Intermediate Algebra

Gregg Waterman
Oregon Institute of Technology

Contents

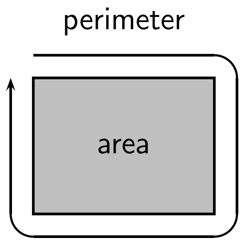
5 Applications Of Equations

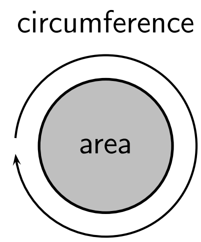
5.1 Using Formulas

5. (a) Use formulas to solve applied problems.

Geometry

We will be working with three geometric shapes: rectangles, triangles and circles. When we discuss the amount of surface that a shape covers, we are talking about its **area**. Some standard sorts of practical uses of areas are for measuring the amount of carpet needed to cover a floor, or the amount of paint to paint a certain amount of surface. The distance around a shape is usually called its **perimeter**, except in the case of a circle. The distance around a circle is called its **circumference**.





A **formula** is an equation that describes the relationship between several (two or more) values. Here are some formulas you are likely familiar with:

Area and Perimeter/Circumference Formulas

- For a rectangle with width w and length l, the perimeter P and area A are given by

$$P = 2w + 2l \qquad \text{and} \qquad A = lw$$

- For a circle with radius r, the circumference C and area A are given by

$$C = 2\pi r \qquad \text{and} \qquad A = \pi r^2$$

When computing the circumference or area of a circle it is necessary to use a special number called **pi**. Pi is a little over 3, but in decimal form it never ends or repeats. We use the symbol π for it, and its "exact" value is

$$\pi = 3.141592654...$$

Sometimes we round this to 3.14, but to use it most accurately one should use more places past the decimal. We might not care to type in more, but we don't need to - pi is so important that it has its own key on our calculators! Find it on yours.

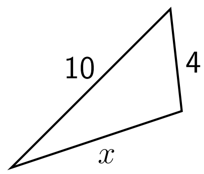
Let's look at some examples that use these formulas.

◇ **Example 5.1(a):** A rectangle that is 7 feet long has an area of 31.5 square feet. (The units of areas are always square somethings.) What is the width of the rectangle?

Solution: $A = lw \quad \Rightarrow \quad 31.5 = 7w \quad \Rightarrow \quad w = \dfrac{31.5}{7} = 4.5 \text{ feet}$

◇ **Example 5.1(b):** The shortest side of a triangle has a length of 4.5 inches, the longest side has a length of 10.0 inches, and the perimeter is 21.0 inches. What is the length of the middle side?

Solution:

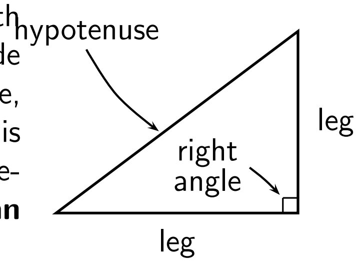


$$\begin{aligned} 4.5 + x + 10 &= 21 \\ x + 14.5 &= 21 \\ x &= 6.5 \text{ inches} \end{aligned}$$

◇ **Example 5.1(c):** A circle has a radius of 6.5 centimeters. What is the area of the circle?

Solution: $A = \pi r^2 = \pi(6.5)^2 = 132.7 \text{ square centimeters}$

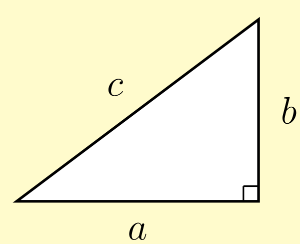
There is a special kind of triangle that comes up often in applications, called a **right triangle**. A right triangle is a triangle with one of its angles being a right angle ("square corner"). The side opposite the right angle is called the **hypotenuse** of the triangle, and the other two sides are called the **legs** of the triangle. There is a special relationship between the sides of the right triangle; this relationship is expressed by a famous "rule" called the **Pythagorean Theorem**.



Pythagorean Theorem

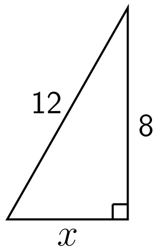
If a right triangle has legs with lengths a and b and hypotenuse of length c, then

$$a^2 + b^2 = c^2$$



◇ **Example 5.1(d):** One of the legs of a right triangle is 8 inches long and the hypotenuse is 12 inches long. How long is the other leg?

