Intermediate Algebra

Gregg Waterman Oregon Institute of Technology

©2017 Gregg Waterman



This work is licensed under the Creative Commons Attribution 4.0 International license. The essence of the license is that

You are free to:

- Share copy and redistribute the material in any medium or format
- Adapt remix, transform, and build upon the material for any purpose, even commercially.

The licensor cannot revoke these freedoms as long as you follow the license terms.

Under the following terms:

• Attribution - You must give appropriate credit, provide a link to the license, and indicate if changes were made. You may do so in any reasonable manner, but not in any way that suggests the licensor endorses you or your use.

No additional restrictions ? You may not apply legal terms or technological measures that legally restrict others from doing anything the license permits.

Notices:

You do not have to comply with the license for elements of the material in the public domain or where your use is permitted by an applicable exception or limitation.

No warranties are given. The license may not give you all of the permissions necessary for your intended use. For example, other rights such as publicity, privacy, or moral rights may limit how you use the material.

For any reuse or distribution, you must make clear to others the license terms of this work. The best way to do this is with a link to the web page below.

To view a full copy of this license, visit https://creativecommons.org/licenses/by/4.0/legalcode.

Contents

9	More Exponents and Roots, Complex Numbers		131
	9.1	Fractional Exponents	131
	9.2	Rationalizing Denominators	133
	9.3	Complex Numbers and Their Arithmetic	135
	9.4	Solving Quadratic Equations, Again	138
Α	Solutions to Exercises		141
	A.9	Chapter 9 Solutions	141

9 More Exponents and Roots, Complex Numbers

9.1 Fractional Exponents

9. (a) Evaluate numerical expressions containing fractional exponents.

When we first saw exponents, the exponents we encountered were the counting numbers 1, 2, 3, 4, and so on. Later we gave meaning to negative and zero exponents. Here we will briefly introduce fractional exponents, which are useful in certain situations. Here is how we define fractional exponents:



- We define $a^{\frac{1}{n}}$ to mean $\sqrt[n]{a}$ $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$
- ♦ **Example 9.1(a):** Find $16^{\frac{1}{4}}$

Solution:
$$16^{\frac{1}{4}} = \sqrt[4]{16} = 2$$
, since $2^4 = 16$

 \diamond **Example 9.1(b):** Find $(-25)^{\frac{1}{2}}$ and $-25^{\frac{1}{2}}$ if possible. If not possible, say so.

Solution: $(-25)^{\frac{1}{2}} = \sqrt{-25}$, which does not exist. $-25^{\frac{1}{2}} = -\sqrt{25} = -5$

 \diamond **Example 9.1(c):** Find $(-27)^{\frac{1}{3}}$ and $-27^{\frac{1}{3}}$ if possible. If not possible, say so.

Solution:

$$(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3$$
 because $(-3)^3 = -27$. $-27^{\frac{1}{3}} = -\sqrt[3]{27} = -3$ also.

♦ **Example 9.1(d):** Find $9^{\frac{3}{2}}$.

Solution:

$$9^{\frac{3}{2}} = (\sqrt{9})^3 = 3^3 = 27$$

We could have done this as $9^{\frac{3}{2}} = \sqrt{9^3}$, but 9^3 is hard to find without a calculator, and we probably wouldn't know the square root of the resulting large number either!

♦ **Example 9.1(e):** Find $(-27)^{\frac{2}{3}}$ and $27^{-\frac{2}{3}}$ if possible. If not possible, say so.

Solution: $(-27)^{\frac{2}{3}} = (\sqrt[3]{-27})^2 = (-3)^2 = 9$

To compute $27^{-\frac{2}{3}}$ we use the definition of a negative exponent first:

$$27^{-\frac{2}{3}} = \frac{1}{27^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{27})^2} = \frac{1}{3^2} = \frac{1}{9}$$

Section 9.1 Exercises

To Solutions

- 1. Find each of the following, if possible. If not, write DNE for "does not exist."
 - (a) $4^{\frac{1}{2}}$ (b) $1000^{\frac{1}{3}}$ (c) $-4^{\frac{1}{2}}$ (d) $(-16)^{\frac{1}{4}}$
- 2. Find each of the following, if possible. If not, write DNE for "does not exist."
 - (a) $(-8)^{\frac{1}{3}}$ (b) $27^{\frac{2}{3}}$ (c) $(-9)^{\frac{3}{2}}$ (d) $16^{\frac{1}{4}}$
- 3. Give the domain of each of the following functions.

