Intermediate Algebra

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Contents

6 Equations Relating Two Variables

6.1 Graphs of Equations in Two Unknowns

6. (a) Graph the solution set of an equation in two unknowns.

Consider the equation $5x + 2 = 17$. The solution to this equation is $x = 3$ because if we substitute three for x in the expression on the left side of the equation, the equation becomes a true statement. *Remember that we don't have to actually solve an equation to determine whether something is a solution - it suffices to show that the value makes the statement true.* As we know, this equation has only one solution, so three is the only solution.

Now suppose that we have the equation $3x - y = 8$, which contains two unknowns, x and y. (**NOTE:** When we use two different letters for unknowns it is implied that they likely have different numerical values, but *they can be the same*.) What do we mean by a solution to such an equation? In this case, where the equation contains two unknowns, a solution consists of a *pair* of numbers that make the equation true when substituted in. One solution to the equation is $x = 3$ and $y = 1$, because $3(3) - 1 = 8$.

◇ **Example 6.1(a):** Is $x = -1,\ y = -11$ a solution to $3x - y = 8$? Is $x = 5,\ y = 3$? Is $x = 2,\ y = -2$?

Solution: $3(-1) - (-11) = -3 + 11 = 8$, so $x = -1,\ y = -11$ is a solution.

$3(5) - (3) = 15 - 3 = 12 \neq 8$, so $x = 5,\ y = 3$ is not a solution.

$3(2) - (-2) = 6 + 2 = 8$, so $x = 2,\ y = -2$ is a solution.

As you might guess, the equation $3x - y = 8$ has infinitely many solutions.

At some point (maybe already!) we will tire of writing $x =$ and $y =$ for every solution pair. To eliminate this annoyance, mathematicians have developed the following convention. In order to indicate $x = 5$, $y = -4$, we write $(5, -4)$. This is called an **ordered pair**. Note three important things:

- We write the value of x, *then* y, always in that order.
- The two values are separated by a comma.
- The values are enclosed by parentheses. *Do not write* $\{5, -4\}$ *or* $[5, -4]$*; those mean something else in mathematics.*

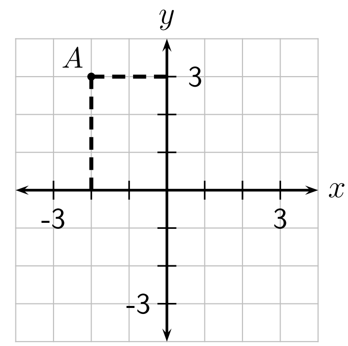
◇ **Example 6.1(b):** Give some solutions to the equation $y = x^2 - 3$ in the form just shown.

Solution: Here we can see that if $x = 1$, $y = 1^2 - 3 = -2$, so $(1, -2)$ is a solution pair. Substituting 0, -1 and 5 for x, we get the solution pairs $(0, -3)$, $(-1, -2)$, $(5, 22)$.

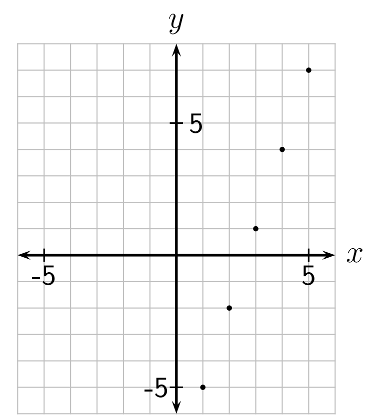
For the last example you may have noticed that to find solutions for the equation $y = x^2 - 3$ you can simply choose any value for x, substitute it into the right hand side of the equation and evaluate, giving you the value that y must have for that particular choice of x.

At this point you have three solution pairs for the equation $3x - y = 8$, $(3, 1), (-1, -11)$ and $(2, -2)$. It should be clear to you that there are many more **solution pairs**, as we call them. Rather than try to find all of them, we'll draw a 'picture' of them. We will 'plot' the points we have on what we call a **coordinate grid**, or **Cartesian plane**, named after the philosopher and mathematician René Descartes. We then try to guess (correctly) where *all* other points would be.

The Cartesian plane consists of two number lines, placed at right angles to each other, with zero on one line placed on zero on the other. The picture to the right shows such a coordinate grid, with a point A on it. The point represents two numbers, an x and a y. To get the x value represented by A we read from the point straight down to the the horizontal line, which we call the x-**axis**. So for A we get -2, which we call the x-**coordinate** of the point. The value on the y-**axis** associated A is called its y-**coordinate**; in this case it is $y = 3$. In this manner, every point on the Cartesian plane represents an ordered pair, and every ordered pair has a spot on the Cartesian plane.



Going back now to the equation $3x - y = 8$ with solution pairs including $(3, 1)$ and $(2, -2)$, we would find that $(4, 4)$, $(1, -5)$ and $(5, 7)$ are also solutions. We can plot those points on a coordinate grid, as shown to the right. Note that it appears that all five points lie on a line. (In mathematics, when we talk about a line it means a *straight* line.) That line is significant in that

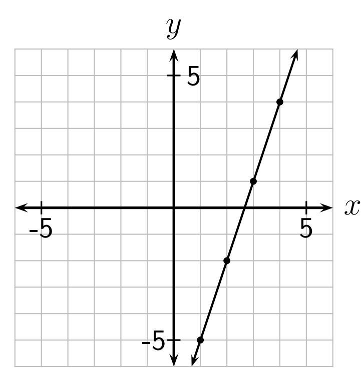


- every (x, y) that is a solution to the equation, when plotted on the coordinate grid, will lie on that line,
- the x- and y-coordinates of any point on the line are a solution pair for the equation.

We call the line the **solution set** of the equation. We will often just call it the graph of the equation, which is slightly incorrect, but we all know what we mean.

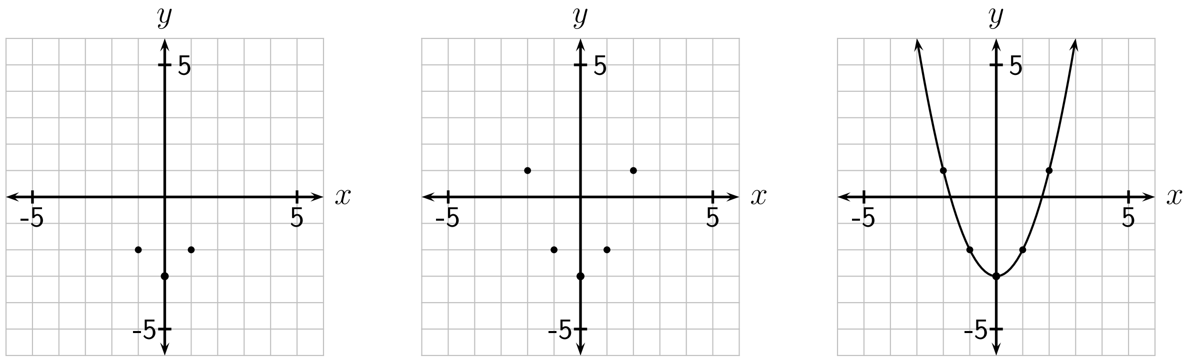
◇ **Example 6.1(c):** Graph the equation $3x - y = 8$.

Solution: Once we have found four or five solution pairs we plot each one as a point, as shown above. We then draw a curve or line that smoothly connects all the points, as shown to the right. The arrowheads on each end of the line indicates that the line keeps going in the direction of the arrows.



