - \frac{x^2}{1}\cdot\frac{1}{x} &= \frac{12}{x^2}\cdot\frac{x^2}{1} \\ x^2 - x &= 12 \end{aligned}$$

$$\begin{aligned} x^2 - x - 12 &= 0 \\ (x-4)(x+3) &= 0 \\ x &= 4, -3 \end{aligned}$$

◇ **Example 3.3(b):** Solve $\frac{5}{y+1} = \frac{4}{y+2}$

$y \neq -1, -2$

$$\begin{aligned} \frac{(y+1)(y+2)}{1}\cdot\frac{5}{y+1} &= \frac{4}{y+2}\cdot\frac{(y+1)(y+2)}{1} \\ 5(y+2) &= 4(y+1) \\ 5y + 10 &= 4y + 4 \\ y &= -6 \end{aligned}$$

The process used in the previous two examples is often called "clearing the denominators" of the fractions. Note carefully how this is done in the next example.

◇ **Example 3.3(c):** Solve $\dfrac{y+3}{y^2-y} = \dfrac{8}{y^2-1}$

Solution: When we factor the denominators of both sides of the equation we get

$$\frac{y+3}{y(y-1)} = \frac{8}{(y+1)(y-1)}$$

Solution: so, clearly, $y \neq 0, 1, -1$. To clear the denominators we only need *ONE* factor of $y-1$ even though it occurs in the denominators of both sides, because anything we multiply one side by, we must multiply the other side by as well.

$$\begin{aligned}
\frac{y(y+1)(y-1)}{1} \cdot \frac{y+3}{y(y-1)} &= \frac{8}{(y+1)(y-1)} \cdot \frac{y(y+1)(y-1)}{1} \\
(y+1)(y+3) &= 8y \\
y^2+4y+3 &= 8y \\
y^2-4y+3 &= 0 \\
(y-1)(y-3) &= 0 \\
y &= \cancel{1}, 3
\end{aligned}$$

The solution $y=1$ is not valid because y cannot be one in the original equation. We note this by simply crossing it out, as shown above.

◇ **Example 3.3(d):** Solve $\dfrac{x-4}{x^2+2x-15} = 2 - \dfrac{2}{x-3}$

$$\begin{aligned}
\frac{x-4}{(x+5)(x-3)} &= 2 - \frac{2}{x-3} \qquad x \neq -5, 3 \\
\frac{(x+5)(x-3)}{1} \cdot \frac{x-4}{(x+5)(x-3)} &= \left(2 - \frac{2}{x-3}\right) \frac{(x+5)(x-3)}{1} \\
x-4 &= 2(x+5)(x-3) - \frac{2}{x-3} \cdot \frac{(x+5)(x-3)}{1} \\
x-4 &= 2(x^2+2x-15) - 2(x+5) \\
x-4 &= 2x^2+4x-30-2x-10 \\
x-4 &= 2x^2+2x-40 \\
0 &= 2x^2+x-36 \\
0 &= (2x+9)(x-4)
\end{aligned}$$

At this point we can easily see that $x=4$ is one solution. To obtain the other solution we set $2x+9=0$ and solve to get $x=-\frac{9}{2}$. Neither of these causes a problem with the original equation, so both are solutions.

◇ **Example 3.3(e):** Solve $\dfrac{x}{x+5} = \dfrac{x}{x-2}$

Solution: We see that $x \neq -5, 3$. Multiplying both sides by $(x+5)(x-2)$ we get

$$\begin{aligned} \frac{(x+5)(x-2)}{1} \cdot \frac{x}{x+5} &= \frac{x}{x-2} \cdot \frac{(x+5)(x-2)}{1} \\ x(x-2) &= x(x+5) \\ x^2 - 2x &= x^2 + 5x \\ -2x &= 5x \\ 0 &= 7x \\ x &= 0 \end{aligned}$$

Looking back at the original equation, it is clear that $x = 0$ is a solution, but it is *NOT* clear that it is the only solution.

◇ **Example 3.3(f):** Solve $\dfrac{x-2}{x-6} - \dfrac{4}{x} = \dfrac{24}{x^2 - 6x}$

Solution: Factoring the denominator of the right hand side, we get $\dfrac{x-2}{x-6} - \dfrac{4}{x} = \dfrac{24}{x(x-6)}$. From this we can see that $x \neq 0, 6$.

$$\begin{aligned} \frac{x(x-6)}{1} \cdot \left(\frac{x-2}{x-6} - \frac{4}{x} \right) &= \frac{24}{x(x-6)} \cdot \frac{x(x-6)}{1} \\ x(x-2) - 4(x-6) &= 24 \\ x^2 - 2x - 4x + 24 &= 24 \\ x^2 - 6x &= 0 \\ x(x-6) &= 0 \\ x &= \not 0, \not 6 \end{aligned}$$

Because the procedure for solving the equation leads to only the two values that x is not allowed to have, the equation has no solution.

