Intermediate Algebra

Gregg Waterman
Oregon Institute of Technology

Contents

8 Functions

8.1 Introduction To Functions

8. (a) Evaluate a function for a given numerical or algebraic value.
   (b) Find all numerical ("input") values for which a function takes a certain ("output") value.

◇ **Example 8.1(a):** Find some solutions to $y = \frac{2}{3}x - 5$ and sketch the graph of the equation.

Solution: Let x be some multiples of 3:

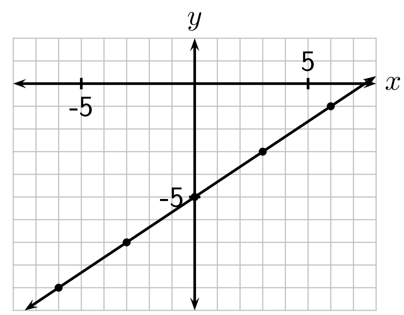
$x = 0: \quad y = \frac{2}{3}(0) - 5 = -5$

$x = 3: \quad y = \frac{2}{3}(3) - 5 = -3$

$x = 6: \quad y = \frac{2}{3}(6) - 5 = -1$

$x = -3: \quad y = \frac{2}{3}(-3) - 5 = -7$

$x = -6: \quad y = \frac{2}{3}(-6) - 5 = -9$



◇ **Example 8.1(b):** Find some solutions to $y = x^2 - 3x + 1$ and sketch the graph of the equation.

$x = 0: \quad y = 0^2 - 3(0) + 1 = 1$

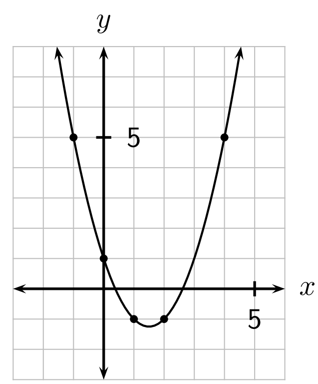
$x = 1: \quad y = 1^2 - 3(1) + 1 = -1$

$x = 2: \quad y = 2^2 - 3(2) + 1 = -1$

$x = 3: \quad y = 3^2 - 3(3) + 1 = 1$

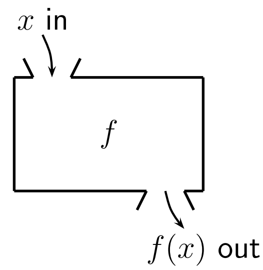
$x = 4: \quad y = 4^2 - 3(4) + 1 = 5$

$x = -1: \quad y = (-1)^2 - 3(-1) + 1 = 5$



Notice that in both of the above examples we chose values for x, put them into the equations, and 'got out' values for y. A mathematical machine that takes in an 'input' and gives us an 'output" is called a function; the equations $y = \frac{2}{3}x - 5$ and $y = x^2 - 3x + 1$ are functions.

There is an alternative notation for functions that is very efficient once a person is used to it. Suppose we name the function $y = \frac{2}{3}x - 5$ with the letter f. We then think of f as a sort of 'machine' that we feed numbers into and get numbers out of. We denote by $f(3)$ the value that comes out when 3 is fed into the machine, so $f(3) = -3$. This process can be shown pictorially in the manner seen to the right. f is the name of the machine, with input x and output $f(x)$. The notation $f(x)$ represents the result when 3 is put into the machine. In general, $f(x) = \frac{2}{3}x - 5$ for any value of x.



◇ **Example 8.1(c):** For $f(x) = \frac{2}{3}x - 5$, find $f(-6)$ and $f(1)$.

Solution: When $x = -6$, $\frac{2}{3}(-6) - 5 = -4 - 5 = -9$, so $f(-6) = -9$. We can think of replacing every x in $f(x) = \frac{2}{3}x - 5$ with -6 to get

$$f(-6) = \tfrac{2}{3}(-6) - 5 = -4 - 5 = -9.$$

Similarly,

$$f(1) = \tfrac{2}{3}(1) - 5 = \tfrac{2}{3} - \tfrac{15}{3} = -\tfrac{13}{3}.$$

◇ **Example 8.1(d):** Let $g(x) = x^2 - 3x + 1$, and find $g(4)$ and $g(-1)$.

$$g(4) = (4)^2 - 3(4) + 1 = 16 - 12 + 1 = 5$$
$$g(-1) = (-1)^2 - 3(-1) + 1 = 1 = 3 = 1 = 5$$

What we did in the previous two examples is find values of the function for given *numerical* inputs. We can also find values of the function for inputs that contain unknown values. This is shown in the following two examples.

◇ **Example 8.1(e):** For $g(x) = x^2 - 3x + 1$, find $g(a)$.

Solution: To find $g(a)$ we simply replace x with a:

$$g(a) = (a)^2 - 3(a) + 1 = a^2 - 3a + 1$$

◇ **Example 8.1(f):** For $g(x) = x^2 - 3x + 1$, find and simplify $g(a - 2)$.

Solution: To find $g(a-2)$ we replace x with $a - 2$. To simplify, we then 'FOIL' $(a-2)^2$, distribute and combine like terms:

$$\begin{aligned} g(a-2) &= (a-2)^2 - 3(a-2) + 1 \\ &= (a-2)(a-2) - 3a + 6 + 1 \\ &= a^2 - 4a + 4 - 3a + 7 \\ &= a^2 - 7a + 11 \end{aligned}$$

Therefore $g(a-2) = a^2 - 7a + 11$.

Here is an important fact: *We will usually use x for the "input" value of a function, but there is no reason that we couldn't use some other letter instead.* For example, the function g could be written as

$$g(x) = 2x^2 + x - 5 \qquad \text{or} \qquad g(t) = 2t^2 + t - 5 \qquad \text{or} \qquad g(a) = 2a^2 + a - 5$$

The letter used has no impact on how we compute the output for a given input.

So far we have found outputs of functions for given inputs. Sometimes we would like to find an input that gives a particular output. For example, we might wish to find a value of x that gives an output of 7 for the function $f(x) = 3x - 8$; that is, we wish to find a value of x such that $f(x) = 7$. To do this we simply replace $f(x)$ with 7 and solve:

$$\begin{aligned} 7 &= 3x - 8 \\ 15 &= 3x \\ 5 &= x \end{aligned}$$

Thus $f(5) = 7$, and we have found the desired value of x.

