Intermediate Algebra

Gregg Waterman
Oregon Institute of Technology

Contents

7 Systems of Two Linear Equations

7.1 Solving Systems of Two Linear Equations

7. (a) Solve a system of two linear equations by addition.
   (b) Solve a system of two linear equations by substitution.

Consider the two equations

$$\begin{aligned} 2x - 4y &= 18 \\ 3x + 5y &= 5 \end{aligned}$$

Taken together, we call them a **system of equations**. In particular, this is a system of *linear* equations. From past experience it should be clear that each equation has infinitely many solution pairs. When faced with a system of linear equations, our goal will be to *solve the system*. This means find a value of x and a value of y which, when taken together, make *BOTH* equations true. In this case you can easily verify that the values $x = 5,\ y = -2$ make both equations true. We say that the ordered pair $(5, -2)$ is a solution to the system of equations; it turns out it is the *only* solution.

You may be wondering how that solution pair could be obtained from the equations themselves. There are two methods for solving a system of equations, the **addition method** and the **substitution method**.

The Addition Method

We'll begin with the addition method, going from the easiest scenario to the most difficult (which still isn't too hard).

◇ **Example 7.1(a):** Solve the system $\begin{aligned} 3x - y &= 5 \\ 2x + y &= 15 \end{aligned}$.

Solution: The basic idea of the addition method is to add the two equations together so that one of the unknowns goes away. In this case, as shown below and to the left, nothing fancy need be done. The remaining unknown is then solved for and placed back into *either* equation to find the other unknown as shown below and to the right.

$$\begin{aligned} 3x - y &= 5 \\ 2x + y &= 15 \\ \hline 5x &= 20 \\ x &= 4 \end{aligned} \qquad \longrightarrow \qquad \begin{aligned} 3(4) - y &= 5 \\ 12 - y &= 5 \\ 12 &= y + 5 \\ 7 &= y \end{aligned}$$

The solution to the system is $(4, 7)$.

What made this work so smoothly is the $-y$ in the first equation and the $+y$ in the second; when we add the two equations, the sum of these is zero and y "has gone away." In the next example we see what to do in a slightly more difficult situation.

◇ **Example 7.1(b):** Solve the system $\begin{array}{rcl} 3x+4y & = & 13 \\ x+2y & = & 7 \end{array}$.

Solution: We can see that if we just add the two equations together we get $4x+6y=20$, which doesn't help us find either of x or y. The trick here is to multiply the second equation by -3 so that the the first term of that equation becomes $-3x$, the opposite of the first term of the first equation. When we then add the two equations the x terms go away and we can solve for y:

$$\begin{array}{rcl} 3x+4y & = & 13 \\ x+2y & = & 7 \end{array} \quad \begin{array}{c} \Longrightarrow \\ \underset{\text{times } -3}{\Longrightarrow} \end{array} \quad \begin{array}{rcl} 3x+4y & = & 13 \\ -3x-6y & = & -21 \\ \hline -2y & = & -8 \\ y & = & 4 \end{array} \quad \longrightarrow \quad \begin{array}{rcl} x+2(4) & = & 7 \\ x+8 & = & 7 \\ x & = & -1 \end{array}$$

The solution to the system of equations is $(-1,4)$. Note that we could have eliminated y first instead of x:

$$\begin{array}{rcl} 3x+4y & = & 13 \\ x+2y & = & 7 \end{array} \quad \begin{array}{c} \Longrightarrow \\ \underset{\text{times } -2}{\Longrightarrow} \end{array} \quad \begin{array}{rcl} 3x+4y & = & 13 \\ -2x-4y & = & -14 \\ \hline x & = & -1 \end{array} \quad \longrightarrow \quad \begin{array}{rcl} -1+2y & = & 7 \\ 2y & = & 8 \\ y & = & 4 \end{array}$$

The solution to the system of equations is of course the same, $(-1,4)$.

◇ **Example 7.1(c):** Solve the system $\begin{array}{rcl} 2x-4y & = & 18 \\ 3x+5y & = & 5 \end{array}$.

Solution: This is the same system as last time, but we will show how we can eliminate y instead of x if we want. We'll multiply the second equation by -2 so that the the y term of that equation becomes $-4y$, the opposite of the y term of the first equation. When we then add the two equations the y terms go away and we can solve for x:

$$\begin{array}{rcl} 2x-4y & = & 18 \\ 3x+5y & = & 5 \end{array} \quad \begin{array}{c} \overset{\text{times } 5}{\Longrightarrow} \\ \underset{\text{times } 4}{\Longrightarrow} \end{array} \quad \begin{array}{rcl} 10x-20y & = & 90 \\ 12x+20y & = & 20 \\ \hline 22x & = & 110 \\ x & = & 5 \end{array} \quad \longrightarrow \quad \begin{array}{rcl} 2(5)-4y & = & 18 \\ 10-4y & = & 18 \\ -4y & = & 8 \\ y & = & -2 \end{array}$$

The solution to the system of equations is $(5,-2)$.

