College Algebra

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Contents

2 Introduction to Functions

Outcome/Performance Criteria:

2. Understand general concepts relating to functions and their graphs.

    (a) Evaluate a function, and find all values for which a function takes a particular value, from the equation of the function.

    (b) Graph a function; evaluate a function, and find all values for which a function takes a particular value, from the graph of the function.

    (c) Describe a set using set builder notation or interval notation.

    (d) Give the domain and range of a function from the equation of the function.

    (e) Give the domain and range of a function from its graph.

    (f) Given the graph of a function,

    - give the intervals on which the function is increasing (decreasing)
    - give the intervals on which the function is positive (negative)
    - give the maxima and minima of the function and their locations, and determine whether each is relative or absolute
    - identify the function as even, odd, or neither

    (g) Given the definitions of even and odd functions, determine whether a function is even, odd or neither from its equation.

    (h) Use a linear model to solve problems; create a linear model for a given situation.

    (i) Interpret the slope and intercept of a linear model.

    (j) Solve a problem using a function whose equation is given.

    (k) Determine the equation of a function modeling a situation.

    (l) Give the feasible domain of a function modeling a situation.

    (m) Given the graph of a function, sketch or identify translations of the function.

    (n) Given the graph of a function, sketch or identify vertical reflections of the function.

2.1 Functions and Their Graphs

Performance Criteria:

2. (a) Evaluate a function, and find all values for which a function takes a particular value, from the equation of the function.

   (b) Graph a function; evaluate a function, and find all values for which a function takes a particular value, from the graph of the function.

Introduction

Consider the three equations

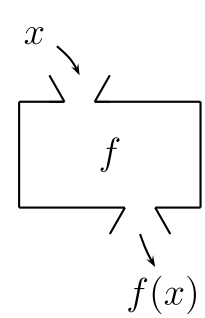
$$y = 3x - 8 \qquad\qquad y = \sqrt{25 - x^2} \qquad\qquad y = 2x^2 + x - 5$$

For each equation, if we choose a value for x and substitute it into the equation, we can find a corresponding value for y, and *there is no doubt what that value of* y *is.* (In the second equation we have to be a bit careful about what value of x we choose, but we'll get into that later!) Whenever we have an equation for which any allowable value of x leads to only one value of y we say that "y is a function of x."

Of course the choice of letters doesn't really matter. If we were to rewrite the last equation as $v = 2w^2 + w - 5$ we would say that v is a function of w, but it is really the same function because v is obtained from w in the same way that y is obtained from x. Functions are a very fundamental concept of math and science, and we will spend most of this term studying them and their applications. Many of you will then take trigonometry after this course, and most of that course is devoted to studying special kinds of functions called **trigonometric functions**.

Rather than saying "y is a function of x" each time, when dealing with such an equation we will just say it is a function. Notice that finding y in these three functions amounts to simply substituting a value for x and doing a little arithmetic. Each of these functions can be thought of as a "machine" that we simply feed an x value into and get out a corresponding value for y. There is an alternative notation for functions that is very efficient once a person is used to it. Suppose that we name the function $y = 3x - 8$ with the letter f; we write this as $f(x) = 3x - 8$. We then think of f as a sort of "black box" that we feed numbers into and get numbers out of. We denote by $f(-2)$ the value that comes out when -2 is fed into the machine, so $f(-2) = -14$.

In general, the "input" of a function f is x, and the output is $f(x)$. The picture to the right illustrates this idea of a function as a "machine." If we put a number like 5 into the machine, the symbolism $f(5)$ represents the "output," the number that will come out when 5 is put in. *It is important to keep in mind that the symbolism* $f(x)$ *really stands for one number.* Technically speaking, a function is simply a way to associate to any "input" value a corresponding "output" value. We will always want to describe how to get the output of a function for any given input, and the two main ways we will do this in this course are by



i) giving an equation describing how the output is obtained from the input, as we have been doing so far.

ii) giving a graph (picture) of how the outputs relate to the inputs.

Let's take a look at the first way of describing a function.

Equations of Functions

Consider again the equation $y = 3x - 8$. We think of x as the input for this equation, and y is the output. Using the function notation described on the previous page, we can replace y with $f(x)$ to get $f(x) = 3x - 8$. Our first two concerns are these:

- Given an input for a function, what is the corresponding output?
- Given an output for a function what are the corresponding inputs?

You will note the way these two are written. Given an input for a function, there will be at most one output - this is what makes a function different from some of the other equations we will work with. However, *one output can come from more than one input.* A simple example of this is the function $y = x^2$; the output 25 could occur when the input is either of 5 or -5.

There are very specific ways that you will be asked the above two questions, and it is important that you get used to them and can quickly and easily know what you are being asked to do. Let's continue looking at the function $f(x) = 3x - 8$, and suppose you are asked to find $f(-2)$. Recalling that $f(x)$ represents the output for an input of x, in this case you are being asked to find the output when the input is -2. Since the number -2 is in the place of x in the expression $f(x)$, we need to replace the x on the other side of the equal sign with -2 and compute the result.