◇ **Example 8.1(g):** Let $f(x) = 2x^2 - x$. Find all values of x for which $f(x) = 3$.

Solution: Note that $\underline{x}$ is not three, $\underline{f(x)}$ is! (One clue that we shouldn't put 3 in for x is that we are asked to find x.) We replace $f(x)$ with 3 and find x:

$$\begin{aligned} 3 &= 2x^2 - x \\ 0 &= 2x^2 - x - 3 \\ 0 &= (2x - 3)(x + 1) \end{aligned}$$

$$x = -1 \quad \text{or} \quad \begin{aligned} 2x - 3 &= 0 \\ 2x &= 3 \\ x &= \tfrac{3}{2} \end{aligned}$$

$f(x) = 3$ when $x = -1$ or $x = \frac{3}{2}$.

Section 8.1 Exercises To Solutions

1. For $f(x) = x^2 - 3x$, find each of the following, writing each of your answers in the form $f(5) = number$.

   (a) $f(5)$ (b) $f(-1)$ (c) $f(-3)$ (d) $f(0)$

2. Still letting $f(x) = x^2 - 3x$, find $f(\frac{1}{2})$, giving your answer in fraction form.

3. We don't always use the letter f to name a function; sometimes we use g, h or another letter. Consider the function $g(x) = \sqrt{3 - x}$.

   (a) Find $g(2)$ (b) Find $g(-6)$

   (c) Find $g(-5)$, giving your answer in simplified square root form.

(d) Find $g(-2)$, using your calculator. Your answer will be in decimal form and you will have to round it somewhere - round to the hundredth's place. This is two places past the decimal - the first place past the decimal is the tenth's, the next place is the hundredth's, the next is the thousandth's, and so on.

4. Let the function h be given by $h(x) = \sqrt{25 - x^2}$.

   (a) Find $h(-4)$.

   (b) Find $h(2)$, giving your answer in decimal form rounded to the thousandth's place.

   (c) Why can't we find $h(7)$?

   (d) For what values of x *CAN* we find $h(x)$? Be sure to consider negative values as well as positive.

5. Let $f(x) = x^2 - 3x$.

   (a) Find $f(s)$. (b) Find and simplify $f(s+1)$.
   (c) Find all values of x for which $f(x) = 10$.

6. For the function $h(x) = \frac{2}{3}x - 5$, find and simplify

   (a) $h(a-3)$ (b) $h(a+2)$

7. Still for $h(x) = \frac{2}{3}x - 5$, find all values of x for which $h(x) = 2$. Give your answer(s) in exact form - no decimals!

8. For the function $g(x) = \sqrt{3-x}$, find and simplify $g(s+1)$. You will not be able to get rid of the square root.

9. Consider again the function $h(x) = \sqrt{25 - x^2}$.

   (a) Find and simplify $h(a-2)$.

   (b) Find all values of x for which $h(x) = 2$. Give your answer(s) in exact form - no decimals!

10. Solve each of the following systems by the substitution method.

    (a) $\begin{aligned} x + y &= 3 \\ 2x + 3y &= -4 \end{aligned}$ (b) $\begin{aligned} 2x + y &= 13 \\ -5x + 3y &= 6 \end{aligned}$ (c) $\begin{aligned} 2x - 3y &= -6 \\ 3x - y &= 5 \end{aligned}$

11. Solve each of the systems from Exercise 10 using the addition method.

8.2 Compositions Of Functions

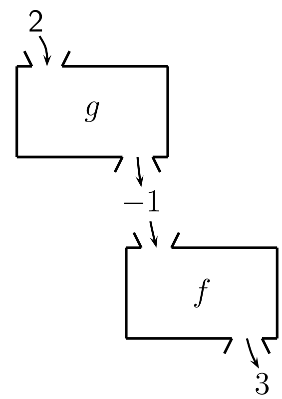
8. (c) Evaluate the composition of two functions for a given "input" value; determine a simplified composition function for two given functions.

Recall that we can think of a function as a "machine" that takes in a number, works with it, and outputs another number that depends on the input number. For example, when we input the number five to the function $f(x) = x^2 - 2x$ we get out

$$f(5) = 5^2 - 2(5) = 25 - 10 = 15.$$

Similarly, if we input $x = -1$ we get

$$f(-1) = (-1)^2 - 2(-1) = 1 + 2 = 3.$$



Now suppose that we have a second function $g(x) = x - 3$ and we wish to give a number to g as input, then use the output as an input for f. The picture you can think of is shown to the right. If we put two into g we get $g(2) = 2 - 3 = -1$ out; we then put that result into f to get three out, as demonstrated at the end of the previous paragraph.

For now, the notation that we will use for what we did is $f[g(2)]$. To think about this, start with the number inside the parentheses, then apply the function that is closest to the number; in this case it is g. Take the result of that computation and apply the function f to that number.

◇ **Example 8.2(a):** For the same two functions $f(x) = x^2 - 2x$ and $g(x) = x - 3$, find $g[f(4)]$.

Solution: First $f(4) = 4^2 - 2(4) = 16 - 8 = 8$. Now we apply g to the result of 8 to get $g(8) = 8 - 3 = 5$, so $g[f(4)] = 5$. We will often combine both of these computation into one as follows:

$$\begin{aligned} g[f(4)] &= g[4^2 - 2(4)] \\ &= g[16 - 8] \\ &= g(8) \\ &= 8 - 3 \\ &= 5 \end{aligned}$$

Think carefully about the above example. $f(4)$ becomes $4^2 - 2(4)$ because of how f is defined, and that simplifies to 8. Then g acts on 8, resulting in $8 - 3 = 5$.