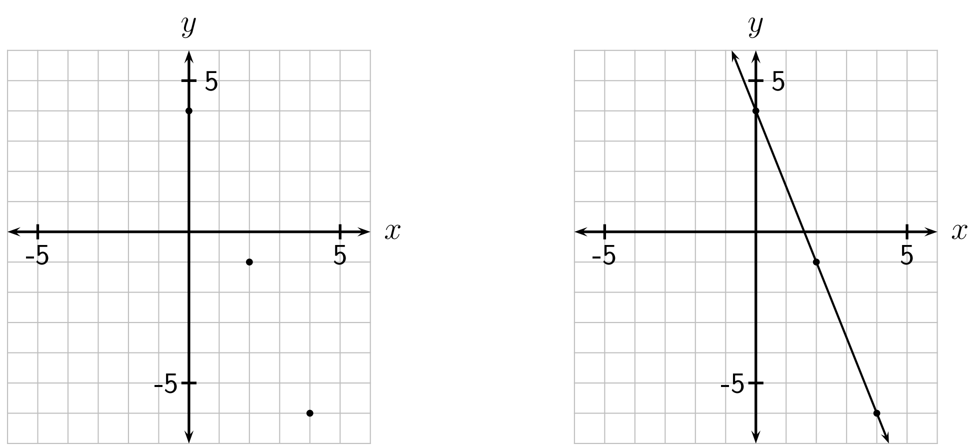
◇ **Example 6.1(d):** Graph the equation $y = x^2 - 3$. Another Example

Solution: In Example 6.1(b) we found the solution pairs $(1,-2)$, $(0,-3)$, $(-1,-2)$ and $(5,22)$. Plotting the first three of those points gives us the graph shown below and to the left. Those are not enough to give a clear indication of what the graph would look like, so we let $x=-2$ and $x=2$ to get two more ordered pairs $(-2,1)$ and $(2,1)$, which are added to the graph shown in the middle below. We can then see that the graph is a U-shape, called a **parabola**, as shown in the third graph below.



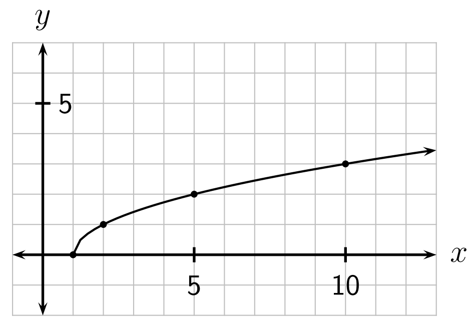
◇ **Example 6.1(e):** Graph the equation $y = -\frac{5}{2}x + 4$.

Solution: As in the previous example, we want to choose some values for x and substitute them into the equation to find the corresponding y value for each x. It should be easily seen that one solution pair is $(0,4)$. If we substitute one for x we get $y = -\frac{5}{2}(1)+4 = \frac{3}{2} = 1\frac{1}{2}$. Now the ordered pair $(1, 1\frac{1}{2})$ can be plotted, but it is not as easy to work with as a pair in which both numbers are integers. Note that if instead of choosing $x=1$ we use $x=2$, we get $y = -\frac{5}{2}(2)+4 = -5+4 = -1$, giving us the ordered pair $(2,-1)$. What made this work out better is that the value $x=2$ cancels with the two in the bottom of $-\frac{5}{2}$. The same sort of this happens if we choose x to be other multiples of two, like four, which gives us the ordered pair $(4,-6)$. Plotting the three points we now have gives us the graph shown below and to the left. The three points appear to be in a line, and in fact they are; the graph is shown below and to the right.



◇ **Example 6.1(f):** Graph the equation $y = \sqrt{x-1}$.

Solution: As before, we want to substitute values for x and find the corresponding y values. It is nicest if we choose values like $x = 5$, because then $y = \sqrt{5-1} = \sqrt{4} = 2$, an integer. This shows that we want to choose values of x resulting in finding square roots of $0, 1, 4, 9, 16$, ..., the perfect squares. From this we get the solution pairs $(1,0)$, $(2,1)$, $(5,2)$, $(10,3)$, $(17,4)$, etc. Plotting the few of those that can fit on the grid to the right, we get the graph shown there.



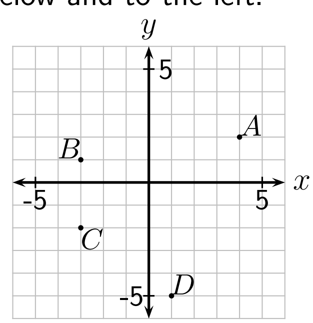
Section 6.1 Exercises

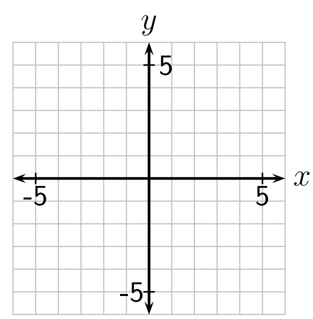
To Solutions

1. Consider again the equation $3x - y = 8$. Solve for y and find five solutions pairs, including some with negative values of x.

2. Solve the equation $x^2 + y = 5$ for y, then find five solution pairs. **Use some negative values for x as well as positive values.** Once you find one solution, it should be easy to get another - explain.

3. (a) Give the coordinates for each of the points A through D plotted on the coordinate grid below and to the left.





   (b) Plot and label (with letters) the points $A(4,-1)$, $B(-3,-2)$, $C(1,5)$, $D(5,1)$ on the coordinate grid above and to the right.

4. Plot the solution pairs that you found for the equation from Exercise 2. *They should not lie on a line*, but there should be a pattern in their locations. If you can't see it, find a few more. Then draw a curve that you think goes through all of them.

5. There is no reason that we need to solve for y and substitute values for x. Sometimes it will be much easier to solve for x and substitute values for y to get solution pairs.

   (a) Do this to find three solutions for $x - y^2 = 1$. *Make sure that you give solution pairs with x first, y second!*

(b) Can you use the idea from Exercise 3 to get more solutions from the three that you found? You should be able to get at least one more.

(c) Plot your solution pairs and draw a curve that you think contains all solutions to the equation.

6. Consider the equation $2x + 3y = 15$.

   (a) Solve the equation for y.

   (b) To find solution pairs, we should choose values of x that are multiples of what number?

   (c) Find four solution pairs.

   (d) Graph the equation.

7. Find three solution pairs to $y = \frac{3}{5}x - 2$ and plot them. Do you think you might know what the graph of the equation is? Draw it.

8. Use the equation $x = \sqrt{y+5}$ for the following.

   (a) Find four integer solutions to the equation. This will require values of y that result in $\sqrt{0}$, $\sqrt{1}$, $\sqrt{4}$, $\sqrt{9}$, etc.

   (b) Is there a largest or smallest value that you can use for y? What is the solution pair in that case?

   (c) There is no reason that we have to use values of y that give perfect square values for $y + 5$. Give the solution pair corresponding to $y = 7$, *in exact form*.

   (d) Give the solution pair corresponding to $y = 7$, *in decimal form, rounded to the hundredth's place*.

   (e) Graph the equation.

9. Consider the equation $y + 3 = \sqrt{1 - x}$

   (a) Solve the equation for y.

   (b) Is there a largest or smallest value that you can use for x? What is the solution in that case?

   (c) Find three more solutions and graph the equation.

10. Multiply and simplify.

    (a) $(1 - \sqrt{2})(1 + \sqrt{2})$ (b) $(3 - 5\sqrt{6})^2$ (c) $(5 + 3\sqrt{10})(5 - 3\sqrt{10})$

11. A retailer adds 40% of her cost for an item to get the price she sells it at in her store. Find her cost for an item that she sells for $\$59.95$. **Write an equation that can be used to solve this problem, and solve the equation.**

12. The hypotenuse of a right triangle is two more than one of the legs, and the other leg has length 8. What are the lengths of the sides of the triangle? **Write an equation that can be used to solve this problem, and solve the equation.**

6.2 x- and y-Intercepts

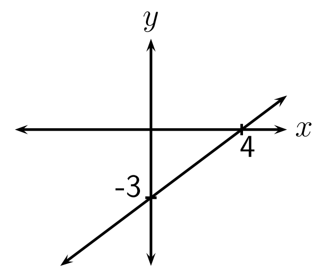
6. (b) Find x- and y-intercepts of an equation in two variables.

In this section we begin with one simple observation: zero is usually the easiest number with which to do computations!