Let's summarize the steps for the addition method, which you've seen in the above examples.

The Addition Method

To solve a system of two linear equations by the addition method,

1) Multiply each equation by something as needed in order to make the coefficients of either x or y the same but opposite in sign.

2) Add the two equations and solve the resulting equation for whichever unknown remains.

3) Substitute that value into either original equation and solve for the other unknown.

The Substitution Method

We will now describe the substitution method, then give an example of how it works.

The Substitution Method

To solve a system of two linear equations by the substitution method,

1) Pick one of the equations in which the coefficient of one of the unknowns is either one or negative one. Solve that equation for that unknown.

2) Substitute the expression for that unknown into *the other* equation and solve for the unknown.

3) Substitute that value into the equation from (1), or into either original equation, and solve for the other unknown.

◇ **Example 7.1(d):** Solve the system of equations $\begin{aligned} x - 3y &= 6 \\ -2x + 5y &= -5 \end{aligned}$ using the substitution method.

Solution: Solving the first equation for x, we get $x = 3y + 6$. We now replace x in the second equation with $3y + 6$ and solve for y. Finally, that result for y can be substituted into $x = 3y + 6$ to find x:

$$\begin{aligned} -2(3y+6) + 5y &= -5 \\ -6y - 12 + 5y &= -5 \\ -y - 12 &= -5 \\ -y &= 7 \\ y &= -7 \end{aligned}$$

$$\begin{aligned} x - 3(-7) &= 6 \\ x + 21 &= 6 \\ x &= -15 \end{aligned}$$

The solution to the system of equations is $(-15, -7)$.

Section 7.1 Exercises To Solutions

1. Solve each of the following systems by the addition method.

(a) $\begin{aligned} 7x-6y &= 13 \\ 6x-5y &= 11 \end{aligned}$ (b) $\begin{aligned} 5x+3y &= 7 \\ 3x-5y &= -23 \end{aligned}$ (c) $\begin{aligned} 5x-3y &= -11 \\ 7x+6y &= -12 \end{aligned}$

2. Solve each of the following systems by the substitution method.

(a) $\begin{aligned} x-3y &= 6 \\ -2x+5y &= -5 \end{aligned}$ (b) $\begin{aligned} 2x-3y &= -6 \\ -3x+y &= -5 \end{aligned}$ (c) $\begin{aligned} 4x-y &= 9 \\ 2x+3y &= -27 \end{aligned}$

3. Consider the system of equations $\begin{aligned} 2x-3y &= 4 \\ 4x+5y &= 3 \end{aligned}$.

   (a) Solve for x by using the addition method to eliminate y. Your answer should be a fraction.

   (b) Ordinarily you would substitute your answer to (a) into either equation to find the other unknown. However, dealing with the fraction that you got for part (a) could be difficult and annoying. Instead, use the addition method again, but eliminate x to find y.

4. Consider the system of equations $\begin{aligned} \tfrac{1}{2}x-\tfrac{1}{3}y &= 2 \\ \tfrac{1}{4}x+\tfrac{2}{3}y &= 6 \end{aligned}$. The steps below indicate how to solve such a system of equations.

   (a) Multiply both sides of the first equation by the least common denominator to "kill off" all fractions.

   (b) Repeat for the second equation.

   (c) You now have a new system of equations without fractional coefficients. Solve that system by the addition method.

5. Attempt to solve the following two systems of equations. With each, something will go wrong at some point. This illustrates that there is a little more to this subject than we have seen so far! We will find out soon what is going on here.

(a) $\begin{aligned} 2x-5y &= 3 \\ -4x+10y &= 1 \end{aligned}$ (b) $\begin{aligned} 2x-5y &= 3 \\ -4x+10y &= -6 \end{aligned}$

6. Solve each system of equations by the addition method.

(a) $\begin{aligned} 3x-5y &= -2 \\ 2x-3y &= 1 \end{aligned}$ (b) $\begin{aligned} 3x-5y &= 11 \\ 2x-6y &= 2 \end{aligned}$ (c) $\begin{aligned} 7x-6y &= 13 \\ 6x-5y &= 11 \end{aligned}$

7. The length of a rectangle is three more than twice the width. The area is 44. Find the length and width. **Write an equation that can be used to solve this problem, and solve the equation.**

8. The sales tax in a particular city is 6.5%. You pay $41.37 for an electric can opener, including tax. What was the price of the can opener? **Write an equation that can be used to solve this problem, and solve the equation.**

9. Give the slope of each line whose equation is given.

   (a) $y = -\frac{4}{7}x + 2$ (b) $x = 2$ (c) $3x + 5y = 2$ (d) $y = -1$

10. The equation $F = \frac{9}{5}C + 32$ gives the Fahrenheit temperature F corresponding to a given Celsius temperature C. This equation describes a line, with C playing the role of x and F playing the role of y.