◇ **Example 8.2(b):** Again using $f(x) = x^2 - 2x$ and $g(x) = x - 3$, find $f[g(4)]$.

$$\begin{aligned} f[g(4)] &= f[4-3] \\ &= f(1) \\ &= 1^2 - 2(1) \\ &= 1 - 2 \\ &= -1 \end{aligned}$$

Note that $f[g(4)] \neq g[f(4)]$; this shows that *when we apply two functions one after the other, the result depends on the order in which the functions are applied.*

In Section 8.1 we saw how to apply functions to algebraic quantities. With a little thought we can see that for $f(x) = x^2 - 2x$ and $g(x) = x - 3$,

$$\begin{aligned} f[g(x)] &= f[x-3] \\ &= (x-3)^2 - 2(x-3) \\ &= x^2 - 6x + 9 - 2x + 6 \\ &= x^2 - 8x + 15 \end{aligned}$$

This final expression $x^2 - 8x + 15$ can thought of as a function in its own right. It was built by putting f and g together in the way that we have been doing, with g acting first, then f. We call this function the **composition** of f and g, and we denote it by $f \circ g$. So

$$(f \circ g)(x) = x^2 - 8x + 15$$

◇ **Example 8.2(c):** For $g(x) = 2x - 3$ and $h(x) = x^2 - 5x + 2$, find $(g \circ h)(x)$.

$$\begin{aligned} (g \circ h)(x) &= g[h(x)] \\ &= g[x^2 - 5x + 2] \\ &= 2(x^2 - 5x + 2) - 3 \\ &= 2x^2 - 10x + 1 \end{aligned}$$

Therefore $(g \circ h)(x) = 2x^2 - 10x + 1$.

Section 8.2 Exercises

To Solutions

1. Consider again the functions $f(x) = x^2 - 2x$ and $g(x) = x - 3$.

   (a) Find $g[f(1)]$.

   (b) Find $g[f(-3)]$.

2. Once again consider the functions $f(x) = x^2 - 2x$ and $g(x) = x - 3$.

   (a) Use the same kind of process to find $f[g(1)]$, noting that **the order in which f and g are applied has been reversed**.

(b) Find $f[g(-3)]$.

3. For the functions $f(x) = x^2 - 2x$ and $g(x) = x - 3$ we found that $(f \circ g)(x) = x^2 - 8x + 15$. Find the value of $(f \circ g)(-3)$ and compare the result with the answer to Exercise 2(b).

4. Continue to let $f(x) = x^2 - 2x$ and $g(x) = x - 3$

   (a) Find the other composition function $(g \circ f)(x)$, noting that it is just $g[f(x)]$.

   (b) Find $(g \circ f)(1)$ and make sure it is the same as the answer to Exercise 1(a).

5. Consider the functions $f(x) = x^2 + 3x$ and $g(x) = x - 7$.

   (a) Find $f[g(2)]$.
   (b) Find $(g \circ f)(2)$.
   (c) Find $(g \circ f)(x)$.
   (d) Find $(f \circ g)(x)$.

6. (a) Give the equation of the horizontal line graphed in Exercise 8(a).

   (b) Give the equation of the vertical line graphed in Exercise 8(a).

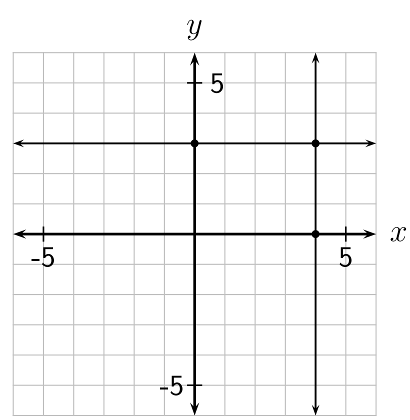
   (c) The lower of the two lines graphed in Exercise 8(b) passes through the point $(-6, 3)$. Give the equation of the line.

7. Solve each of the following systems of equations if possible. If it is not possible, tell whether the system has no solution or infinitely many solutions.

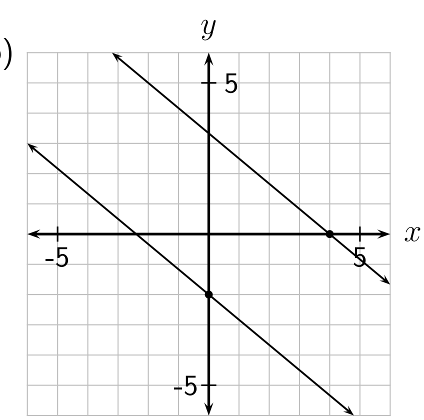
   (a) $\begin{aligned} 5x + y &= 12 \\ 7x - 2y &= 10 \end{aligned}$
   (b) $\begin{aligned} 3x - 9y &= 12 \\ -2x + 6y &= -8 \end{aligned}$
   (c) $\begin{aligned} x - 3y &= -2 \\ 5x - 15y &= 3 \end{aligned}$

8. Each graph below shows the graphs of two linear equations. For each one, give the solution to the system if there is one. If there is no solution or if there are infinitely many solutions, say so.

(a)



(b)



9. For the function $h(x) = 2x - 7$, find all values of x such that $h(x) = 12$.

10. Let $g(x) = x - \sqrt{3x+1}$. Find all values of x for which $g(x) = 3$.

11. Let $f(x) = 2x^2 - 2x$.

    (a) Find $f(-3)$.

    (b) Find all values of x for which $f(x) = 60$.

    (c) Find all values of x for which $f(x) = 1$. *You will need the quadratic formula for this.*

8.3 Sets of Numbers

8. (d) Describe sets of numbers using inequalities or interval notation.

Consider the function $h(x) = \sqrt{25 - x^2}$, from Exercise 4 of Section 8.1. In part (d) of that exercise you were to find the values of x for which the function can be evaluated. For example,

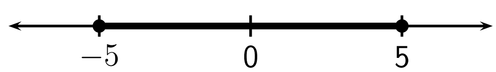
$$h(-4) = \sqrt{25 - 16} = \sqrt{9} = 3, \qquad h(2) = \sqrt{25 - 4} = \sqrt{21}, \qquad h(7) = \sqrt{25 - 49} = \sqrt{-24}$$

Clearly the function can be evaluated for $x = -4$, and for $x = 2$ as well; even though $\sqrt{21}$ is not a "nice" number, we could use our calculators to find an approximate decimal value for it. On the other hand, we cannot take the square root of a negative number, so h cannot be evaluated for $x = 7$.