◇ **Example 6.2(a):** For the equation $3x-4y=12$, let $x=0$ and $y=0$ to get two solution pairs. Given that the graph of the equation is a line, give the graph. Another Example

Solution: Letting each of x and y be zero, we get

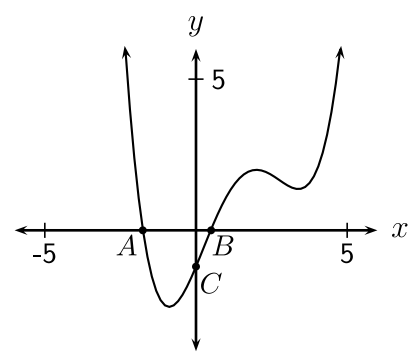
$$\begin{aligned} 3(0)-4y &= 12 \\ -4y &= 12 \\ y &= -3 \end{aligned} \qquad \begin{aligned} 3x-4(0) &= 12 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$



This gives us the two points $(0,-3)$ and $(4,0)$. Having been told that the graph is a line, the graph must be as shown to the right.

We call point $(4,0)$ the x-**intercept** of the equation and $(0,-3)$ the y-**intercept** of the equation. One or both of these two points are often of interest to us, so this section is devoted entirely to finding the intercepts.

Consider the graph shown to the right, for some unknown equation. Points A and B are *both* x-intercepts, and point C is a y-intercept. *It is possible to have more than one intercept on each axis.* The points are called intercepts because they are where the graph of the equation intercepts the axes.



Since I put no scale on either axis, you don't know the x-coordinates of A and B, but it should be clear that *they both have y-coordinates of zero.* Similarly, C has an x-coordinate of zero. This indicates the following.

Finding Intercepts

- To find the x-intercepts for an equation, let $y=0$ and solve for x.
- To find the y-intercepts for an equation, let $x=0$ and solve for y.
- We sometimes give the intercepts as single numbers rather than ordered pairs, because we know that the y-coordinates of x-intercepts must be zero, and vice-versa.

◇ **Example 6.2(b):** Find the intercepts of $x = 4 - y^2$

Solution: As just discussed, to find the x-intercept(s) we set $y = 0$ and solve, and vice-versa for the y-intercept(s):

$$\begin{aligned} x &= 4-(0)^2 \\ x &= 4 \end{aligned} \qquad\qquad \begin{aligned} 0 &= 4-y^2 \\ y^2 &= 4 \\ y &= \pm 2 \end{aligned}$$

The x-intercept is $(4,0)$ and there are two y-intercepts, $(0,2)$ and $(0,-2)$.

Let's take a moment to note a couple of things:

- If you can remember that the intercepts are found by letting x and y be zero and solving for the other variable, you can just find all ordered pairs you can this way. Then think about where each would be plotted in order to determine whether it is an x-intercept or y-intercept.
- For an x-intercept of $(a,0)$ we will often just say the x-intercept is a, since we know that the y-value at an x-intercept must be zero, and similarly for a y-intercept. So for the above example, we would say that there is an x-intercept of 4 and y-intercepts of 2 and -2.

Section 6.2 Exercises

To Solutions

1. Find the intercepts of $-5x + 3y = 15$.

2. Find the intercepts of $y = x^2 - 2x - 3$.

3. This exercise will illustrate that intercepts are not always integers! (Remember that integers are positive or negative whole numbers, or zero.) Find the intercepts of $2x + 3y = 9$.

4. This exercise illustrates that the an x-intercept can also be a y-intercept.

   (a) Find the x-intercepts of $y = x^2 - 5x$. *Give your answers as ordered pairs.*

   (b) Find the y-intercepts of $y = x^2 - 5x$. *Give your answers as ordered pairs.*

5. Find just the y-intercept of each of the following. y-intercepts of equations like these will be important to us soon.

   (a) $y = \frac{3}{5}x - 2$ (b) $y = -\frac{4}{3}x + 5$ (c) $y = -3x + 1$

6. Find the intercepts for each of the following equations.

   (a) $3x - 2y = 2$ (b) $x = y^2 - 2y$ (c) $y = \sqrt{x+4}$

   (d) $\sqrt{x+4} = y + 1$ (e) $3x + 5y = 30$ (f) $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

7. For each equation, tell what values x is not allowed to have, then solve the equation.

   (a) $\dfrac{3}{2} + \dfrac{5}{x-3} = \dfrac{x+9}{2x-6}$ (b) $\dfrac{4x}{x+2} = 4 - \dfrac{2}{x-1}$

8. For each of the following, use the quadratic formula to solve the given equation. Simplify your answers.

   (a) $x^2 + 4x = 41$ (b) $25x^2 + 7 = 30x$

9. Solve each equation for y. Give your answers in $y = mx + b$ form.

   (a) $2x - 3y = 6$ (b) $3x + 5y + 10 = 0$

10. A manufacturer of small calculators knows that the number x of calculators that it can sell each week is related to the price per calculator p by the equation $x = 1300 - 100p$. The weekly revenue (money they bring in) is the price times the number of calculators: $R = px = p(1300 - 100p)$. What price should they set for the cartridges if they want the weekly revenue to be \$4225?

11. For each of the following equations, find at least five solution pairs, then draw a graph of the solution set for the equation.

    (a) $y = x^2 - 2x$ (b) $y = (x+3)^2$ (c) $x + 2y = 4$

6.3 Slopes of Lines

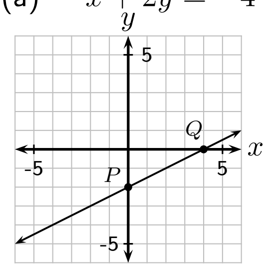
> 6. (c) Find the slope of a line, including vertical and horizontal.
>   (d) Know and apply the relationship between slopes of parallel and perpendicular lines.

Consider the following equations. Note that each has the form $Ax + By = C$, where any of A, B or C could be negative. For example, for equation (a) we have $A = -1$, $B = 2$ and $C = -4$.

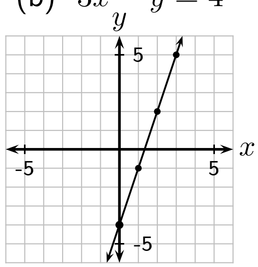
(a) $-x + 2y = -4$ (b) $3x - y = 4$ (c) $2x - 3y = 6$ (d) $2x + 3y = 3$

Because neither the x or the y is squared, under a square root, or in the bottom of a fraction, the graphs of all of these equations will be lines. Here are their graphs:

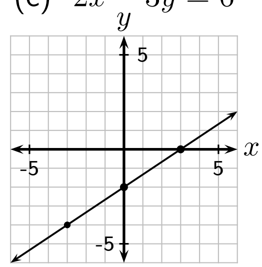
(a) $-x + 2y = -4$



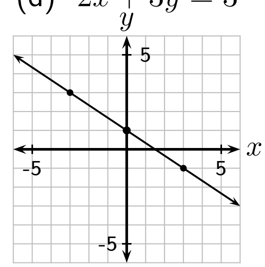
(b) $3x - y = 4$



(c) $2x - 3y = 6$



(d) $2x + 3y = 3$



When dealing with lines, it will be convenient to define a concept called the **slope** of a line. For any line, the slope is a number that should describe the "steepness" of the line, with steeper lines having larger numbers for their slopes. So, for example, the slope of the line in (a) above should be less than the slope of the line in (b). A horizontal line has no steepness, so its slope should be zero.

Take a close look at the lines in (c) and (d) above - both have the same steepness even though they look quite different. We want to distinguish those two from each other, so we will say that the line in (c) has a positive slope, and the line in (d) has a negative slope. *Any line going upward from left to right has a positive slope*, and any line going downward from left to right has a negative slope.

This gives us a general idea what slope is about, but we now need a way to determine the actual slope of a line. The general idea is this: We find two points on the line and consider the vertical difference between the two points, which we call the **rise**, and the horizontal difference between the two points, which we call the **run**. We then define the slope to be the *rise over the run*, with the appropriate sign as described above.

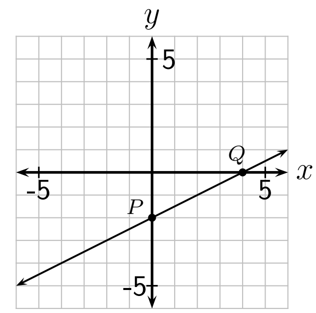
◇ **Example 6.3(a):** Find the slope of the line in graph (a) above.