Solution:



$$
\begin{aligned}
x^2 + 8^2 &= 12^2 \\
x^2 + 64 &= 144 \\
x^2 &= 80 \\
x &= \sqrt{80} \\
x &= 8.9 \text{ inches}
\end{aligned}
$$

Projectile Motion

> **Projectile Motion**
>
> Suppose that an object is projected straight upward from a height of h_0 feet, with an initial velocity of v_0 feet per second. Its height (in feet) at any time t seconds after it is projected (until it hits the ground) is
>
> $$h = -16t^2 + v_0 t + h_0$$

◇ **Example 5.1(e):** A ball is thrown upward from a height of 5 feet, with an initial velocity of 60 feet per second. How high is the ball after 1.5 seconds?

Solution: The values of v_0 and h_0 are 60 and 5, respectively, so the equation becomes

$$h = -16t^2 + 60t + 5.$$

At time $t = 5$ seconds,

$$h = -16(1.5)^2 + 60(1.5) + 5 = 59 \text{ feet}$$

◇ **Example 5.1(f):** For the same ball, at what time or times is the ball 50 feet above the ground? **Round your answers to the nearest hundredth of a second.**

Solution: Here we know that $h = 50$ and we want to find t. This gives us the equation below and to the left. Adding $16t^2$ to both sides and subtracting $60t$ and 5 from both sides gives the equation below and to the right.

$$50 = -16t^2 + 60t + 5 \qquad \Rightarrow \qquad 16t^2 - 60t + 45$$

We need to solve the second equation for t. That equation looks difficult to factor, so let's just use the quadratic formula to solve it:

$$t = \frac{-(-60) \pm \sqrt{(-60)^2 - 4(16)(45)}}{2(16)} = \frac{60 \pm \sqrt{720}}{32} = 1.04,\ 2.71 \text{ seconds}$$

We get two answers because the ball will reach a height of 50 feet on both the way up *and* the way down.

Percents

Many quantities of interest are obtained by taking a certain percentage of other quantities. For example, some salespeople earn a certain percentage of their sales. (In this case the percentage that they earn is called their **commission**.) Various taxes are computed by a percentage of the item bought (or owned, in the case of property tax).

Percent means "out of one hundred," so seven percent means seven out of 100. If a person bought something for $200 and they had to pay 7% sales tax, they would pay seven dollars for every $100, or $14. When doing math with a percent, it is necessary to change a percent to its decimal form. Since seven percent means seven out of one hundred,

$$7\% = \frac{7}{100} = 0.07$$

This last value is called the **decimal form of the percent**. Note that the decimal point in $7 = 7.0$ has been moved two places to the left.

◇ **Example 5.1(g):** Change 4.25% to decimal form.

Solution: $$4.25\% = 0.0425$$

> **Working With Percents**
>
> To find p percent of an amount A, change p to decimal form and multiply it times A.

◇ **Example 5.1(h):** The standard tip for waiters and waitresses is 15-20% of the cost of the meal. If you and some friends go out to eat and the total bill is $87.50, how much of a tip should you give your waitress if you wish to give 15%?

15% of $\$87.50$ is $(0.15)(87.50) = \$13.13$, so maybe give a tip of $\$13.25$ or $\$14$.

Simple Interest

Suppose that a **principal** of P dollars is invested or borrowed at an annual interest rate of r percent (in decimal form) for t years. The amount A that is then had or owed at the end of t years is found by

$$A = P + Prt$$

◇ **Example 5.1(i):** You borrow $1000 at 8.5% interest for five years. How much money do you owe at the end of the five years?

Solution: $A = 1000 + 1000(0.085)(5) = \$1425$ owed

Note that what is owed at the end of the period is the original $1000 borrowed *and* $425 of interest.

◇ **Example 5.1(j):** You are going to invest $400 for 5 years, and you would like to have $500 at the end of the five years. What percentage rate would you need to have, to the nearest hundredth of a percent?

Solution:

$$\begin{aligned} 500 &= 400 + 400(5)r \\ 100 &= 2000r \\ \tfrac{100}{2000} &= r \\ r &= 0.05 = 5\% \end{aligned}$$

Section 5.1 Exercises To Solutions

Give correct units with all answers!

1. A rectangle has a length of 19 inches and a perimeter of 50 inches. What is the width of the rectangle?

2. Find the area of a circle with a radius of 5.3 inches. Round your answer to the tenths place.

3. The hypotenuse of a right triangle has length 13 inches and one of the legs has length 12 inches. What is the length of the other leg?