After a bit of thought, one should realize that h can be evaluated for any number between -5 and 5, including both of those. Taken together, all of those numbers constitute something we call a **set** of numbers. Clearly there are infinitely many of them because we can use numbers that are not whole numbers, as long as whatever we get under the square root is positive. So far we have described the set of values that the function can be evaluated for verbally, using words: 'all the numbers between -5 and 5, including both of those.' We will now see a way top describe this set graphically, and two ways to describe the set symbolically.

First let us recall inequalities. When we write $a < b$ for two numbers a and b, it means that the number a is less than the number b. $b > a$ says the same thing, just turned around. $a \leq b$ means that a is less than, or possibly equal to, b. This, of course, is equivalent to $b \geq a$. Using this notation, we know that we want x to be less than or equal to 5, so we write $x \leq 5$. At the same time we want $x \geq -5$, which is equivalent to $-5 \leq x$. We can combine the two statements into one inequality: $-5 \leq x \leq 5$. This is one of our two ways to describe the set symbolically.

The statement $-5 \leq x \leq 5$ then describes a set of numbers, the numbers that are allowed for x when considering the function $h(x) = \sqrt{25 - x^2}$. This set can be described graphically by shading it in on a number line:



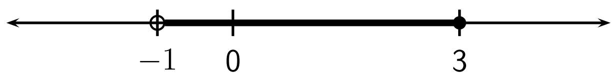
The solid dots at either end of the set indicate that those numbers are included. If we wanted to show all the numbers less than three, we would do it like this:

In this case the small circle at three indicates that three is not included, but every number up to three is.

Another way to describe a set symbolically is to use something called **interval notation**. The interval notation for the set $-5 \leq x \leq 5$ is formed by first writing the number at the smaller end of the set, then a comma, then the number at the larger end of the set: $-5, 5$ Then, if the number at the smaller end is to be included we put a square bracket to its left: $[-5, 5$. *If the* *number was not to be included we would use a parenthesis*. We then do the same thing for the larger number to get the final result $[-5, 5]$.

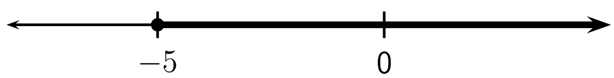
◇ **Example 8.3(a):** Give the interval notation for the set of all numbers greater than negative one and less than or equal to three.

Solution: For those of us who process visually, it might be helpful to graph this set on the number line:

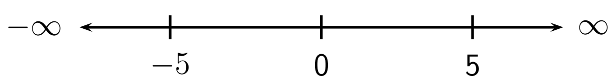


The two endpoints of our interval are -1 and 3, so we put them in that order, separated by a comma: $-1, 3$ Next, because the set is for numbers *greater* than -1, we enclose the -1 of our pair with a parenthesis: $(-1, 3$. Finally, because the number 3 *is* in the set, we put a bracket on the right side of our pair: $(-1, 3\,]$.

Consider the set with this graph:



The smallest number in this set is clearly -5, but there is no largest number in the set. To deal with this kind of thing we invent two symbols, ∞ and $-\infty$, **infinity** and **negative infinity**. These are not really numbers, but can be thought of as "places" beyond the two ends of the number line:



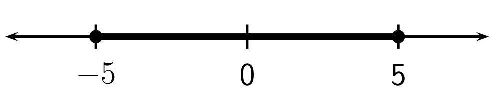
The symbols $-\infty$ and ∞ allow us to use interval notation to describe sets like the one shown above, all numbers greater than or equal to negative five. The interval notation for this set is $[-5, \infty)$. *We always use a parenthesis with either of the infinities, since they are not numbers that can be included in a set.*

◇ **Example 8.3(b):** Give the interval notation for the set of all numbers less than four.

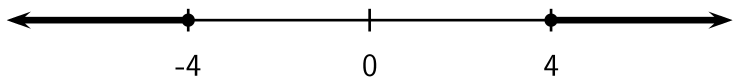
Solution: The graph of this set on the number line is

The interval notation for this set is $(-\infty, 4)$.

Returning again to our function $h(x) = \sqrt{25 - x^2}$, the set of numbers that x is allowed to be can be shown graphically by



If instead we consider the function $f(x) = \sqrt{x^2 - 16}$, a little thought would show us that the set of numbers for which f can be computed is the following:



The right hand part of this graph can be described using interval notation as $[4, \infty)$, and the left hand side as $(-\infty, -4]$. The whole set, consisting of the two parts together, is then described by the notation $(-\infty, -4] \cup [4, \infty)$. The symbol $\cup$ is "union," which means the two sets put together.

Section 8.3 Exercises To Solutions

1. For each part below, a set is described in words. Give the same set using inequalities. Sometimes you will need two inequalities combined, like you just saw, sometimes one will be enough. Then give the same set with a graph of a number line.

   (a) All numbers x less than ten.

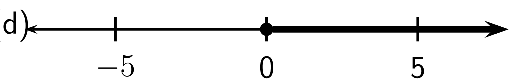
   (b) All numbers t greater than or equal to zero.

   (c) All numbers x between -3 and 3, not including either.

   (d) All numbers y greater than -2 and less than or equal to 4.

2. A set can be described with (1) words, (2) inequalities, (3) a graph on a number line, (4) interval notation. For each of the following, a set of numbers is described in one of these four ways. Describe the set in each of the other three ways. When using words, make it clear whether "boundary" numbers are included. When using interval notation and more than one interval is needed, use union, $\cup$.

   (a) All numbers greater than five. (b) $(-\infty, 3]$

   (c) $x < -4$ or $x \geq 1$ (d) 

   (e) $(-\infty, -7) \cup [1, \infty)$ (f) $x < 17$

   (g) All numbers greater than -10 and less than or equal to -3.

3. Find the intercepts for each of the following equations algebraically.

   (a) $4x - 5y = 20$ (b) $x^2 + y^2 = 25$ (c) $x = \sqrt{1 - y}$

4. Give the slope of a line that is perpendicular to the line with equation $5x - 2y = 7$.

5. Find the equation of the line through the two points **algebraically**.

   (a) $(-8, -7)$, $(-4, -6)$ (b) $(2, -1)$, $(6, -4)$ (c) $(-2, 4)$, $(1, 4)$

6. Solve each of the following systems of equations if possible. If it is not possible, tell whether the system has no solution or infinitely many solutions.