◇ **Example 2.1(a):** For the function $f(x) = 3x - 8$, find $f(-2)$ and $f(1.4)$.

Solution: $f(-2) = 3(-2) - 8 = -6 - 8 = -14$, $f(1.4) = 3(1.4) - 8 = 4.2 - 4 = -3.8$

When working with functions, I would like you to make it clear what any result you obtain represents. In the above example we arrive at the values -14 and -3.8, and if a person follows the string of steps backward in either computation they can see that -14 is $f(-2)$ and -3.8 is $f(1.4)$.

The second question, "Given an output for a function what are the corresponding inputs?," will be asked like this:

◇ **Example 2.1(b):** For the function $f(x) = 3x - 8$, find all values of x such that $f(x) = 5$.

Solution: There are two clues here as to what we are supposed to be doing. The first is the words "find all values of x," which says that we are supposed to be finding input(s) x. The second clue is that we are given that $f(x)$, which represents an output, is 5. to carry out the task we are asked to do, we replace $f(x)$ with 5 and solve the resulting equation:

$$\begin{aligned} 5 &= 3x - 8 \\ 13 &= 3x \\ \tfrac{13}{3} &= x \end{aligned}$$

Note that the final result is labeled as what we are supposed to find, x.

In this example we were asked to find all values of x for which $f(x) = 5$, and we can verify that our answer is such an x by putting it into the function:

$$f(\tfrac{13}{3}) = 3(\tfrac{13}{3}) - 8 = 13 - 8 = 5$$

◇ **Example 2.1(c):** Let $g(x) = x^2 - 3x$. Find $g(-4)$ and find all values of x such that $g(x) = 10$.

Solution: Finding $g(-4)$ is not difficult, but we need to take a bit of care with signs:

$$g(-4) = (-4)^2 - 3(-4) = 16 + 12 = 28$$

To find all values of x for which $g(x) = 10$, we replace $g(x)$ with 10 in the equation of the function and solve:

$$\begin{aligned} 10 &= x^2 - 3x \\ 0 &= x^2 - 3x - 10 \\ 0 &= (x-5)(x+2) \\ x = 5 & \quad \text{and} \quad x = -2 \end{aligned}$$

The process of finding a value of a function for a given input value is called *evaluating* the function. Sometimes you will be asked to evaluate a function for an input that is not a number. This can be a bit confusing, and the algebra is often a bit more involved. Suppose that we have the function $g(x) = x^2 - 3x$, and we are then asked to find $g(x-2)$. The x in the second of these *IS NOT* the same as the x in the first. Here it is helpful to recognize that we could rewrite the description of the function g as $g(u) = u^2 - 3u$; to find $g(x-2)$ we then replace all occurrences of u with $x - 2$ and simplify. The following example shows this.

◇ **Example 2.1(d):** Find $g(x-2)$ for the function $g(x) = x^2 - 3x$.

Solution:

$$\begin{aligned} g(x-2) &= (x-2)^2 - 3(x-2) \\ &= (x^2 - 4x + 4) - 3x + 6 \\ &= x^2 - 7x + 10 \end{aligned}$$

Note where we stop in the above example. We simply put $x-2$ into g, and then simplify and that's it. The end result is the output of a function, and it has an unknown value x in it because the input value $x-2$ had an x in it.

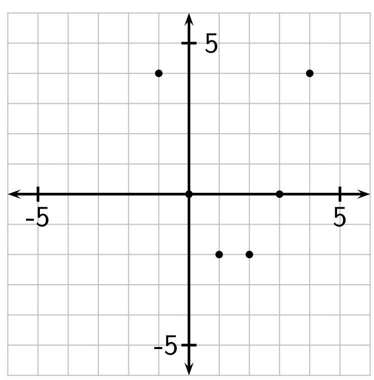
Graphs of Functions

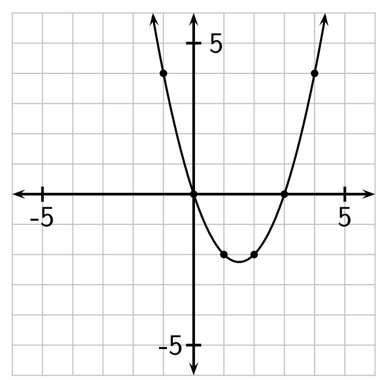
Consider the function $g(x) = x^2 - 3x$. This is simply an equation that lets us find outputs of the function for various inputs. We can graph a function just like any other equation, with the inputs being the x values and the outputs the $g(x)$, or y, values.

◇ **Example 2.1(e):** Graph the function $g(x) = x^2 - 3x$.