4. The circumference of a circle is 16.4 inches. Find the radius of the circle, to the nearest tenth of an inch.

5. Sales tax in a particular city is 5.5%, and you buy an item with a *pre-tax* price of $19.95.

   (a) How much sales tax will you have to pay for the item?

   (b) How much will you have to pay for the item, including tax?

6. A salesperson in an art gallery gets a monthly salary of $1000 plus 3% of all sales over $50,000. How much do they make in a month that they sell $112,350 worth of art?

7. You invest $1200 at 4.5% simple interest.

   (a) How much money will you have if you take it out after 4 years?

   (b) How many years, to the nearest tenth, would it take to “double your money?”

8. A baseball is hit upward from a height of four feet and with an initial velocity of 96 feet per second.

   (a) When is it at a height of 84 feet?

   (b) When is it at a height of 148 feet?

   (c) When is it at a height of 57 feet? Use your calculator and the quadratic formula, and round your answer(s) to the nearest hundredth of a second. (The hundredth's place is two places past the decimal.)

   (d) When is the ball at a height of 200 feet, to the nearest hundredth of a second?

   (e) When does the ball hit the ground, to the nearest hundredth of a second?

9. A company that manufactures ink cartridges for printers knows that the number x of cartridges that it can sell each week is related to the price per cartridge p by the equation $x = 1200 - 100p$. The weekly revenue (money they bring in) is the price times the number of cartridges: $R = px = p(1200 - 100p)$. What price should they set for the cartridges if they want the weekly revenue to be $3200?

10. Suppose that a cylinder has a height of h inches and a radius of r inches. The surface area S (in square inches) of the cylinder is given by $S = 2\pi rh + 2\pi r^2$. Find the height of a cylinder that has a a radius of 4.3 inches and a surface area of 465 square inches. **Round your answer to the tenth's place.**

11. For each of the following, determine whether the root exists. If it doesn't exist, say so (write DNE for “does not exist”) and you are done. If the root does exist, give its value without using your calculator.

    (a) $\sqrt[4]{-16}$ (b) $\sqrt{-16}$ (c) $-\sqrt{16}$ (d) $\sqrt[3]{1000}$ (e) $\sqrt{36}$

12. Simplify by combining like terms, *if possible*.

(a) $12 - 9\sqrt{5} + 4\sqrt{5} - 15$

(b) $25 + 10\sqrt{3} + 10\sqrt{3} + 12$

13. Multiply and simplify.

(a) $(7 + 2\sqrt{3})(1 - \sqrt{5})$

(b) $(4 + \sqrt{7})(3 - 2\sqrt{7})$

(c) $(2 - \sqrt{3})^2$

14. Solve each equation. Be sure to check all solutions to see if they are valid.

(a) $\sqrt[3]{6x + 9} = 3$

(b) $\sqrt{-2x + 1} = -3$

(c) $\sqrt{2x + 11} - 4 = x$

(d) $\sqrt[3]{x - 3} = -2$

5.2 Solving Formulas

5. (b) Solve formulas for given unknowns.

Recall the simple interest formula $A = P + Prt$. Suppose that we wanted to find out how long \$300 must be invested at 4.5% in order to have \$500. We would replace P, r and A with 300, 0.045 and 500, respectively, and solve for t:

$$\begin{aligned} 500 &= 300 + 300(0.045)t \\ 200 &= 300(0.045)t \\ t &= \frac{200}{300(0.045)} = 14.8 \text{ years} \end{aligned}$$

Now suppose instead we wanted to know how long \$2000 must be invested at 7.25% in order to have \$7000. We'd find out like this:

$$\begin{aligned} 7000 &= 2000 + 2000(0.0725)t \\ 5000 &= 2000(0.0725)t \\ t &= \frac{5000}{2000(0.0725)} = 34.5 \text{ years} \end{aligned}$$

What if we wanted to know how long \$800 must be invested at 5% in order to have \$1000? We'd simply repeat the process that we just went through twice already, which is getting a bit boring! If we wanted to find things like this out many times over, it would be simpler to do all of the above steps with the equation $A = P + Prt$, *THEN* substitute the values of A, r and P into the resulting equation to calculate t. The next example shows us how to do this sort of thing.

◇ **Example 5.2(a):** Solve $A = P + Prt$ for t, then use the result to find out how long \$800 must be invested at 5% in order to have \$1000.