(a) $\begin{aligned} 4x - 5y &= 20 \\ -3x + 6y &= -15 \end{aligned}$ (b) $\begin{aligned} 3x - 9y &= -3 \\ 4x - 12y &= -4 \end{aligned}$ (c) $\begin{aligned} -6x + 3y &= -1 \\ 10x - 5y &= 2 \end{aligned}$

7. For $g(x) = 2x - 3$ and $h(x) = x^2 - 5x + 2$, find $(h \circ g)(x)$.

8. Consider the functions $f(x) = 3 - x^2$ and $g(x) = 2x + 1$.

(a) Find $f[g(-1)]$. (b) Find $(g \circ f)(4)$.

(c) Find $(g \circ f)(x)$. (d) Find $(f \circ g)(x)$.

8.4 Domains of Functions

8. (e) Determine the domains of rational and radical functions.

In the previous section we discussed at some length the fact that the only values for which we can compute the function $h(x) = \sqrt{25 - x^2}$ are the numbers in the interval $[-5, 5]$. The set of numbers for which a function can be computed is called the **domain** of the function. When finding the domain of a function we need to make sure that

1. We don't try to compute the square root of a negative number.
2. We don't have zero in the denominator (bottom) of a fraction.

◇ **Example 8.4(a):** Find the domain of $f(x) = \dfrac{x-1}{x+4}$.

If $x = -4$ we would get zero in the denominator, which can't be allowed. The domain is all real numbers *except* -4. We usually just say the domain is $x \neq -4$; because the only number we say x can't be is -4, this implies that x can be any other number.

◇ **Example 8.4(b):** Find the domain of $g(x) = \sqrt{3 - x}$.

Solution: Here there is a problem if x is greater than three, so we must require that x is less than or equal to three. The domain is then $x \leq 3$ or, using interval notation, $(-\infty, 3]$.

◇ **Example 8.4(c):** Find the domain of $h(x) = \dfrac{1}{\sqrt{3-x}}$.

Solution: There is no problem computing $\sqrt{3-x}$ as long as x is less than or equal to three. However, we can't allow x to be three, because that would cause a zero in the denominator. The domain is therefore $(-\infty, 3)$.

Section 8.4 Exercises To Solutions

1. Give the domain of each of the following functions.

(a) $y = \dfrac{1}{x+4}$ (b) $g(x) = x^2 - 5x + 6$ (c) $h(x) = \dfrac{1}{x^2 - 5x + 6}$

(d) $y = \sqrt{x+5}$ (e) $f(t) = \dfrac{1}{\sqrt{t-3}}$ (f) $g(x) = \dfrac{1}{x^2 - 16}$

2. Let $h(x) = x^2 - 2x - 3$.

   (a) Find all values of x such that $h(x) = 5$.

   (b) Find all values of x such that $h(x) = -3$

   (c) Find $h(-3)$.

   (d) Find all values of x such that $h(x) = 2$. Give your answers in both in decimal form to the nearest hundredth and simplified exact form.

8.5 Graphs of Functions

> 8. (f) Graph quadratic, polynomial and simple root and rational functions.

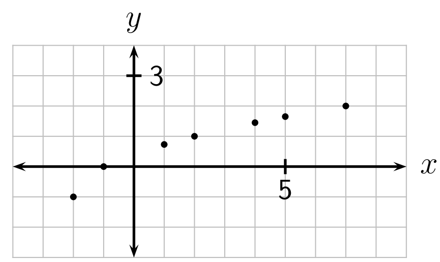
Look back at Examples 8.1(a) and 8.1(b). There we found some solutions to $y = \frac{2}{3}x - 5$ and $y = x^2 - 3x + 1$, then graphed two equations. Later we saw how to think of these two equations a little differently, as functions $f(x) = \frac{2}{3}x - 5$ and $g(x) = x^2 - 3x + 1$. This shows us that functions have graphs; for a function f we simply evaluate the function for a number of x values, and plot the corresponding $f(x)$ values as y's corresponding to those x's.

◇ **Example 8.5(a):** Graph the function $h(x) = \sqrt{x+2} - 1$.

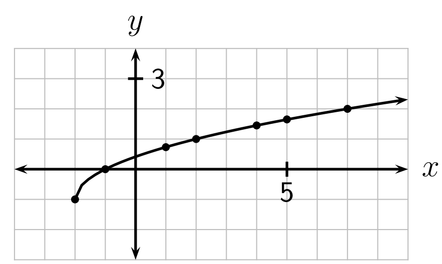
Solution: First we note the domain of this function. x *can* be negative, like $x = -1$, but it can't be less than -2 or we'll get a negative under the square root. Therefore the domain is $x \geq 2$, or $[-2, \infty)$. We can let x take a few values for which it is easy to calculate the square root, as shown in the table below and to the left. We can even use a calculator to evaluate $h(x)$ for other values of x, shown in the center below.

x	$h(x)$
-2	-1
-1	0
2	1
7	2

x	$h(x)$
1	0.4
4	1.45
5	1.65



The graph above and to the right shows the points found and recorded in the tables. We now need to connect them to form the graph of the function. Before doing so, lets return again to the domain of the function, $[-2, \infty)$. This tells us that the smallest value of x that we can have is -2, so the graph should not extend to the left of -2. The domain also tells us that x can be as large as we want, so the graph extends forever in the positive x direction. The final graph is shown to the right.

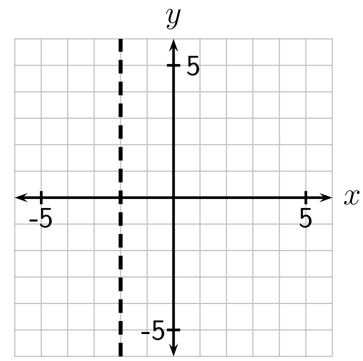


Let's reiterate: When graphing something like $h(x) = \sqrt{x+2} - 1$, the process is exactly the same as graphing $y = \sqrt{x+2} - 1$. The only difference is that the y values now represent the function values $h(x)$.