(a)
$$s(t) = \frac{t+2}{7-t}$$
 (b) $h(x) = \frac{x^2}{5} + 2$ (c) $y = \frac{1}{(x-3)^2}$

4. For each of the following quadratic equations, first determine whether its graph will be a parabola opening upward or opening downward. Then find the coordinates of the vertex.

(a)
$$h(x) = -x^2 - 2x - 3$$
 (b) $s(t) = \frac{1}{2}t^2 + 3t + \frac{9}{2}$

9.2 Rationalizing Denominators

9. (b) Rationalize denominators of rational expressions with roots in their denominators.

When working with fractions, we always expect final results to be given in lowest terms; that is, reduced as much as possible. A fraction not in lowest terms is considered "undesirable" in a sense. When working with roots it is considered undesirable to have a root in the denominator of a fraction. There is a simple method called **rationalizing the denominator** for solving this problem. Before going into it, let's examine how we *really* reduce a fraction. (We usually don't show all the steps I'm going to show, but they really take place.) We'll use $\frac{15}{20}$ as an example.

$$\frac{15}{20} = \frac{3 \cdot 5}{4 \cdot 5} = \frac{3}{4} \cdot \frac{5}{5} = \frac{3}{4} \cdot 1 = \frac{3}{4}$$

Note how we essentially 'take out' the fraction $\frac{5}{5}$, which is really just the number one.

To rationalize the denominator of a fraction we instead 'put in' an extra fraction that is really equal to one. When considering square roots the fraction to be used is the root in the denominator over itself. Let's demonstrate with the fraction $\frac{5}{\sqrt{3}}$:

$$\frac{5}{\sqrt{3}} = \frac{5}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{\sqrt{9}} = \frac{5\sqrt{3}}{3}$$

If we have an expression like $\frac{10x}{15}$ we can reduce it to $\frac{2x}{3}$. We do something similar when rationalizing the denominator, as shown in the next example.

♦ **Example 9.2(a):** Rationalize the denominator: $\frac{15}{\sqrt{6}}$.

Solution:
$$\frac{15}{\sqrt{6}} = \frac{15}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{15\sqrt{6}}{\sqrt{36}} = \frac{\frac{5}{15}\sqrt{6}}{\frac{9}{2}} = \frac{5\sqrt{6}}{2}$$

Sometimes there will be a root in the numerator as well, and we might have to simplify a square root, as shown in the next example.

Example 9.2(b): Rationalize the denominator:
$$\frac{3\sqrt{2}}{\sqrt{10}}$$
.

Solution:
$$\frac{3\sqrt{2}}{\sqrt{10}} = \frac{3\sqrt{2}}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{3\sqrt{20}}{\sqrt{100}} = \frac{3\sqrt{4}\sqrt{5}}{10} = \frac{3\sqrt{5}}{\frac{3\sqrt{5}}{5}} = \frac{6\sqrt{5}}{10} = \frac{3\sqrt{5}}{5}$$

Occasionally we run into numbers like $\frac{5}{3+\sqrt{2}}$. This of course has a square root in the denominator, so we need to rationalize the denominator. If we multiply both the numerator and denominator by $\sqrt{2}$ the denominator will become $3\sqrt{2}+2$, so there will still be a root in the denominator. The trick here is to multiply both the numerator and denominator by $3-\sqrt{2}$, which we call the **conjugate** of $3+\sqrt{2}$.

♦ **Example 9.2(c):** Rationalize the denominator of
$$\frac{5}{3+\sqrt{2}}$$
.

Solution:
$$\frac{5}{3+\sqrt{2}} = \frac{5}{3+\sqrt{2}} \cdot \frac{3-\sqrt{2}}{3-\sqrt{2}} = \frac{15-5\sqrt{2}}{9-3\sqrt{2}+3\sqrt{2}-\sqrt{2}\sqrt{2}} = \frac{15-5\sqrt{2}}{7}$$

NOTE: If the denominator in the above example had been $7 - 3\sqrt{2}$, we would have multiplied the numerator and denominator each by $7 + 3\sqrt{2}$.