Solution:

$$\begin{aligned} A &= P + Prt \\ A - P &= Prt \\ \frac{A-P}{Pr} &= t \end{aligned}$$

$$t = \frac{1000 - 800}{800(0.05)} = 5 \text{ years}$$

We will now look at an example of a computation that we will find valuable later.

◇ **Example 5.2(b):** Solve $-2x-7y=14$ for y. Give your answer in $y=mx+b$ form.

Solution:

$$\begin{aligned}
-2x-7y &= 14 && \text{original equation}\\
-7y &= 2x+14 && \text{add } 2x \text{ to both sides}\\
y &= \frac{2x+14}{-7} && \text{divide both sides by } -7\\
y &= \frac{2x}{-7}+\frac{14}{-7} && \text{split the fraction into two fractions}\\
y &= -\tfrac{2}{7}x-2 && \text{simplify fractions}
\end{aligned}$$

Suppose now that we wish to solve the equation $ax-8=bx+3$ for x. The problem here is that there are two terms with x in them, one on each side of the equation. The key is to get both those terms on one side of the equation, then factor the x out:

◇ **Example 5.2(c):** Solve $ax-8=bx+3$ for x.

Solution:

$$\begin{aligned}
ax-8 &= bx+3 && \text{original equation}\\
ax-bx &= 3+8 && \text{subtract } bx \text{ from, and add } 8 \text{ to, both sides}\\
(a-b)x &= 11 && \text{add } 3 \text{ and } 8, \text{ factor } x \text{ out of the left side}\\
x &= \frac{11}{a-b} && \text{divide both sides by } a-b
\end{aligned}$$

Section 5.2 Exercises To Solutions

1. (a) Use the result of the example to determine how long \$1000 must be invested at 3.5% to have \$1800.

   (b) Solve the equation $A=P+Prt$ for r.

2. Solve $PV=nRT$ for R.

3. Solve $ax+3=bx-5$ for b.

4. Solve $P=2w+2l$ for l.

5. Solve $C=2\pi r$ for r. Here the symbol π is for the special number pi. When solving for r you can treat π just as you would any letter.

6. Solve each equation for y. Give your answers in $y=mx+b$ form.

   (a) $3x+4y=-8$ (b) $5x+2y=-10$ (c) $3x-2y=-5$

   (d) $3x-4y=8$ (e) $3x+2y=5$

7. (a) Solve $8x + 3 = 5x - 7$ for x.

   (b) Try solving $ax + 7 = bx + 3$ for x. You may have some trouble getting x alone on one side (and not on the other side). *If you can't figure out how to do this, take a look at Example 5.2(c).*

8. Solve each equation for x.

   (a) $ax + b = cx + d$

   (b) $x + 1.065x = 8.99$

   (c) $ax + bx = c$

   (d) $ax + 3 = cx - 7$

9. Solve $A = P + Prt$ for P.

10. For each of the following, simplify the root if possible.

   (a) $\sqrt{17}$

   (b) $\sqrt{\frac{20}{9}}$

   (c) $-\sqrt{72}$

11. For each of the following, use the quadratic formula to solve the given equation. Simplify your answers.

   (a) $x^2 - 19 = 6x$

   (b) $x^2 = 2x + 17$

12. A projectile is shot upward from ground level with an initial velocity of 48 feet per second.

   (a) When is the projectile at a height of 20 feet?

   (b) When does the projectile hit the ground?

   (c) What is the height of the projectile at 1.92 seconds? **Round your answer to the nearest tenth of a foot.**

   (d) When is the projectile at a height of 32 feet?

   (e) When is the projectile at a height of 36 feet? Your answer here is a bit different than your answers to (a) and (d). What is happening physically with the projectile?

   (f) When is the projectile at a height of 40 feet?

5.3 Applications of Equations

> 5. (c) Create equations whose solutions are answers to applied problems.

In this section we will attempt to solve "word problems." The process we will use is the following:

- Get to know the situation by guessing a value for the thing we are supposed to find, then doing the appropriate arithmetic to check our guess. (If we are supposed to find several quantities, we just guess the value of one of them.) Do this a few times if necessary.
- Let x be the correct value that we are looking for. Do everything to x that we did to our guess in order to check it, but set the result equal to what it is really supposed to be. This gives us an equation.
- Solve the equation to get the value asked for. If asked for more than one value, find the others as well.

Let's look at some examples to see how this process works.