Now we'll graph a kind of function that you didn't see until this chapter, called a **rational function**.

◇ **Example 8.5(b):** Graph the function $f(x) = \dfrac{4}{x+2}$.

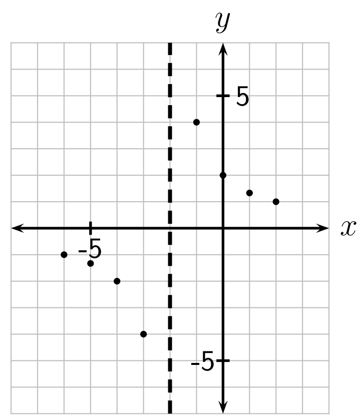
Solution: The domain of the function is $x \neq -2$, so the graph cannot have any points with x-coordinates of $x = -2$. All such points are indicated by the vertical dashed line shown on the grid to the right. Even though those points are not on the graph of the function, it is customary to show them as a vertical dashed line. Such a line is called a **vertical asymptote**. The asymptote causes the graph to appear as two pieces, one to the left of the asymptote and one to the right, with neither piece crossing the asymptote.

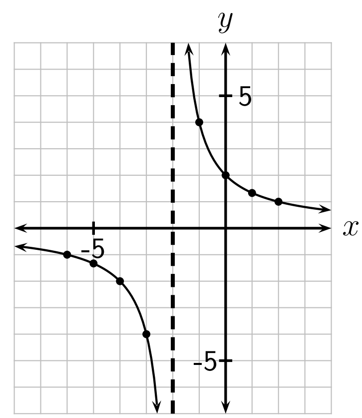


To find what the two pieces of the graph look like, we need to find some input-output pairs for the function. The first x values chosen should be on either side of the value that x is not allowed to have; in this case we begin with $x = -3$ and $x = -1$, as shown in the first table to the left below. I've included a DNE for the value of $f(x)$ when $x = -2$ to give our table a central point to work out from. The next table to the right shows some additional x values and their corresponding function values $f(x)$.

x	$f(x)$
-3	-4
-2	DNE
-1	4

x	$f(x)$
-6	-1
-5	$-\frac{4}{3}$
-4	-2
-3	-4
-2	DNE
-1	4
0	2
1	$\frac{4}{3}$
2	1





Solution: The leftmost of the two graphs above shows the points we found, plotted. We can see that they arrange themselves into two groups, one on either side of the asymptote. The graph above and to the right shows how each set of dots is connected to get the two parts of the graph.

Section 8.5 Exercises

To Solutions

1. Graph each of the following functions. See this example.

(a) $y = \dfrac{4}{x+4}$ (b) $g(x) = x^2 - 5x + 6$ (c) $h(x) = \dfrac{6}{2-x}$

(d) $y = \sqrt{x+5}$ (e) $f(x) = \dfrac{8}{x^2}$ (f) $h(x) = \sqrt{9-x^2}$

2. Let $h(x) = x^2 - 2x - 3$ and let $f(x) = 3x - 1$. Find $(f \circ h)(x)$ and $(h \circ f)(x)$.

3. Give the domain of each of the following functions.

(a) $h(x) = \sqrt{x^2 - 9}$ (b) $y = \sqrt{9 - x^2}$ (c) $f(x) = \dfrac{1}{\sqrt{9 - x^2}}$

8.6 Quadratic Functions

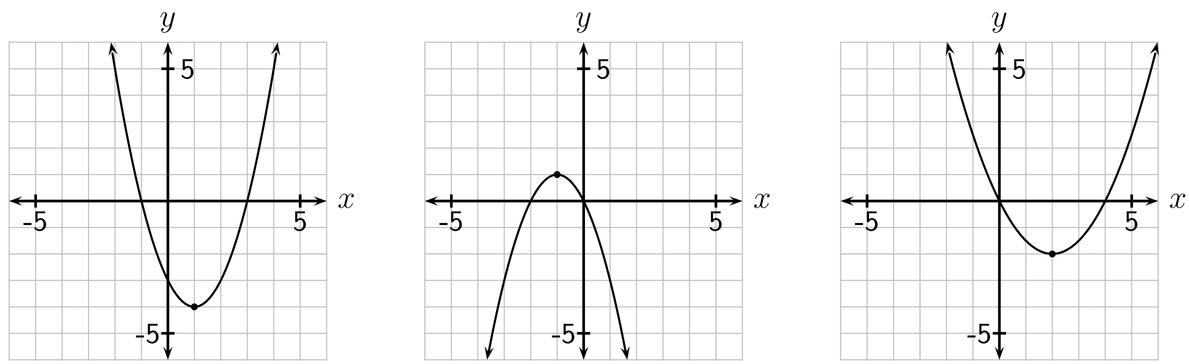
> 8. (g) Determine whether the graph of a quadratic function is a parabola opening up or down, and find the coordinates of the vertex of the parabola.

Quadratic functions are functions of the form

$$f(x) = ax^2 + bx + c \ ,$$

where a, b, and c are real numbers and $a \neq 0$. You have seen a number of these functions already, of course, including some in 'real world' situations. We will refer to the value a, that x^2 is multiplied by, as the **lead coefficient**, and c will be called the **constant term**. From previous work you should know that the graph of a quadratic function is a U-shape that we call a **parabola**. In this section we will investigate the graphs of quadratic functions a bit more.

Let's begin by looking at some graphs. Below are the graphs of three quadratic functions, with the equation for each below its graph.



$f(x) = x^2 - 2x - 3$ $\qquad$ $g(x) = -x^2 - 2x$ $\qquad$ $h(x) = \frac{1}{2}x^2 - 2x$

The graphs and corresponding equations illustrate the following:

> The effect of a on the graph of $f(x) = ax^2 + bx + c$ is as follows: If a is positive, the graph of the function is a parabola that "opens upward." If a is negative, the graph of the function is a parabola that opens downward.