Solution: The slope is positive, with a rise of two (from point P to point Q) and a run of four. The slope is therefore $\frac{2}{4} = \frac{1}{2}$.

◇ **Example 6.3(b):** Find the slope of the line in graph (d).

Solution: The slope is negative in this case, with a rise of two and a run of three, so the slope is $-\frac{2}{3}$.

Graph (a) is shown again to the right. Note that point A has coordinates $(0,-2)$ and point B has coordinates $(4,0)$. Now the y-coordinate of each point tells its "height" in some sense, so if we subtract the y-coordinates of point A from the y-coordinate of point B we get the rise between the two points: rise $=2=0-(-2)$. If we subtract the x-coordinate of point A from the x-coordinate of point B we get the run between the two points: run $=4=4-0$. We can then take the rise over the run to get a slope of $\frac{2}{4}=\frac{1}{2}$.



The upshot of all this is that to find the slope of the line through two points, we can simply subtract the y-coordinates of the points, subtract the x-coordinates, and divide the results. Letting the letter m represent slope, we can summarize this as follows. If we have two points (x_1,y_1) and (x_2,y_2), the slope of the line through the two points is

$$m=\frac{y_2-y_1}{x_2-x_1}$$

Given two ordered pairs, it doesn't matter which pair we call (x_1,y_1) and which pair we call (x_2,y_2), but x_1 and y_1 have to come from the same ordered pair, as do x_2 and y_2.

◇ **Example 6.3(c):** Find the slope of the line through $(2,3)$ and $(-4,5)$. Another Example

Solution: Subtracting the y's we get $5-3=2$ and subtracting the x's *in the same order* we get $-4-2=-6$. the slope is then

$$m=\frac{\text{rise}}{\text{run}}=\frac{5-3}{-4-2}=\frac{2}{-6}=-\frac{1}{3}$$

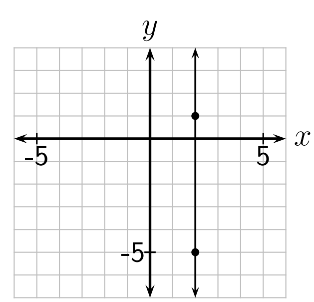
Note that if we were to subtract in the other order we would get the same result in the end:

$$m=\frac{3-5}{2-(-4)}=\frac{-2}{6}=-\frac{1}{3}\quad\text{again}$$

The next example illustrates an important idea.

◇ **Example 6.3(d):** Find the slope of the line through $(2,1)$ and $(2,-5)$.Another Example

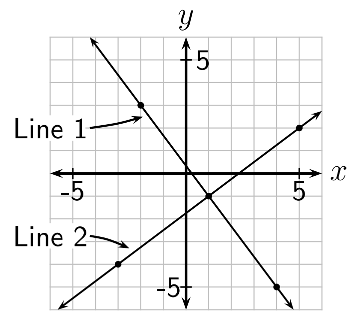
Solution: $m=\dfrac{-5-1}{2-2}=\dfrac{-6}{0}$, which is not defined. When we plot the two points and the line containing them, shown to the right, we see that the line is vertical.



The last example shows us that *the slope of a vertical line is undefined*. With a little thought it should be clear that a horizontal line has a rise of zero for any value of run we want, resulting in a fraction of the form $\frac{0}{a} = 0$. Therefore *horizontal lines have slopes of zero*.

It should also be clear that if two lines are parallel, they have the same slope. What about if the two lines are perpendicular?

◇ **Example 6.3(e):** The two lines shown to the right are perpendicular. Find each of their slopes.



Solution: The slope of Line 1 is $m = -\dfrac{4}{3}$ and the slope of Line 2 is $m = \dfrac{3}{4}$.

The above example seems to indicate that if two lines are perpendicular, their slopes have opposite signs and are reciprocal fractions of each other. This is in fact always the case.

Let's summarize what we now know about slopes of lines.

Slopes of Lines

- The slope of a line is a single number that indicates how steep the line is.
- Lines that go up from left to right have positive slopes and lines that go down from left to right have negative slopes.
- The slope of a horizontal line is zero and the slope of a vertical line is undefined, or is said to not exist.
- The slope of a line is $\dfrac{\text{rise}}{\text{run}}$
- **Slope Formula:** The slope of the line through two points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
- Parallel lines have the same slope.
- Slopes of perpendicular lines are negative reciprocals of each other.

Section 6.3 Exercises

To Solutions

1. Find the slope of the line in graph (b) at the start of the section.

2. Find the slope of the line in graph (c) at the start of the section.

3. Find the slope of the line through the given pair of points.

   (a) $(1, 0)$ and $(5, 2)$ (b) $(3, -4)$ and $(7, -4)$

4. The slope of Line 1 is $-\frac{3}{4}$.

   (a) Line 2 is parallel to Line 1. What is the slope of Line 2?

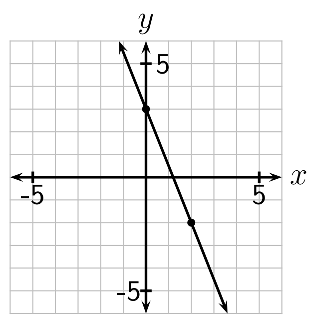
   (b) Line 3 is perpendicular to Line 1. What is the slope of Line 3?

5. Find the slopes of the lines through the following pairs of points.

   (a) $(5, -2)$ and $(3, 2)$ (b) $(-3, 7)$ and $(7, 11)$ (c) $(3, 2)$ and $(-5, 2)$

   (d) $(1, 7)$ and $(1, -1)$ (e) $(-4, -5)$ and $(-1, 4)$ (f) $(-1, 3)$ and $(2, -3)$

6. Find the slope of each line.

(a)


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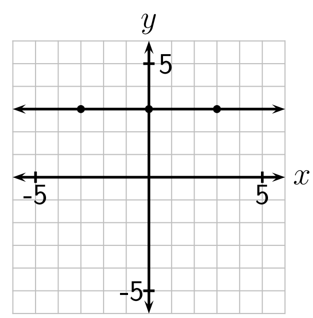

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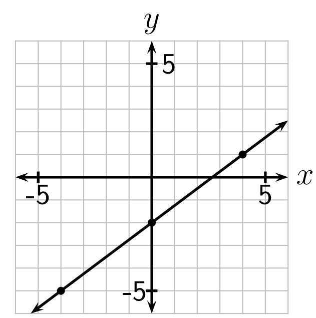

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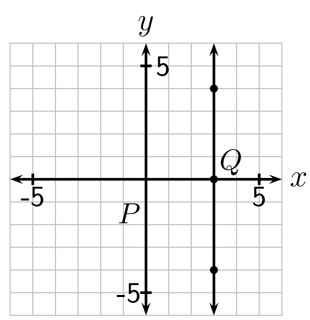

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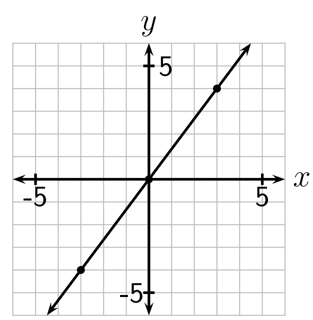
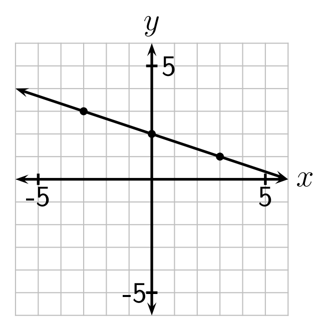


(f)


7. Line 1 has slope $-\frac{1}{3}$, and Line 2 is perpendicular to Line 1. What is the slope of Line 2?

8. Line 1 has slope $\frac{1}{2}$. Line 2 is perpendicular to Line 1, and Line 3 is perpendicular to Line 2. What is the slope of Line 3?