◇ **Example 5.3(a):** The longest side of a triangle is five more than the medium side, shortest side is half the medium side. The perimeter is $27\frac{1}{2}$. Find the lengths of the sides of the triangle. Another Example

Solution: Let's guess the medium side has length 8. Then the longest side is five more than the medium side, or $8+5=13$. The shortest side is half the medium side, or $\frac{1}{2}(8)=4$. The perimeter (distance around, remember) is then $4+8+13=25$. Since the perimeter is supposed to be 27.5, our guess is close but incorrect.

Rather than adjusting our guess and trying again, let's say that the medium side has length x. The longest side is then five more, or $x+5$. Notice that this repeats what we did above, but with x instead of 8. The shortest side is half the medium side, or $\frac{1}{2}x$. The perimeter is then $\frac{1}{2}x+x+(x+5)$, and must equal 27.5. We set these equal and solve:

$\frac{1}{2}x+x+(x+5)$	$=$	27.5	original equation
$x+2x+2(x+5)$	$=$	55	multiply both sides by two
$3x+2x+10$	$=$	55	add first two terms, distribute 2
$5x$	$=$	45	combine x terms, subtract 10 from both sides
x	$=$	9	divide both sides by 5

x represents the length of the medium side, so the length of the longest side is $9+5=14$ and the length of the shortest side is $\frac{1}{2}(9)=4\frac{1}{2}$.

◇ **Example 5.3(b):** In a certain city sales tax is 4.5%. If the sales tax on an item was $1.80, what was the price of the item?

Solution: Guess the price was $30: 4.5% of $30 = (0.045)(30) = \$1.35$. Now that we see what to do with the price, let's suppose that the correct price is x:

$$4.5\% \text{ of } x = 0.045x = \$1.80 \qquad \Rightarrow \qquad x = \frac{1.80}{0.045} = \$40$$

The price of the item was $\$40$.

Read the next example, and make sure you see the difference between what it is asking, versus the last example.

◇ **Example 5.3(c):** In the same city you buy something for $78.37, including tax. What was the price of the thing you bought?

Solution: Guess the price was $\$70$. Then the tax was $(0.045)(70) = \$3.15$. The cost, with tax, is then $70 + 3.15 = \$73.15$. Now let x be the correct price, so it takes the place of 70 in the above two calculations. The tax is then $0.045x$ and the cost with tax is $x + 0.045x$. Note that x represents the price of the item, which must be paid, and $0.045x$ represents the tax paid. So we set $x + 0.045x$ equal to the total cost of $\$78.37$ and solve:

$$\begin{aligned} x + 0.045x &= 78.37 && \text{the equation} \\ (1 + 0.045)x &= 78.37 && \text{factor } x \text{ out of the left side} \\ 1.045x &= 78.37 && \text{add } 1 + 0.045 \\ x &= \$75.00 && \text{divide both sides by } 1.045 \end{aligned}$$

◇ **Example 5.3(d):** The length of a rectangle is one more than three times the width, and the area is 520. Write an equation and use it to find the length and width of the rectangle.

Solution: Suppose that the width of the rectangle is $w = 5$. Then the length is one more than three times the width, or $l = 1 + 3(5) = 16$. The area would then be $lw = (16)(5) = 80$. This is far too low, but it doesn't matter! We've used our guess to see what needs to be done. Now suppose that the width is just w units. Then the length is $l = 1 + 3w$ and the area is $lw = (1 + 3w)w$. We set this equal to 520 and solve:

$$\begin{aligned} (1 + 3w)w &= 520 \\ w + 3w^2 &= 520 \\ 3w^2 + w - 520 &= 0 \end{aligned}$$

It turns out that the left side of the last equation can be factored, but not in a way that is obvious to most of us! Let's use the quadratic formula instead:

$$w = \frac{-1 \pm \sqrt{1^2 - 4(3)(-520)}}{2(3)} = \frac{-1 \pm \sqrt{6241}}{6} = \frac{-1 \pm 79}{6} = \frac{78}{6}, -\frac{80}{6} = 13, -\frac{40}{3}$$

The width cannot be negative, so the only possible value of the width is $13,$ giving a corresponding length of 40 units.

Section 5.3 Exercises To Solutions

For exercises one through five, write an equation that can be used to solve the problem, then solve.

1. The length of a rectangle is 3.2 inches less than twice its width, and its perimeter is 57.1 inches. Find the length and width of the rectangle.

2. Sales tax in a state is 5.5%. If the tax on an item was $10.42, what was the price of the item?

3. One leg of a right triangle is twice as long as the other leg, and the hypotenuse has a length of 15 feet. How long are the legs?