The **vertex** of a parabola is the point at the bottom or top of the parabola, depending on whether it opens upward or downward. For the three parabolas graphed above, the coordinates of the vertices (plural of vertex) are $(1, -4)$, $(-1, 1)$ and $(2, -2)$, respectively. *The vertex of a parabola is the most important point on its graph.* In fact, if we can find the vertex of a parabola and we know whether it opens upward or downward, we have a pretty good idea what the graph of the parabola looks like. The following will help us find the coordinates of the vertex of a parabola:

Vertex of a Parabola

The x-coordinate of the vertex of any parabola is always given by $x = \dfrac{-b}{2a}$.

The y-coordinate is then found by substituting this value of x into the equation of the parabola.

◇ **Example 8.6(b):** Find the vertices of $f(x) = -3x^2+6x-10$ and $g(x) = \frac{1}{2}x^2-2x+5$.

Solution: The x-coordinate of the vertex of $f(x)$ is given by $x = \dfrac{-6}{2(-3)} = \dfrac{-6}{-6} = 1$. The y-coordinate is found by substituting this value in for x:

$$f(1) = -3(1)^2 + 6(1) - 10 = -3 + 6 - 10 = -7$$

The vertex of $f(x)$ is $(1, -7)$.

Solution: The x-coordinate of the vertex of $g(x)$ is $x = \dfrac{-(-2)}{2(\frac{1}{2})} = \dfrac{2}{1} = 2$. The y-coordinate is

$$g(2) = \frac{1}{2}(2)^2 - 2(2) + 5 = 2 - 4 + 5 = 3$$

The vertex of $g(x)$ is $(2, 3)$.

Section 8.6 Exercises

To Solutions

1. Use the above method to find the coordinates of the vertices for each of the parabolas whose graphs are shown on the previous page. Make sure your answers agree with what you see on the graphs!

2. For each of the following quadratic equations, first determine whether its graph will be a parabola opening upward or opening downward. Then find the coordinates of the vertex.

   (a) $f(x) = 3x^2 + 6x + 2$ (b) $y = x^2 - 6x - 7$

   (c) $g(x) = \frac{1}{2}x^2 + 2x + 5$ (d) $g(x) = 10 + 3x - x^2$

3. Solve each of the following systems by the addition method.

   (a) $\begin{aligned} 2x - 3y &= -7 \\ -2x + 5y &= 9 \end{aligned}$ (b) $\begin{aligned} 2x - 3y &= -6 \\ 3x - y &= 5 \end{aligned}$ (c) $\begin{aligned} 4x + y &= 14 \\ 2x + 3y &= 12 \end{aligned}$

4. Two of the systems from Exercise 4 can easily be solved using the substitution method. Determine which they are, and solve them with that method.

5. Graph each of the following functions.

   (a) $y = \dfrac{6}{x-1}$ (b) $f(x) = -\dfrac{2}{3}x + 4$

A Solutions to Exercises

A.8 Chapter 8 Solutions

Section 8.1 Solutions Back to 8.1 Exercises

1. (a) $f(5) = 10$ (b) $f(-1) = 4$ (c) $f(-3) = 18$ (d) $f(0) = 0$

2. $f(\frac{1}{2}) = -\frac{5}{4}$

3. (a) $g(2) = 1$ (b) $g(-6) = 3$ (c) $g(-5) = 2\sqrt{2}$ (d) $g(-2) = 2.24$

4. (a) (a) $h(-4) = 3$ (b) $h(2) = 4.58$ (c) $h(7)$ doesn't exist.
   (d) We must make sure that $25 - x^2 \geq 0$. This holds for all values of x between -5 and 5, including those two.

5. (a) $f(s) = s^2 - 3s$ (b) $f(s+1) = s^2 - s - 2$ (c) $x = -2, 5$

6. (a) $h(a-3) = \frac{2}{3}a - 7$ (b) $h(a+2) = \frac{2}{3}a - \frac{11}{3}$

7. $x = \frac{21}{2}$ 8. $g(s+1) = \sqrt{2-s}$

9. (a) $h(a-2) = \sqrt{21 + 4a - a^2}$ (b) $x = \sqrt{21}, -\sqrt{21}$

10. (a) $(13, -10)$ (b) $(3, 7)$ (c) $(3, 4)$

Section 8.2 Solutions Back to 8.2 Exercises

1. (a) $g[f(1)] = g[(1)^2 - 2(1)] = g[1-2] = g[-1] = -1 - 3 = -4$
   (b) $g[f(-3)] = g[(-3)^2 - 2(-3)] = g[9+6] = g[15] = 15 - 3 = 12$

2. (a) $f[g(1)] = f[1-3] = f[-2] = (-2)^2 - 2(-2) = 4 + 4 = 8$
   (b) $f[g(-3)] = f[-3-3] = f[-6] = (-6)^2 - 2(-6) = 36 + 12 = 48$

3. $(f \circ g)(-3) = (-3)^2 - 8(-3) + 15 = 9 + 24 + 15 = 48$

4. (a) $(g \circ f)(x) = g[f(x)] = g[x^2 - 2x] = x^2 - 2x - 3$
   (b) $(g \circ f)(4) = 4^2 - 2(4) - 3 = 16 - 8 - 3 = 5$

5. (a) $f[g(2)] = 10$ (b) $(g \circ f)(2) = 3$ (c) $(g \circ f)(x) = x^2 + 3x - 7$
   (d) $(f \circ g)(x) = x^2 - 11x + 28$

6. (a) $y = 3$ (b) $x = 4$ (c) $y = -\frac{5}{6}x - 2$

7. (a) $(2, 2)$ (b) infinitely many solutions (c) no solution

8. (a) $(4, 3)$ (b) no solution 9. $x = \frac{19}{2}$ 10. $x = 8$ ($x = 1$ doesn't check)

11. (a) $f(-3) = 24$ (b) $x = -5, 6$ (c) $x = \frac{1}{2} + \frac{\sqrt{3}}{2},\ x = \frac{1}{2} - \frac{\sqrt{3}}{2}$