Section 9.2 Exercises

To Solutions

1. Rationalize each denominator, and don't forget to reduce and/or simplify.

(a)
$$\frac{5}{\sqrt{6}}$$
 (b) $\frac{10}{\sqrt{15}}$ (c) $\frac{2}{1-\sqrt{5}}$ (d) $\frac{12}{\sqrt{3}}$
(e) $\frac{3\sqrt{2}}{\sqrt{6}}$ (f) $\frac{\sqrt{3}}{7+\sqrt{3}}$

- 2. Consider the functions f(x) = 3 x and $g(x) = x^2 3x$.
 - (a) Find f[g(3)] without actually finding $(f \circ g)(x)$.
 - (b) Find $(f \circ g)(x)$.
 - (c) Use your answer to (b) to find $(f \circ g)(3)$. It should give the same result as you got for (a) if not, check your work.
 - (d) Find $(g \circ f)(x)$.
- 3. Graph each of the following functions.
 - (a) g(x) = x + 1 (b) $y = \frac{8}{(x-3)^2}$
- 4. Find each of the following, if possible. If not, write DNE for "does not exist." Use the facts that $a^{\frac{1}{n}} = \sqrt[n]{a}$ and $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$.
 - (a) $4^{-\frac{5}{2}}$ (b) $9^{\frac{3}{2}}$ (c) $-25^{\frac{3}{2}}$ (d) $(-1)^{\frac{5}{3}}$

- 9. (c) Add, subtract and multiply complex numbers.
 - (d) Divide complex numbers numbers.
 - (e) Simplify square roots of negative numbers.

Consider trying to solve the equation $x^2 - 4x + 13 = 0$. The first thing that one would be inclined to try is to factor the left side, but that won't go anywhere. Next we might try the quadratic formula:

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2}$$

But since $\sqrt{-36}$ is undefined, we can't solve the equation using the quadratic formula. The equation therefore has no solution.

At one point in the past this was not satisfactory to everyone, so some people tried to find a way around the problem here. In doing so, they invented a 'number' i that has the special property that $i^2 = -1$. Now we know that any number like the kind we are used to (what are called **real numbers**) gives a number greater than or equal to zero when squared. Therefore this thing i is not a real number.

A number made up of a real number times i, like 3i, -5i, $\frac{2}{3}i$ or even $i\sqrt{2}$ is called an **imaginary number**. A real number plus or minus an imaginary number, like 2+3i, -7+3i, $7-i\sqrt{5}$ or $\frac{2}{3}-\frac{1}{3}i$ is called a **complex number**. Note that *it is customary to put the real* number first, then the imaginary number. We sometimes refer to this as a + bi form.

We can add, subtract or multiply complex numbers by simply treating i as we would x:

Example 9.3(a): Combine like terms: 3-5i-2+7i+6.

Solution: 3-5i-2+7i+6 = (3-2+6) + (-5i+7i) = 7+2i

♦ **Example 9.3(b):** Subtract (2 - 6i) - (4 - 3i).

Solution:
$$(2-6i) - (4-3i) = 2 - 6i - 4 + 3i = -2 - 3i$$

When multiplying complex numbers we again treat i as we would x, but replace i^2 with -1 everywhere that it occurs. The following examples demonstrate this.

♦ **Example 9.3(c):** Multiply (5i)(1-3i).

Solution:

 $(5i)(1-3i) = (5i)(1) - (5i)(3i) = 5i - 15i^2 = 5i - 15(-1) = 5i + 15 = 15 + 5i$

Note that the answer was put in the standard form a + bi - you should always do this.

♦ **Example 9.3(d):** Multiply (-3+5i)(2-i).

Solution: $(-3+5i)(2-i) = -6+3i+10i-5i^2 = -6+13i+5 = -1+13i$

The next example shows that when we multiply two complex numbers the result is not necessarily a complex number.

♦ Example 9.3(e): Multiply (5 - 2i)(5 + 2i).
Solution: (5 - 2i)(5 + 2i) = 25 + 10i - 10i - 4i² = 25 - 4(-1) = 25 + 4 = 29

The two numbers 5-2i and 5+2i are called **complex conjugates**. Computations like the one in this last example are important when dividing complex numbers, as the next example shows.

 \diamond **Example 9.3(f):** Divide 4 + 3i by 5 - 2i, giving your answer in a + bi form.

Solution: The key here is to write the division as a fraction, and multiply the fraction by one, in the form of the complex conjugate of the denominator over itself:

$$(4+3i) \div (5-2i) = \frac{4+3i}{5-2i} = \frac{4+3i}{5-2i} \cdot \frac{5+2i}{5+2i} = \frac{20+8i+15i+6i^2}{25+10i-10i-4i^2} = \frac{14+23i}{29} = \frac{14}{29} + \frac{23}{29}i$$

Now think about this: $\sqrt{4} = 2$ because $2^2 = 4$, $\sqrt{9} = 3$ because $3^2 = 9$. So since $i^2 = -1$, it must be the case that $\sqrt{-1} = i$. This allows us to take square roots of negative numbers. We do this in the same way that we simplify square roots of positive numbers, using the fact that $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$. Here are some examples:

♦ **Example 9.3(g):** Simplify $\sqrt{-16}$, allowing complex numbers.