    (a) What is the F-intercept of the line, and what does it tell us?

    (b) What is the slope of the line, and what does it tell us?

7.2 More on Systems of Linear Equations

> 7. (c) Recognize when a system of two linear equations has no solution or infinitely many solutions.
>   (d) Solve a system of two linear equations by graphing.

Let's begin with two examples:

◇ **Example 7.2(a):** Solve the system $\begin{aligned} 2x - 5y &= 3 \\ -4x + 10y &= 1 \end{aligned}$.

Solution:

$$\begin{array}{rcl} 2x - \;\; 5y & = & 3 \\ -4x + 10y & = & 1 \end{array} \quad \begin{array}{c} \overset{\text{times } 2}{\Longrightarrow} \\ \Longrightarrow \end{array} \quad \begin{array}{rcl} 4x - 10y & = & 6 \\ -4x + 10y & = & 1 \\ \hline 0 & = & 7 \end{array}$$

We are not able to find a solution as we did before.

The next example is just like the last, except that we will change the right side of the second equation to -6.

◇ **Example 7.2(b):** Solve the system $\begin{aligned} 2x - 5y &= 3 \\ -4x + 10y &= -6 \end{aligned}$.

Solution:

$$\begin{array}{rcl} 2x - \;\; 5y & = & 3 \\ -4x + 10y & = & -6 \end{array} \quad \begin{array}{c} \overset{\text{times } 2}{\Longrightarrow} \\ \Longrightarrow \end{array} \quad \begin{array}{rcl} 4x - 10y & = & 6 \\ -4x + 10y & = & -6 \\ \hline 0 & = & 0 \end{array}$$

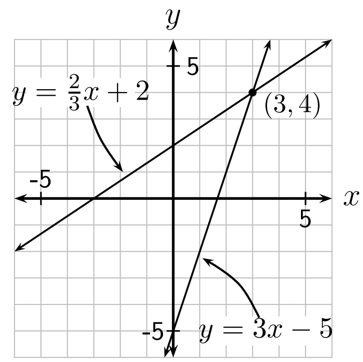
We are still not able to find a solution.

You might recognize these as being the systems in Exercise 5 of the previous section, and clearly something has "gone wrong" here. Before going into the above examples further, lets consider the system of equations $\begin{aligned} 2x - 3y &= -6 \\ 3x - y &= 5 \end{aligned}$ from Exercise 1(b) of the previous section, which you should have found to have the solution $(3, 4)$. If we solve each of those equations for y we get the equations

$$y = \tfrac{2}{3}x + 2 \qquad \text{and} \qquad y = 3x - 5$$

If we graph the two equations on the same graph, we get the result shown at the top of the next page.

Note that the two lines cross at the point $(3,4)$, which is the solution to the system. This is because every point on a line is a solution to the equation of the line, so the point where the two lines cross is a solution to both equations!



Now consider the system $\begin{aligned} 2x - 5y &= 3 \\ -4x + 10y &= 1 \end{aligned}$ from Example 7.2(a) of the previous section. When we tried using the addition method to solve this, we ended up with $0 = 7$, which is clearly not true! This is trying to tell us something; let's see what it is. If we solve each of the two equations for y we get

$$y = \tfrac{2}{5}x - \tfrac{3}{5} \qquad \text{and} \qquad y = \tfrac{2}{5}x + \tfrac{1}{10}$$

It would be fairly hard to graph these two equations, but we can see from their equations that they have the same slope, but different y-intercepts. This means that *they are parallel lines*, so they don't intersect. Therefore they have no point in common, so the system has no solution.

In the case of the other system from Exercise 5, $\begin{aligned} 2x - 5y &= 3 \\ -4x + 10y &= -6 \end{aligned}$, when trying to solve the equation we arrived at $0 = 0$. This is also trying to tell us something, but it is not the same thing as before. This time, if we solve each of the two equations for y we get

$$y = \tfrac{2}{5}x - \tfrac{3}{5} \qquad \text{and} \qquad y = \tfrac{2}{5}x - \tfrac{3}{5}$$

In this case the two equations are for the same line, so every point on that line is a solution to both equations. Thus there are infinitely many solutions.