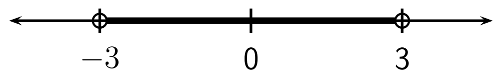
Section 8.3 Solutions

Back to 8.3 Exercises

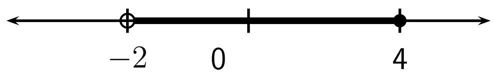
1. (a) $x < 10$ or $10 > x$

(b) $t \geq 0$ or $0 \leq t$

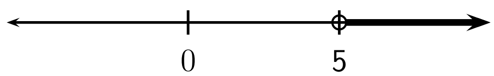
(c) $-3 < x < 3$ or $3 > x > -3$



(d) $-2 < x \leq 4$



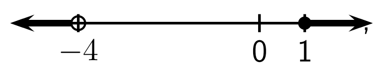
2. (a) $5 < x$ or $x > 5$,



, $(5, \infty)$

(b) All numbers less than or equal to 3, $x \leq 3$,

(c) All numbers less than -4 or greater than or equal to 1, $(-\infty, -4) \cup [1, \infty)$



(d) All numbers greater than 2 and less than or equal to $5\frac{1}{2}$, $2 < x \leq 5\frac{1}{2}$, $(2, 5\frac{1}{2}]$

(e) All numbers less than -7 or greater than or equal to 1, $x < -7$ or $1 \leq x$,

(f) All numbers less than 17,

, $(-\infty, 17)$

(g) $-10 < x \leq -3$,

, $(-10, -3]$

3. (a) x-intercept: 5 y-intercept: -4 (b) x-intercepts: $5, -5$ y-intercepts: $5, -5$

(c) x-intercept: 1 y-intercept: 1

4. $m = -\frac{2}{5}$

5. (a) $y = \frac{1}{4}x - 5$ (b) $y = -\frac{3}{4}x + \frac{1}{2}$ (c) $y = 4$

6. (a) $(5, 0)$ (b) infinitely many solutions (c) no solution

7. $(h \circ g)(x) = h[2x - 3] = (2x - 3)^2 - 5(2x - 3) + 2 = 4x^2 - 22x + 26$

8. (a) $f[g(-1)] = 2$ (b) $(g \circ f)(4) = -25$ (c) $(g \circ f)(x) = 7 - 2x^2$

(d) $(f \circ g)(x) = -4x^2 - 4x + 2$

Section 8.4 Solutions

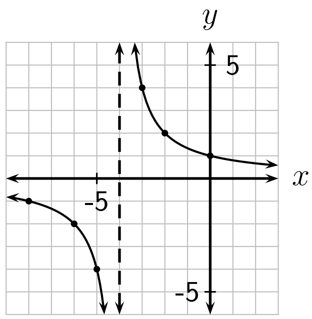
Back to 8.4 Exercises

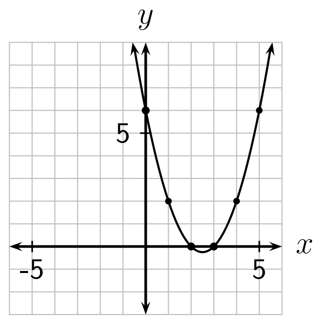
1. (a) All real numbers except -4.
   (b) All real numbers.
   (c) All real numbers except 2 and 3.
   (d) $x \geq -5$ OR $[-5, \infty)$
   (e) $x > 3$ OR $(3, \infty)$
   (f) All real numbers except 4 and -4.

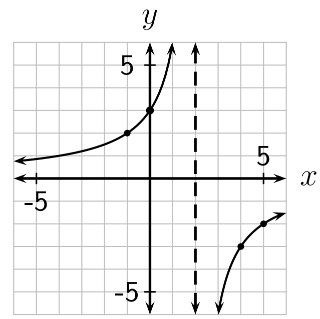
2. (a) $x = -2, 4$
   (b) $x = 0, 2$
   (c) $h(-3) = 12$
   (d) $x = 1 + \sqrt{6}, 1 - \sqrt{6}$, $x = 3.45, \ -1.45$

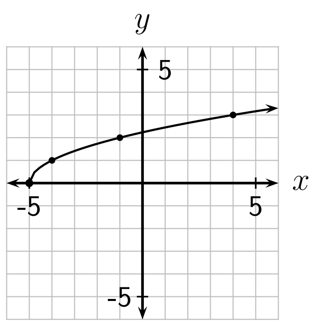
Section 8.5 Solutions

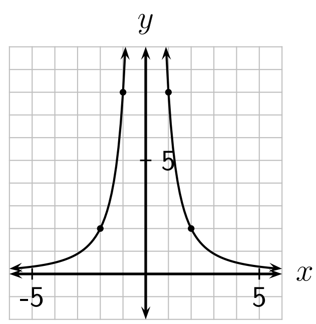
Back to 8.5 Exercises

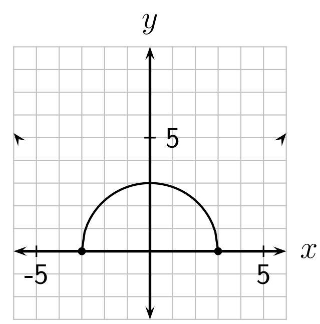
1. (a)


   (b)


   (c)


   (d)


   (e)


   (f)


2. $(f \circ h)(x) = 3x^2 - 6x - 10$ $\qquad$ $(h \circ f)(x) = 9x^2 - 12x$

3. (a) $x \leq -3$ or $x \geq 3$ OR $(-\infty, -3] \cup [3, \infty)$
   (b) $-3 \leq x \leq 3$ OR $[-3, 3]$
   (c) $-3 < x < 3$ OR $(-3, 3)$

Section 8.6 Solutions

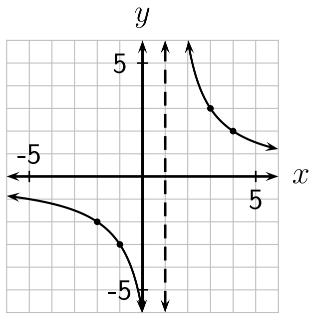
Back to 8.6 Exercises

2. (a) downward, $(-1, -1)$
   (b) upward, $(3, -16)$
   (c) upward, $(-2, 3)$
   (d) downward, $(\frac{3}{2}, \frac{49}{4})$

3. (a) $(-2, 1)$
   (b) $(3, 4)$
   (c) $(3, 2)$

4. The systems in 4. (b) and (c) can easily be solved using the substitution method. See the solutions above.

5. (a)



(b)