9. The width of a rectangle is three more than half the length. The perimeter is 39. How long are the sides of the rectangle? **Write an equation that can be used to solve this problem, and solve the equation.**

10. The cost of a compact disc, with 6% sales tax, was \$10.55. What was the price of the compact disc? **Write an equation that can be used to solve this problem, and solve the equation.**

11. Find the intercepts for each of the following equations.

    (a) $y = x^2 - 2x$ (b) $y = (x+3)^2$ (c) $x + 2y = 4$

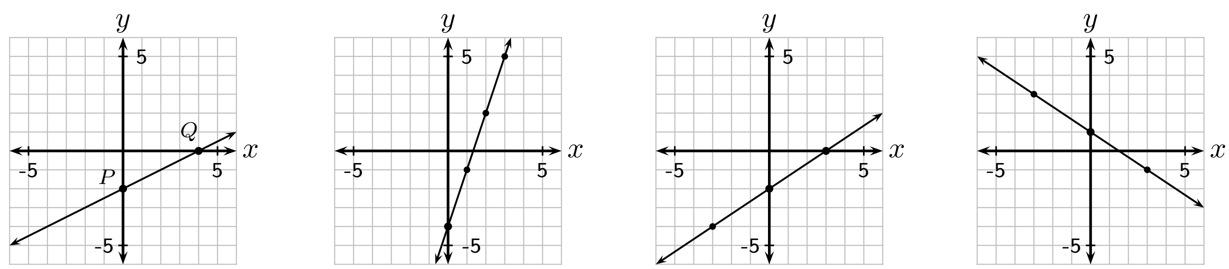
6.4 Equations of Lines

> 6. (e) Graph a line, given its equation; give the equation of a line having a given graph.
>
>   (f) Determine the equation of a line through two points.

We'll begin with three examples, all based on the same four graphs. You may wish to treat the examples as exercises; you should already know how to do all of them yourself.

◇ **Example 6.4(a):** Find the slopes of the four lines graphed below: Another Example

(a) $-x + 2y = -4$ (b) $3x - y = 4$ (c) $2x - 3y = 6$ (d) $2x + 3y = 3$



Solution: The slopes of the lines are (a) $\frac{1}{2}$, (b) 3 or $\frac{3}{1}$, (c) $\frac{2}{3}$, (d) $-\frac{2}{3}$.

◇ **Example 6.4(b):** Give the y-intercepts of the lines whose graphs are shown in Example 6.4(a).

Solution: The y-intercepts of the lines are (a) -2, (b) -4, (c) -2, (d) 1.

◇ **Example 6.4(c):** The equations of the lines graphed in Example 6.4(a) are given above each of the graphs. Solve each equation for y.

Solution:

$$\text{(a)}\quad \begin{aligned} -x + 2y &= -4 \\ 2y &= x - 4 \\ y &= \tfrac{1}{2}x - 4 \end{aligned} \qquad \text{(b)}\quad \begin{aligned} 3x - y &= 4 \\ -y &= -3x + 4 \\ y &= 3x - 4 \end{aligned}$$

$$\text{(c)}\quad \begin{aligned} 2x - 3y &= 6 \\ -3y &= -2x + 6 \\ y &= \tfrac{2}{3}x - 2 \end{aligned} \qquad \text{(d)}\quad \begin{aligned} 2x + 3y &= 3 \\ 3y &= -2x + 3 \\ y &= -\tfrac{2}{3}x + 1 \end{aligned}$$

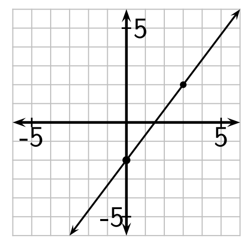
We want to look at the results of the above three examples and see how the equations, when solved for y, are related to the slopes and y-intercepts. First, we note that each equation is of the form $y = mx + b$ for some numbers m and b, with either or both of them perhaps being negative. From the above, the number m always seems to be the slope of the line, and the number b seems to be the y-intercept. Both of these things are in fact true:

Slope-Intercept Form of a Line

A line with slope m and y-intercept b has equation $y = mx + b$.

This can be used two ways, to get the equation of a line from a graph, and to graph a line whose equation is given.

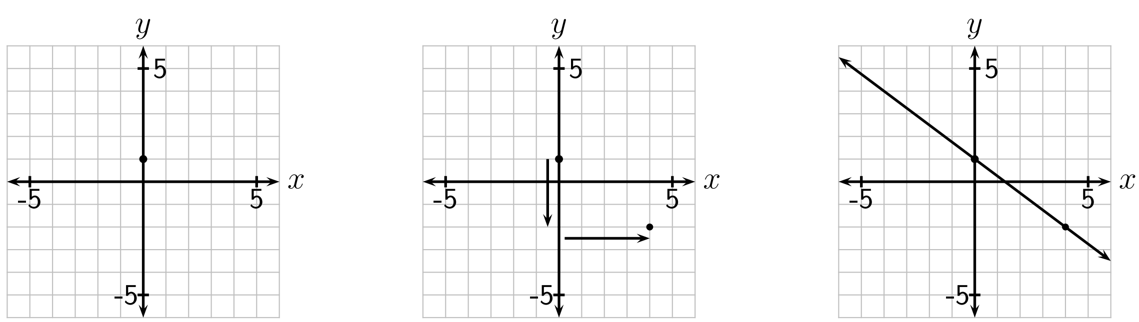
◇ **Example 6.4(d):** Give the equation of the line with the graph shown to the right. Another Example



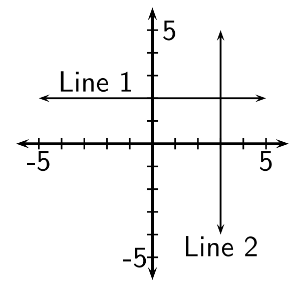
Solution: We can see that the line has a slope of $\frac{4}{3}$ and a y-intercept of -2, so the equation of the line is $y = \frac{4}{3}x - 2$.

◇ **Example 6.4(e):** Graph the line with equation $y = -\frac{3}{4}x + 1$. Another Example Another Example

Solution: First we plot the y-intercept $(0, 1)$, as shown on the graph below and to the left. From there we go down three units and to the right four units, based on the slope of $-\frac{3}{4}$, and plot another point. This is shown o the middle graph below. Finally we draw a line through those two points, as shown on the graph below and to the right.



Consider a horizontal line, like the Line 1 shown to the right. Note that the line has a slope of zero, since it is horizontal, and its y-intercept is 2. Therefore the equation of the line is $y = 0x + 2$, which is really just $y = 2$. Now the equation of a line gives us conditions on how the values of x and y have to be related. In this case nothing is said about x, so we can take it to be anything. y, on the other hand, has to be 2 no matter what x is. So the graph is all the points for which y is 2, which is the horizontal line at $y = 2$.



We can't use $y = mx + b$ to get the equation of a vertical line because the slope of a vertical line is undefined, so we have nothing to put in for m! For Line 2 on the graph above, however, we can note that every point on Line 2 has an x-coordinate of 3, so the equation of the line is $x = 3$. These observations lead us to the following:

Equations of Horizontal and Vertical Lines

A horizontal line has equation $y = a$ and a vertical line has equation $x = b$.

All that we need in order to plot the graph of a line is two points that are on the line. Now we will use two given points and a bit of algebra to find the *equation* of the line through those points. The procedure is given first, but it may not make sense until you follow along with it as you read Example 6.4(f).

Equation of a Line Through Two Points

To find the equation of a line through two points you need to find values for m and b in $y = mx + b$. To do this,

- use the two points to find the slope of the line,
- put the slope into $y = mx + b$,
- put *either* point into $y = mx + b$ (with the slope you found for m) and solve for b,
- write the equation of the line with the m and b that you have found.

◇ **Example 6.4(f):** Find the equation of the line through the points $(-3, 3)$ and $(6, -3)$ algebraically.

Another Example

Solution: The slope of the line is $m = \dfrac{3-(-3)}{-3-6} = \dfrac{6}{-9} = -\dfrac{2}{3}$, so the equation of the line must look like $y = -\frac{2}{3}x + b$. We then insert the coordinates of either of the two points on the line for x and y, and solve for b:

$$\begin{aligned} 3 &= -\tfrac{2}{3}(-3)+b \\ 3 &= 2+b \\ b &= 1 \end{aligned}$$

We insert this value into $y=-\frac{2}{3}x+b$ to get the equation of the line as $y=-\frac{2}{3}x+1$.