In summary, think about taking two infinitely long pieces of dry spaghetti and throwing them randomly on the ground. They could land in any one of three ways: they could cross each other in one place, they could be parallel, or they could land exactly on top of each other. These illustrate the three possibilities when solving a system of two linear equations: the system can have one solution, no solution, or infinitely many solutions. To find out which, we simply try solving the system with the addition method or the substitution method. If everything goes smoothly you will arrive at one solution. If something funny happens, it means there is either no solution or infinitely many solutions. Let's summarize this, along with the graphical interpretation of each.

Systems of Two Linear Equations

- If $a = b$ is obtained when attempting to solve a system of equations, where a and b are different numbers, then the system has no solution. The graphs of the two equations are parallel lines.
- If $a = a$ is obtained, for some number a, when attempting to solve a system of equations, then the system has infinitely many solutions. The graphs of the two equations are the same line.
- If a system of two linear equations has a solution, it is the point where the graphs of the two equations intersect.

Section 7.2 Exercises

To Solutions

1. *Use the addition method* to solve each of the following systems of equations if possible. If it is not possible, tell whether the system has no solution or infinitely many solutions.

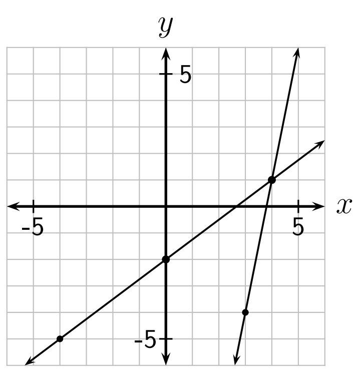
(a) $\begin{aligned} -4x + 6y &= 5 \\ 6x - 9y &= 7 \end{aligned}$ (b) $\begin{aligned} 3x - 4y &= 18 \\ 6x + 4y &= 0 \end{aligned}$ (c) $\begin{aligned} 2x - 3y &= 5 \\ -4x + 6y &= 8 \end{aligned}$

2. *Use the substitution method* to solve each of the following systems of equations if possible. If it is not possible, tell whether the system has no solution or infinitely many solutions.

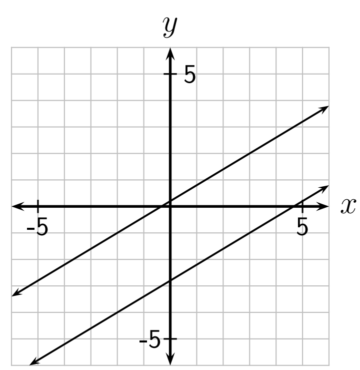
(a) $\begin{aligned} 10x - 2y &= 14 \\ 5x - y &= 7 \end{aligned}$ (b) $\begin{aligned} x + 3y &= 5 \\ 4x + 12y &= 20 \end{aligned}$ (c) $\begin{aligned} 3x + 2y &= 3 \\ x - 5y &= -16 \end{aligned}$

3. Each graph below shows the graphs of two linear equations. For each one, give the solution to the system if there is one. If there is no solution or if there are infinitely many solutions, say so.

(a)



(b)



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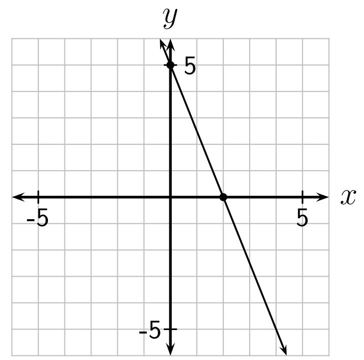

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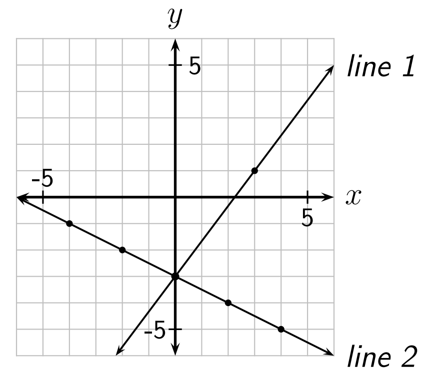


4. Find the intercepts for each of the following equations.

   (a) $x^2 + y = 1$ (b) $x - y^2 = 1$ (c) $y = x^3$

5. (a) Give the slope of a line that is parallel to *line 1* from Exercise 3(d).