4. After a 2% raise your hourly wage is $7.37 per hour. What was it before the raise?

5. The length of the shortest side of a triangle is half the length of the longest side. The middle side is 1.4 inches longer than the shortest side, and the perimeter is 42 inches. Find the lengths of all three sides.

6. Solve each equation. Be sure to check all solutions to see if they are valid.

   (a) $\sqrt{2x+7}-6=-2$

   (b) $\sqrt{5x+9}=x-1$

   (c) $\sqrt[3]{6x+5}=2$

7. Solve each equation for x.

   (a) $ax+3=bx-5$

   (b) $ax+7=b(x+c)$

A Solutions to Exercises

A.5 Chapter 5 Solutions

Section 5.1 Solutions Back to 5.1 Exercises

1. 6 inches
2. 88.2 square inches
3. 5 inches
4. 2.6 inches
5. (a) \$1.10 (b) \$21.05
6. \$2870.50
7. (a) \$1416 (b) 22.2 years
8. (a) 1, 5 sec (b) 3 sec (c) 0.62, 5.38 sec (d) never (e) 6.04 seconds
9. \$4 or \$8
10. 12.9 inches
11. (a) DNE (b) DNE (c) -4 (d) 10 (e) 6
12. (a) $-3-5\sqrt{5}$ (b) $37+20\sqrt{3}$
13. (a) $7-7\sqrt{5}+2\sqrt{3}-2\sqrt{15}$ (b) $-2-5\sqrt{7}$ (c) $7-4\sqrt{3}$
14. (a) $x=3$ (b) no solution ($x=-4$ doesn't check)
    (c) $x=-1$ $x=-5$ doesn't check (d) $x=-5$

Section 5.2 Solutions Back to 5.2 Exercises

1. (a) 22.9 years (b) $r=\dfrac{A-P}{Pt}$
2. $R=\dfrac{PV}{nT}$
3. $b=\dfrac{ax+8}{x}$
4. $l=\dfrac{P-2w}{2}$
5. $r=\dfrac{C}{2\pi}$
6. (a) $y=-\frac{3}{4}x-2$ (b) $y=-\frac{5}{2}x-5$ (c) $y=\frac{3}{2}x+\frac{5}{2}$
   (d) $y=\frac{3}{4}x-2$ (e) $y=-\frac{3}{2}x+\frac{5}{2}$
7. (a) $x=-\frac{10}{3}$ (b) $x=\dfrac{-4}{a-b}$
8. (a) $x=\dfrac{d-b}{a-c}$ (b) $x=\dfrac{8.99}{2.065}=4.35$ (c) $x=\dfrac{c}{a+b}$ (d) $x=\dfrac{-10}{a-c}$
9. $P=\dfrac{A}{1+rt}$
10. (a) $\sqrt{17}$ (b) $\frac{2\sqrt{5}}{3}$ (c) $-6\sqrt{2}$
11. (a) $x=3+2\sqrt{7},\ 3-2\sqrt{7}$ (b) $x=1+3\sqrt{2},\ 1-3\sqrt{2}$
12. (a) $\frac{1}{2},\frac{5}{2}$ seconds (b) 3 seconds (c) 33.2 feet (d) 1, 2 seconds

(e) $\frac{3}{2}$ seconds (f) never

Section 5.3 Solutions

Back to 5.3 Exercises

1. **Equation:** $2w + 2(2w - 3.2) = 57.1$ **Answer:** $w = 10.6$ inches, $l = 18$ inches
2. **Equation:** $0.055x = 10.42$ **Answer:** $x = \$189.95$
3. **Equation:** $x^2 + (2x)^2 = 15^2$ **Answer:** $x = 3\sqrt{5} = 6.7$ feet (one leg), other leg is 13.4 feet or $6\sqrt{5}$ feet
4. **Equation:** $x + 0.02x = 7.37$ **Answer:** $x = \$7.22$
5. **Equation:** $\frac{1}{2}x + (\frac{1}{2}x + 1.4) + x = 42$ **Answer:** $x = 20.3$ (longest side) shortest side is 10.15 and middle side is 11.55
6. (a) $x = \frac{9}{2}$ (b) $x = 8$ $x = -1$ doesn't check (c) $x = \frac{1}{2}$
7. (a) $x = \dfrac{-8}{a - b}$ (b) $x = \dfrac{bc - 7}{a - b}$