◇ **Example 6.4(g):** Find the equation of the line through the points $(-3,3)$ and $(-3,-1)$ algebraically. Another Example

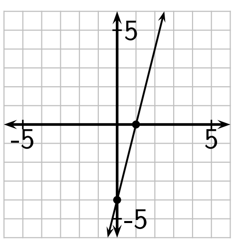
Solution: $m=\dfrac{3-(-1)}{-3-(-3)}=\dfrac{3+1}{-3+3}=\dfrac{4}{0}$. The slope is undefined, so this is a vertical line. We can see that both points have x-coordinates of three, so the equation of the line is $x=3$.

Section 6.4 Exercises

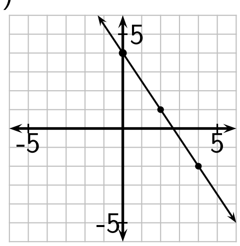
To Solutions

1. Give the equation of each line.

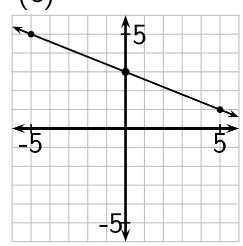
(a)



(b)



(c)



2. Sketch the graph of each line.

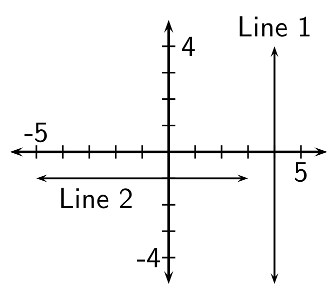
(a) $y=-\frac{1}{2}x+3$ (b) $y=\frac{4}{5}x-1$ (c) $y=3x-4$

3. (a) Give the equation of Line 1 to the right.

   (b) Give the equation of Line 2 to the right.

   (c) Sketch the graph of $x=-2$.

   (d) Sketch the graph of $y=4$.



4. Find the equation of the line through the two points **algebraically**.

(a) $(3,6),\ (12,18)$ (b) $(-3,9),\ (6,3)$ (c) $(3,2),\ (-5,2)$

5. For each of the following equations, find at least five solution pairs, then draw a graph of the solution set for the equation.

   (a) $3x - 2y = 2$ (b) $x = y^2 - 2y$ (c) $y = \sqrt{x+4}$

6. (a) Give the slope of a line that is parallel to the line graphed in Exercise 1(a).
   (b) Give the slope of a line that is perpendicular to the line graphed in Exercise 1(b).
   (c) Give the slope of the line graphed in Exercise 1(c).
   (d) Give the slope of a line that is parallel to the line whose equation is given in Exercise 2(a).
   (e) Give the slope of the line whose equation is given in Exercise 2(b).
   (f) Give the slope of a line that is perpendicular to the line whose equation is given in Exercise 2(c).

6.5 Applications of Linear Equations

6. (g) Use a given linear model of a "real" situation and a given value of either of the two linearly related quantities to find the value of the other.

   (h) Given a linear model of a "real" situation, interpret the values of the slope and intercept.

Using a Linear Model to Solve Problems

An insurance company collects data on amounts of damage (in dollars) sustained by houses that have caught on fire in a small rural community. Based on their data they determine that the expected amount D of fire damage (in dollars) is related to the distance d (in miles) of the house from the fire station. (Note here the importance of distinguishing between upper case variables and lower case variables!) The equation that seems to model the situation well is

$$D = 28000 + 9000d$$

This tells us that the damage D is a function of the distance d of the house from the fire station. Given a value for either of these variables, we can find the value of the other.

◇ **Example 6.5(a):** Determine the expected amount of damage from house fires that are 3.2 miles from the fire station, 4.2 miles from the fire station and 5.2 miles from the fire station.

Solution: For convenience, let's rewrite the equation using function notation, and in the slope-intercept form: $D(d) = 9000d + 28000$. Using this we have

$$D(3.2) = 9000(3.2) + 28000 = 56800, \qquad D(4.2) = 65800, \qquad D(5.2) = 74800$$

The damages for distances of 3.2, 4.2 and 5.2 miles from the fire station are $\$56,800$, $\$65,800$ and $\$74,800$.

Note that in the above example, for each additional mile away from the fire station, the amount of damage increased by $9000, which is the slope of the line with equation $D = 9000d + 28000$.

◇ **Example 6.5(b):** If a house fire caused $47,000 damage, how far would you expect that the fire might have been from the fire station, *to the nearest tenth of a mile*?

Solution: Here we are given a value for D and asked to find a value of d. We do this by substituting the given value of D into the equation and solving for d:

$$\begin{aligned} 47000 &= 9000d + 28000 \\ 19000 &= 9000d \\ 2.1 &= d \end{aligned}$$

We would expect the house that caught fire to be about 2.1 miles from the fire station.

◇ **Example 6.5(c):** How much damage might you expect if your house was right next door to the fire station?

Solution: A house that is right next door to the fire station is essentially a distance of zero miles away. We would then expect the damage to be

$$D(0) = 9000(0) + 28000 = 28000.$$

The damage to the house would be $\$28,000$.

Interpreting the Slope and Intercept of a Linear Model

There are a few important things we want to glean from the above examples.

- When we are given a mathematical relationship between two variables, if we know one we can find the other.
- Recall that slope is rise over run. In this case rise is damage, measured in units of dollars, and the run is distance, measured in miles. Therefore the slope is measured in $\frac{\text{dollars}}{\text{miles}}$, or dollars per mile. The slope of 9000 dollars per mile tells us that for each additional mile farther from the fire station that a house fire is, the amount of damage is expected to increase by \$9000.
- The amount of damage expected for a house fire that is essentially right at the station is \$28,000, which is the D-intercept for the equation.

In general we have the following.

Interpreting Slopes and Intercepts of Lines

When an "output" variable depends linearly on another "input" variable,

- the slope has units of the output variable units over the input variable units, and it represents the amount of increase (or decrease, if it is negative) in the output variable for each one unit increase in the input variable,
- the output variable intercept ("y"-intercept) is the value of the output variable when the value of the input variable is zero, and its units are the units of the output variable. *The intercept is not always meaningful.*

The first of the above two items illustrates what was pointed out after 3.2(a). As the distance increased by one mile from 3.2 miles to 4.2 miles the damage increased by $65800 - 56800 = 9000$ dollars, and when the distance increased again by a mile from 4.2 miles to 5.2 miles the damage again increased by $\$9000$.

When dealing with equations we often call the "input" variable (which is *ALWAYS* graphed on on the horizontal axis) the **independent variable**, and the "output" variable (which is always graphed on the vertical axis) we call the **dependent variable**. Using this language we can reword the items in the previous box as follows.

Slope and Intercept in Applications

For a linear model $y = mx + b$,

- the slope m tells us the amount of increase in the dependent variable for every one unit increase in the independent variable
- the vertical axis intercept tells us the value of the dependent variable when the independent variable is zero

Section 6.5 Exercises

To Solutions

1. The weight w (in grams) of a certain kind of lizard is related to the length l (in centimeters) of the lizard by the equation $w = 22l - 84$. This equation is based on statistical analysis of a bunch of lizards between 12 and 30 cm long.

   (a) Find the weight of a lizard that is 3 cm long. Why is this not reasonable? What is the problem here?

   (b) What is the w-intercept, and why does it have no meaning here?

   (c) What is the slope, with units, and what does it represent?

2. A salesperson earns \$800 per month, plus a 3% commission on all sales. Let P represent the salesperson's gross pay for a month, and let S be the amount of sales they make in a month. (Both are of course in dollars. Remember that to compute 3% of a quantity we multiply by the quantity by 0.03, the decimal equivalent of 3%.)

   (a) Find the pay for the salesperson when they have sales of \$50,000, and when they have sales of \$100,000.

   (b) Find the equation for pay as a function of sales, given that this is a linear relationship.