Solution:
$$\sqrt{-16} = \sqrt{(16)(-1)} = \sqrt{16}\sqrt{-1} = 4i$$

♦ **Example 9.3(h):** Simplify $\sqrt{-18}$, allowing complex numbers.

Solution:
$$\sqrt{-16} = \sqrt{(9)(-1)(2)} = \sqrt{9}\sqrt{-1}\sqrt{2} = 3i\sqrt{2}$$

As shown here, it is customary to write the number, then the i, then the square root.

Section 9.3 Exercises To Solutions

- 1. Add, subtract or just combine like terms as directed.
 - (a) (7+10i) (3-8i) (b) 3i + (5-i)
 - (c) (1+i) (1-i) (d) (2-3i) + (-5+13i)
- 2. Multiply and simplify each of the following. Give your answers in the form a + bi, where either of a or b might be negative.
 - (a) (3i)(5i) (b) (-2i)(7i) (c) 3(6i)
 - (d) 3(5-2i) (e) 2i(4-i) (f) (7+i)(2+5i)
 - (g) (a+bi)(a-bi) (h) (9+15i)(2-i) (i) $(c+di)^2$
- 3. Divide each, giving your answers in a + bi form.
 - (a) $3 \div (2+5i)$ (b) $(1+i) \div (6-3i)$
 - (c) $(2+5i) \div (3+2i)$ (d) $(c+di) \div (c-di)$
- 4. Simplify each of the following, allowing complex numbers.
 - (a) $\sqrt{-25}$ (b) $\sqrt{-24}$ (c) $-\sqrt{8}$
 - (d) $\sqrt{-100}$ (e) $\sqrt{-75}$ (f) $-\sqrt{64}$
- 5. For each of the following quadratic equations, first determine whether its graph will be a parabola opening upward or opening downward. Then find the coordinates of the vertex.
 - (a) $y = -\frac{1}{4}x^2 + \frac{5}{2}x \frac{9}{4}$ (b) $y = 3x^2 + 6x 2$
- 6. Rationalize each denominator, and don't forget to reduce and/or simplify.

(a)
$$\frac{3\sqrt{2}}{\sqrt{10}}$$
 (b) $\frac{2+\sqrt{5}}{2-\sqrt{5}}$

9.4 Solving Quadratic Equations, Again

9. (f) Solve quadratic equations with complex solutions.

We now return to the equation $x^2 - 4x + 13 = 0$. When we first looked at it, it appeared that it had no solution. But

$$(2+3i)^2 = 4 + 6i + 6i + 9i^2 = 4 + 12i - 9 = -5 + 12i,$$

so if we then substitute x = 2 + 3i into $x^2 - 4x + 13$ we get

$$(2+3i)^2 - 4(2+3i) + 13 = -5 + 12i - 8 - 12i + 13 = 0.$$

This shows that 2+3i is a solution to $x^2 - 4x + 13 = 0!$ Two questions should occur to you at this point:

- 1) How do we get that solution?
- 2) What is the other solution, if there is one?

To answer the first question above, recall the following:

Quadratic Formula The solutions to $ax^2 + bx + c = 0$ are obtained from $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Let's see what happens when we try using the quadratic formula to solve $x^2 - 4x + 13 = 0$:

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2}$$

At this point we conclude that there is no solution if we are restricting ourselves to the real numbers, which we often do. However, if we allow complex numbers we can continue to get

$$x = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm \sqrt{(36)(-1)}}{2} = \frac{4 \pm 6i}{2} = \frac{4}{2} \pm \frac{6i}{2} = 2 \pm 3i$$

Thus the equation has the two solutions, 2+3i and 2-3i. We call two complex numbers like this **complex conjugates**. In Exercise 2(g) of the previous section you showed that when we multiply complex conjugates together the result is always a real number. That is one important thing about complex conjugates.

Another important thing about complex conjugates is that what happened when solving the equation $x^2 - 4x + 13 = 0$ was not a fluke; when solving a quadratic equation that has complex solutions, the two solutions will always be complex conjugates.

Example 9.4(a): Solve $x^2 + 16 = 6x$, allowing complex solutions.



 \diamond **Example 9.4(b):** Solve $4x^2 - 20x + 27 = 0$, allowing complex solutions.