Section 3.3 Exercises

To Solutions

1. For each equation, tell what values x is not allowed to have, then solve the equation.

(a) $\dfrac{x+4}{2x} + \dfrac{x+20}{3x} = 3$

(b) $\dfrac{x+1}{x-3} = \dfrac{x+2}{x+5}$

(c) $1 - \dfrac{4}{x+7} = \dfrac{5}{x+7}$

(d) $\dfrac{10}{x-3} - \dfrac{2}{x} = -1$

(e) $\dfrac{1}{x-1} + \dfrac{1}{x+1} = \dfrac{6}{x^2-1}$

(f) $\dfrac{2x-1}{x^2+2x-8} = \dfrac{1}{x-2} - \dfrac{2}{x+4}$

2. Solve each equation.

(a) $15x^2 = 20x$

(b) $\dfrac{2}{3}x - \dfrac{1}{6} = \dfrac{3}{2} - \dfrac{7}{12}x$

(c) $2x^3 + x^2 = 18x + 9$

(d) $5x^2 = 20$

(e) $4 - t = 15t - 20$

(f) $\dfrac{2}{15}x^2 + \dfrac{1}{3}x = \dfrac{1}{5}$

(g) $2x^2 - x - 10 = 0$

(h) $8(a-2) + 3a = 9(1-a)$

A Solutions to Exercises

A.3 Chapter 3 Solutions

Section 3.1 Solutions Back to 3.1 Exercises

1. (a) $x \neq 2, -2$ (b) $x \neq 1, 5$ (c) $x \neq -5, 5$

   (d) $x \neq 4, 5$ (e) $x \neq 0, -5$ (f) $x \neq 2$

2. (a) $\dfrac{x-1}{x-2}$ (b) $\dfrac{x+3}{x-1}$ (c) $\dfrac{5}{x-5}$

   (d) $\dfrac{x-3}{x-5}$ (e) $\dfrac{x+7}{2}$ (f) $\dfrac{x+2}{x-2}$

3. (a) $(x+5)(x-5)$ (b) $(2x+3)(2x-3)$ (c) $(4x+1)(4x-1)$

4. (a) $5x^2(2x+1)(2x-1)$ (b) $3x(5x-4)(2x+3)$

5. (a) $(3x+1)(x+2)(x-2)$ (b) $(x-5)(x+3)(x-3)$ (c) $(2x+7)(x+1)(x-1)$

6. (a) $x = \frac{5}{2}, -\frac{5}{2}$ (b) $x = 0, 11, -2$ (c) $x = -5, -1, 1$

   (d) $x = -\frac{1}{2}, 3$ (e) $x = -3, 7$ (f) $x = 3, -5$

Section 3.2 Solutions Back to 3.2 Exercises

1. (a) $\dfrac{1}{x-3}$ (b) $\dfrac{x+3}{x-2}$ (c) $x+5$ (d) $\dfrac{x+3}{x-2}$

2. (a) $15-2x$ (b) $-8x-4$ (c) $3x^2+22x-35$ (d) $-3x+15$

3. (a) $\frac{1}{6}$ (b) $\dfrac{25}{16}$ (c) 1 (d) $\dfrac{4}{9}$

4. (a) $4x^2-4x+1$ (b) $3x^2-3x+9$

   (c) $x^3+x^2-18x+10$ (d) $-x^2+5x+3$

5. (a) $x \neq -2, 2$ (b) $x \neq 5, -2$ (c) $x \neq -1, -2$

6. (a) $\dfrac{x-5}{x-2}$ (b) $\dfrac{1}{x-5}$ (c) $\dfrac{x-4}{x+2}$

Section 3.3 Solutions Back to 3.3 Exercises

1. (a) $x \neq 0, \quad x = 4$ (b) $x \neq 3, -5, \quad x = -\frac{11}{7}$

   (c) $x \neq -7, \quad x = 2$ (d) $x \neq 3, 0, \quad x = -2, -3$

   (e) $x \neq 1, -1, \quad x = 3$ (f) $x \neq 2, -4, \quad x = 3$

2. (a) $x = 0, \frac{4}{3}$ (b) $x = \frac{4}{3}$ (c) $x = -\frac{1}{2}, -3, 3$ (d) $x = 2, -2$

   (e) $t = \frac{3}{2}$ (f) $x = \frac{1}{2}, -3$ (g) $x = -2, \frac{5}{2}$ (h) $a = \frac{5}{4}$