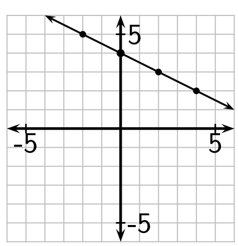
   (b) Give the slope of a line that is perpendicular to *line 2* from Exercise 3(d).

6. Sketch the graph of each line.

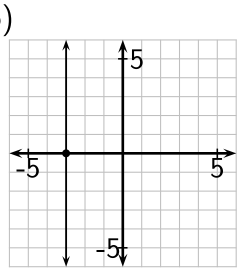
   (a) $y = -3x + 2$ (b) $-x + y = 4$ (c) $y = 2$

7. Give the equation of each line.

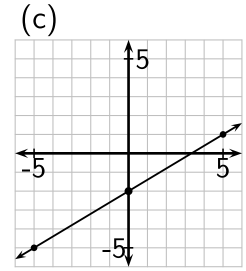
   (a)



   (b)



   (c)



8. Find the equation of the line through the two points **algebraically**.

   (a) $(1, 7), (3, 11)$ (b) $(-6, -2),\ (5, -3)$ (c) $(-2, -5),\ (2, 5)$

9. We again consider the manufacture of Widgets by the Acme Company. The costs for one week of producing Widgets is given by the equation $C = 7x + 5000$, where C is the costs, in dollars, and x is the number of Widgets produced in a week. This equation is clearly linear.

   (a) What is the C-intercept of the line, and what does it represent?

   (b) What is the slope of the line, and what does it represent?

   (c) If they make 1,491 Widgets in one week, what is their total cost? What is the cost for each individual Widget made that week? (The answer to this second question should *NOT* be the same as your answer to (b).)

A Solutions to Exercises

A.7 Chapter 7 Solutions

Section 7.1 Solutions Back to 7.1 Exercises

1. (a) $(1,-1)$ (b) $(-1,4)$ (c) $(-2,\frac{1}{3})$

2. (a) $(-15,-7)$ (b) $(3,4)$ (c) $(0,-9)$

3. $(\frac{29}{22},-\frac{5}{11})$ 4. $(8,6)$ 6. (a) $(11,7)$ (b) $(7,2)$ (c) $(1,-1)$

7. **Equation:** $w(2w+3)=44$ **Answer:** $w=4$ ($-\frac{11}{2}$ is another solution, but w cannot be negative)

8. **Equation:** $x+0.065x=41.37$ **Answer:** $x=\$38.85$

9. Give the slope of each line whose equation is given.

   (a) $m=-\frac{4}{7}$ (b) undefined (c) $m=-\frac{3}{5}$ (d) $m=0$

10. (a) The F-intercept is 32°, which is the Fahrenheit temperature when the Celsius temperature is 0 degrees.

    (b) The slope of the line is $\frac{9}{5}$ degrees Fahrenheit per degree Celsius; it tells us that each degree Celsius is equivalent to $\frac{9}{5}$ degrees Fahrenheit.

Section 7.2 Solutions Back to 7.2 Exercises

1. (a) no solution (b) $(2,-3)$ (c) no solution

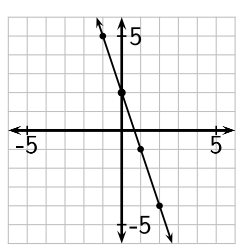
2. (a) infinitely many solutions (b) infinitely many solutions (c) $(-1,3)$

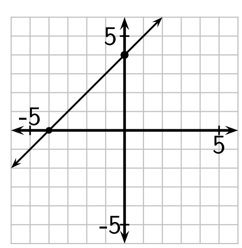
3. (a) $(4,1)$ (b) no solution (c) infinitely many solutions (d) $(0,-3)$

4. (a) x-intercepts: $1,-1$ y-intercept: 1 (b) x-intercept: 1 y-intercepts: none

   (c) x-intercept: 0 y-intercept: 0

5. (a) $m=\frac{4}{3}$ (b) $m=2$

6. (a)


   (b)


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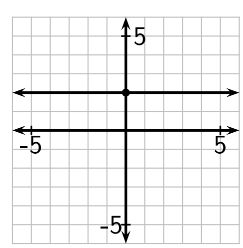

7. (a) $y=-\frac{1}{2}x+4$ (b) $x=-3$ (c) $3x-5y=10$

8. (a) $y = 2x + 5$ (b) $y = -\frac{1}{11}x - \frac{28}{11}$ (c) $y = \frac{5}{2}x$

9. (a) The C-intercept is $5000, which represents the costs when no Widgets are produced.
   (b) The slope is $7 per Widget, and represents the increase in costs for each additional Widget produced. 1,491 widgets cost $15,437 to produce or $\approx \$10.35$ per widget.