   (c) What is the slope of the line, and what does it represent?

   (d) What is the P-intercept of the line, and what does it represent?

3. The cost y (in dollars) of renting a car for one day and driving x miles is given by the equation $y = 0.24x + 30$. Of course this is the equation of a line. Explain what the slope and y-intercept of the line represent, *in terms of renting the car*.

4. Solve each equation for y. Give your answers in $y = mx + b$ form.

   (a) $-5x + 3y = 9$

   (b) $5x + 2y = 20$

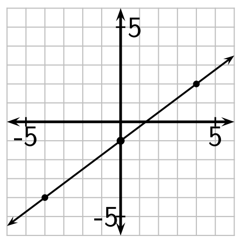
5. For each of the following equations, find at least five solution pairs, then draw a graph of the solution set for the equation.

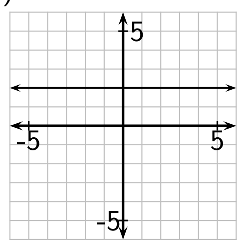
   (a) $x^2 + y = 1$ (b) $x - y^2 = 1$ (c) $y = x^3$

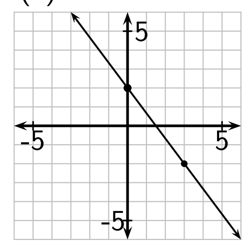
6. Find the intercepts for each of the following equations.

   (a) $4x - 5y = 20$ (b) $x^2 + y^2 = 25$ (c) $x = \sqrt{1 - y}$

7. Give the equation of each line.

   (a)
   

   (b)
   

   (c)
   

8. Sketch the graph of each line.

   (a) $y = -x + 2$ (b) $3x + 5y = 10$ (solve for y, then graph)

9. Find the equation of the line through the two points **algebraically**.

   (a) $(-4, -2),\ (-2, 4)$ (b) $(1, 7),\ (1, -1)$ (c) $(-1, -5),\ (1, 3)$

A Solutions to Exercises

A.6 Chapter 6 Solutions

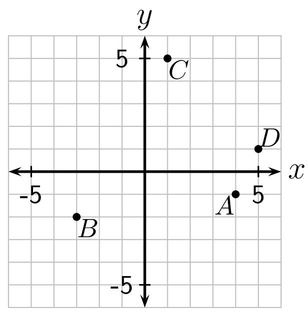
Section 6.1 Solutions Back to 6.1 Exercises

1. Any of $(-4,-20),(-3,-17),(-2,-14),(-1,-11),(0,-8),(1,-5),(2,-2),(3,1),(4,4),\ldots$

2. Any of $(0,5),(1,4),(-1,4),(2,1),(-2,1),(3,-4),(-3,-4),\ldots$ The same value of y results from x and $-x$

3. (a) $A(4,2),\ B(-3,1),\ C(-3,-2),\ D(1,-5)$

3. (b)



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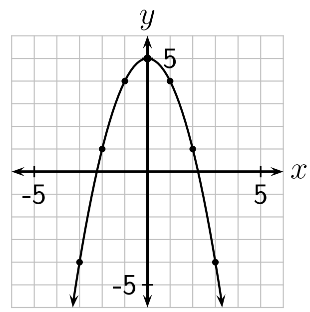

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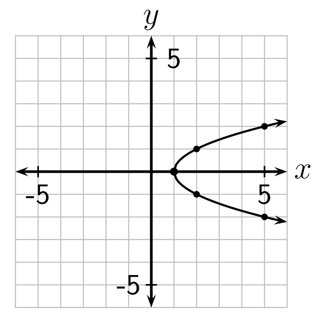
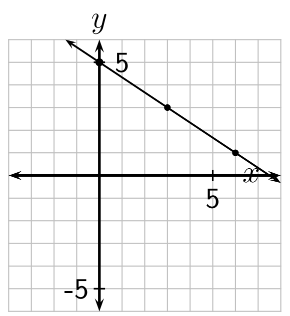
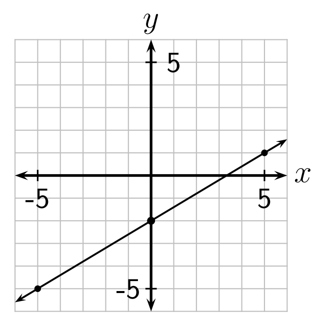


6. (a) $y=-\frac{2}{3}x+5$ (b) multiples of three (c) $(-6,9),(-3,7),(0,5),(3,3),(6,1),(9,-1)$

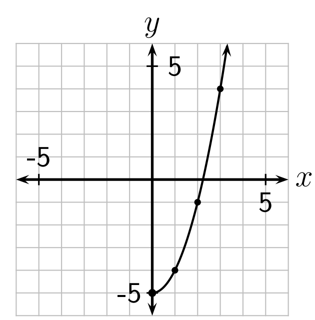
6. (d)



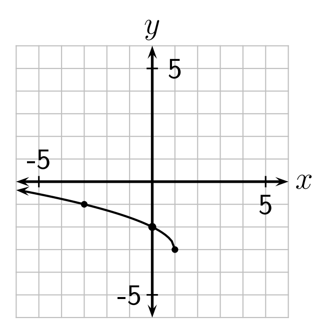
7.



8. (a) $(0,-5)$, $(1,-4)$, $(2,-1)$, $(3,4)$, $(4,11)$, ...
   (b) The smallest value that can be used for y is -5. The solution in that case is $(0,-5)$.
   (c) $(\sqrt{13},7)$
   (d) $(3.61,7)$
   (e) See graph to the right.



9. (a) $y=\sqrt{1-x}-3$
   (b) The *largest* value of x that can be used is $x=1$, giving the solution $(1,-3)$.
   (c) See graph to the right.



10. (a) -1 (b) $159 - 30\sqrt{6}$ (c) -65

11. **Equation:** $x + 0.40x = 59.95$ **Answer:** $x = \$42.82$

12. **Equation:** $x^2 + 8^2 = (x+2)^2$ **Answer:** 8, 15, 17

Section 6.2 Solutions

Back to 6.2 Exercises

1. x-intercept: -3 y-intercept: 5

2. x-intercepts: $-1, 3$ y-intercept: -3

3. x-intercept: $\frac{9}{2}$ or $4\frac{1}{2}$ y-intercept: 3

4. (a) $(0,0),\ (5,0)$ (b) $(0,0)$

5. (a) -2 (b) 5 (c) 1

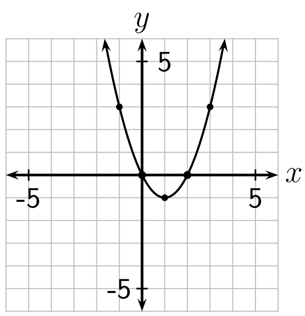
6. (a) x-intercept: $\frac{2}{3}$ y-intercept: -1 (b) x-intercept: 0 y-intercepts: 0,2
   (c) x-intercept: -4 y-intercept: 2 (d) x-intercept: -3 y-intercept: 1
   (e) x-intercept: 10 y-intercept: 6 (f) x-intercepts: $3, -3$ y-intercepts: $4, -4$

7. (a) $x \neq 3,\quad x = 4$ (b) $x \neq -2, 1,\quad x = 2$

8. (a) $x = -2 + 3\sqrt{5},\ -2 - 3\sqrt{5}$ (b) $x = \frac{3}{5} + \frac{\sqrt{2}}{5},\ \frac{3}{5} - \frac{\sqrt{2}}{5}$

9. (a) $y = \frac{2}{3}x - 2$ (b) $y = -\frac{3}{5}x - 2$

10. \$6.50

11. (a)


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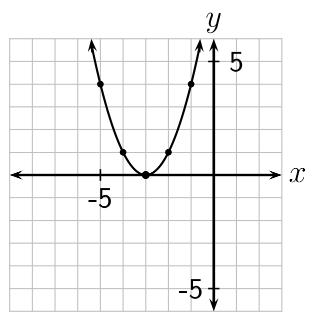
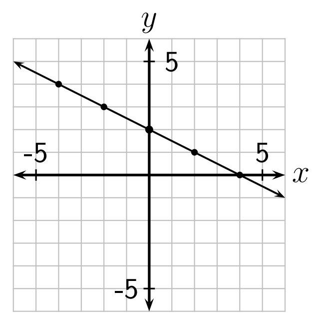