Section 9.4 Exercises

To Solutions

- 1. Solve each of the following quadratic equations, allowing complex solutions.
 - (a) $x^2 + 13 = -6x$ (b) $x^2 + 10x + 26 = 0$ (c) $x^2 + x = 6$ (d) $x^2 + 4 = 0$ (e) $2x^2 + 5 = 2x$ (f) $x^2 + 14x + 53 = 0$
- 2. Give the domains of $f(x) = \sqrt{3-x}$ and $g(x) = \frac{x-1}{x^2 3x 10}$.
- 3. Rationalize each denominator, reduce and/or simplify your answer when possible.

(a)
$$\frac{15}{\sqrt{6}}$$
 (b) $\frac{5+\sqrt{3}}{2-\sqrt{3}}$

- 4. Add, subtract or just combine like terms as directed.
 - (a) (4-7i) (-23-17i) (b) 9+6i-6i-7

5. Multiply and simplify each of the following. Give your answers in the form a + bi, where either of a or b might be negative.

(a)
$$(2+3i)^2$$
 (b) $(4-3i)(4+3i)$ (c) $(-2+3i)(5-4i)$

6. Simplify each square root, allowing complex numbers.

(a)
$$\sqrt{-9}$$
 (b) $\sqrt{-18}$ (c) $-\sqrt{16}$

A Solutions to Exercises

A.9 Chapter 9 Solutions Section 9.1 Solutions Back to 9.1 Exercises 1. (a) 2 (b) 10 (c) -2 (d) DNE 2. (a) -2 (b) 9 (c) DNE (d) 2 3. (a) All real numbers except 7. (b) All real numbers. (c) All real numbers except 3. (b) upward, (-3,9)4. (a) downward, (-1, -2)Section 9.2 Solutions Back to 9.2 Exercises 1. (a) $\frac{5\sqrt{6}}{6}$ (b) $\frac{2\sqrt{15}}{3}$ (c) $-\frac{1+\sqrt{5}}{2}$ (d) $4\sqrt{3}$ (e) $\sqrt{3}$ (f) $\frac{3+7\sqrt{3}}{46}$ 2. (a) f[g(3)] = 3 (b) $(f \circ g)(x) = 3 + 3x - x^2$ (c) $(f \circ g)(3) = 3$ (d) $(g \circ f)(x) = x^2 - 3x$ (b) 3. (a) 5 f • x x-5 -5 4. (a) $\frac{1}{32}$ (c) −125 (b) 27 (d) -1Section 9.3 Solutions Back to 9.3 Exercises 1. (a) 4 + 18i (b) 5 + 2i (c) 2i (d) -3 + 10i

2. (a) -15 (b) 14 (c) 18*i* (d) 15 - 6*i* (e) 2 + 8*i* (f) 9 + 37*i* (g) $a^2 + b^2$ (h) 33 + 21*i* (i) $(c^2 - d^2) + 2cdi$ 3. (a) $\frac{6}{29} + \frac{15}{29}i$ (b) $\frac{1}{15} + \frac{3}{15}i$ (c) $\frac{16}{13} + \frac{11}{13}i$ (d) $\frac{c^2 - d^2}{c^2 + d^2} + \frac{2cd}{c^2 + d^2}i$

- 4. (a) 5i (b) $2i\sqrt{6}$ (c) $-2\sqrt{2}$ (d) 10i (e) $5i\sqrt{3}$ (f) -8
- 5. (a) downward, (5,4) (b) upward, (-1,-5) 6. (a) $\frac{3\sqrt{5}}{5}$ (b) $-9 4\sqrt{5}$

Section 9.4 Solutions Back to 9.4 Exercises

- 1. (a) x = -3 + 2i, x = -3 2i(b) x = -5 + i, x = -5 - i(c) x = 2, x = -3(d) x = 2i, x = -2i(e) $x = \frac{1}{2} + \frac{3}{2}i$, $x = \frac{1}{2} - \frac{3}{2}i$ (f) x = -7 + 2i, x = -7 - 2i
- 2. The domain of f is $(-\infty,3]$ or $x \leq 3$, and the domain of g is $x \neq -2,5$
- 3. (a) $\frac{5\sqrt{6}}{2}$ (b) $19 + 7\sqrt{3}$ 4. (a) 27 + 10i (b) 2 5. (a) -5 + 12i (b) 25 (c) 2 + 23i 6. (a) 3i (b) $3i\sqrt{2}$ (c) -4i