(c)


Section 6.3 Solutions

Back to 6.3 Exercises

1. $\frac{3}{1}$ or 3

2. $\frac{2}{3}$

3. (a) $\frac{1}{2}$ (b) 0

4. (a) $-\frac{3}{4}$ (b) $\frac{4}{3}$

5. (a) -2 (b) $\frac{2}{5}$ (c) 0 (d) undefined (e) 3 (f) -2

6. (a) $-\frac{5}{2}$ (b) 0 (c) $\frac{3}{4}$ (d) undefined (e) $\frac{4}{3}$ (f) $-\frac{1}{3}$

7. 3

8. $\frac{1}{2}$

9. **Equation:** $2l + 2(\frac{1}{2}l + 3) = 39$ **Answer:** $l = 11, w = \frac{1}{2}(11) + 3 = 8.5$

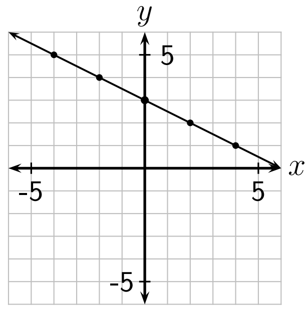
10. **Equation:** $x + 0.06x = 10.55$ **Answer:** $x = \$9.95$

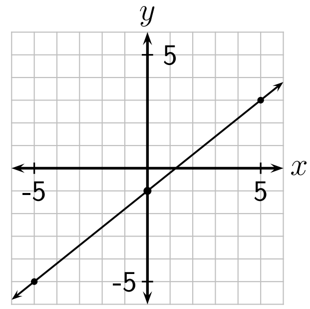
11. (a) x-intercepts: 0,2 y-intercept: 0 (b) x-intercept: -3 y-intercept: 9
    (c) x-intercept: 4 y-intercept: 2

Section 6.4 Solutions

Back to 6.4 Exercises

1. (a) $y = 4x - 4$ (b) $y = -\frac{3}{2}x + 4$ (c) $y = -\frac{2}{5}x + 3$

2. (a)


(b)


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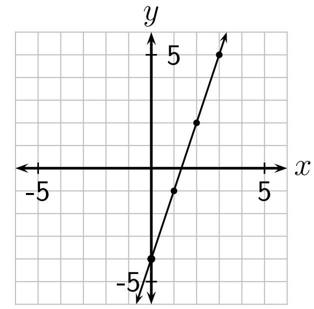
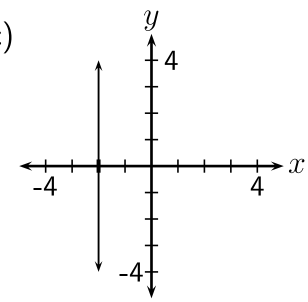
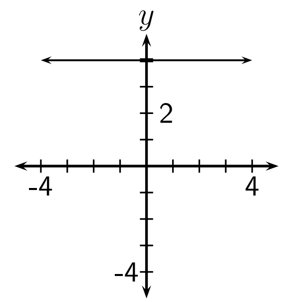
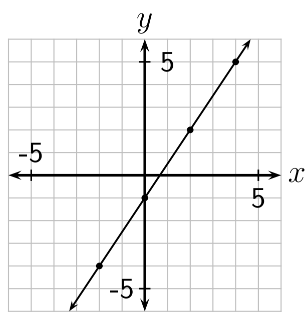
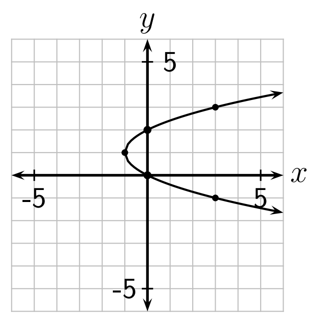


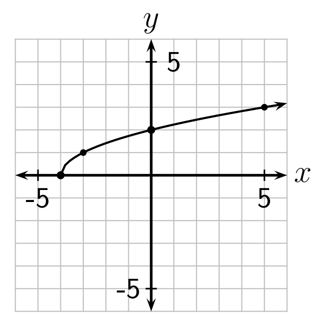
3. (a) $x = 4$ (b) $y = -1$ (c)


(d)


4. (a) $y = \frac{4}{3}x + 2$ (b) $y = -\frac{2}{3}x + 7$ (c) $y = 2$

5. (a)


(b)


(c)


6. (a) $m = 4$ (b) $m = \frac{2}{3}$ (c) $m = -\frac{2}{5}$
(d) $m = -\frac{1}{2}$ (e) $m = \frac{4}{5}$ (f) $m = -\frac{1}{3}$

Section 6.5 Solutions

Back to 6.5 Exercises

1. (a) The weight of a lizard that is 3 cm long is -18 grams. This is not reasonable because a weight cannot be negative. The problem is that the equation is only valid for lizards between 12 and 30 cm in length.
   (b) The w-intercept is -84, which would be the weight of a lizard with a length of zero inches. It is not meaningful for the same reason as given in (a).
   (c) The slope is 22 grams per centimeter. This says that for each centimeter of length gained by a lizard, the weight gain will be 22 grams.

2. (a) \$2300, \$3800 (b) $P = 0.03S + 800$

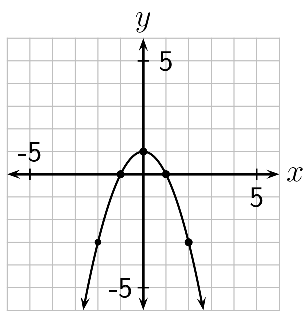
(c) the slope of the line is 0.03, the commission rate (as a decimal percent). (It has no units because it is dollars per dollar, which is one.)

(d) The P-intercept of the line is \$800, the monthly base salary.

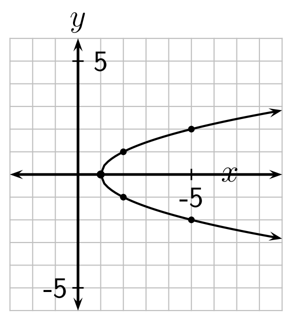
3. The y-intercept is \$30 and represents the daily "flat rate" charge for renting a car. The slope is \$0.24 per mile, the mileage fee for renting the car.

4. (a) $y = \frac{5}{3}x + 3$ (b) $y = \frac{5}{2}x + 10$

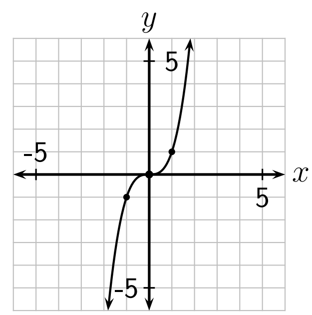
5. (a)



(b)



(c)

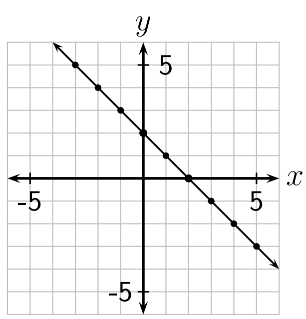


6. (a) x-intercept: 5 y-intercept: -4 (b) x-intercepts: $5, -5$ y-intercepts: $5, -5$

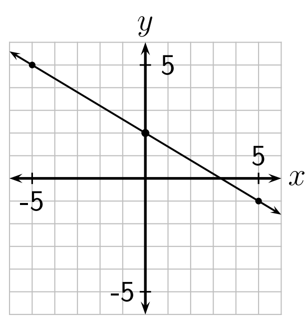
(c) x-intercept: 1 y-intercept: 1

7. (a) $y = \frac{3}{4}x - 1$ (b) $y = 2$ (c) $y = -\frac{4}{3}x + 2$

8. (a)



(c)



9. (a) $y = 3x + 10$ (b) $x = 1$ (c) $y = 4x - 1$