Solution: We recognize this as $y = ax^2 + bx + c$, with the output y simply replaced by $g(x)$, so the graph will be a parabola. $a = 1$ and $b = -3$, so the parabola opens upward and the vertex has x-coordinate $x = -\frac{b}{2a} = -\frac{-3}{2(1)} = \frac{3}{2} = 1\frac{1}{2}$. Rather than evaluating the function at $x = -\frac{3}{2}$, which would require working with fractions, we can simply evaluate for integer values working outward from $x = 1\frac{1}{2}$. $g(1) = -2$ and, because $x = 2$ is the same distance from $x = 1\frac{1}{2}$ but on the other side of the x for the vertex, $g(2)$ will also be -2. Similarly, $g(3) = g(0) = 0$ and $g(-1) = g(4) = 4$. The results are in the table to the left below, and those points are plotted in the first graph below. The second graph is the graph of the function $g(x) = x^2 - 3x$. Note the arrowheads indicating the behavior of the graph as it leaves the edges of the grid.

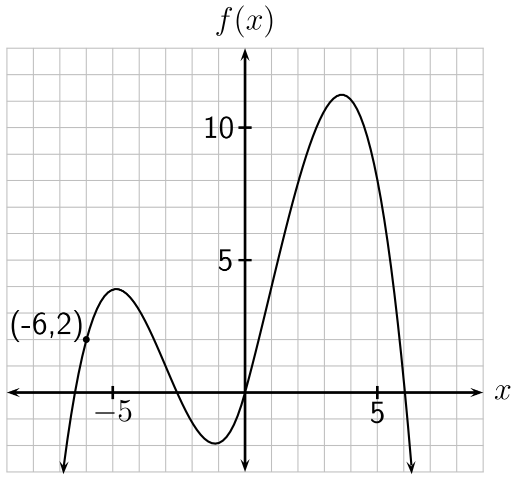
x	$g(x)$
-1	4
0	0
1	-2
2	-2
3	0
4	4

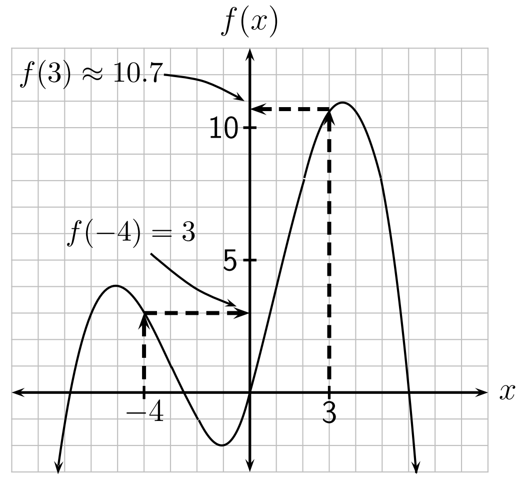




A graph of a function is simply another representation of the function. Just as we found outputs from inputs and inputs from outputs using the equation of a function, we would like to be able to do this from a graph as well. The following example shows how to do this.

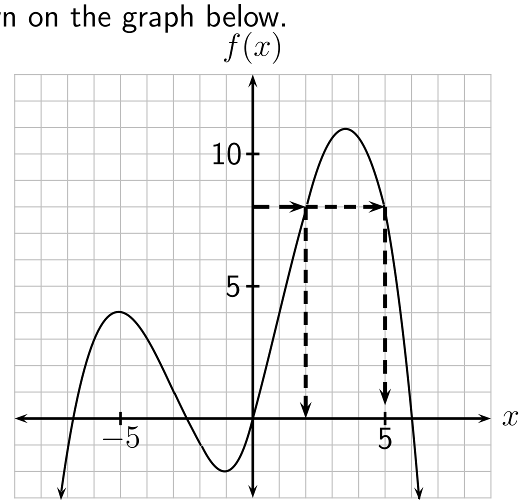
◇ **Example 2.1(f):** Consider the function $y = f(x)$ that has the graph below and to the left. Find $f(-4)$, $f(3)$, and find all values of x such that $f(x) = 8$.





Solution: Note that every point on the graph of the function represents an input-output pair. For example, the pair $(-6, 2)$ indicates that when the input is -6 the output is 2. That is, $f(-6) = 2$. If we want to evaluate the function for a given input value, we simply locate the value on the input (x) axis, go up or down to the graph of the function, then go across to the output (y) axis and read off the output. Using this, $f(-4) = 3$. When working with graphs we often cannot tell exactly what the x or y value corresponding to a particular point is, so we might wish at times to use the approximately equal symbol $\approx$. So we might write $f(3) \approx 10.7$. The way the $f(-4)$ and $f(3)$ are found is shown on the graph above and to the right.

To find all values of x for which $f(x)$ has some value, we "reverse" that process: We locate the given output value on the y-axis, then go across horizontally to all points on the graph with that y value. From each of those we then go straight up or down to find the input (x) value that gives that output. So to find all values of x for which $f(x) = 8$ we go left and right from 8 on the y-axis until we hit points on the graph. From those points we drop down to the x-axis to find the desired x values of 2 and 5. The way these x values are found is shown on the graph below.



Section 2.1 Exercises

To Solutions

1. Let $h(x) = 2x - 7$.

   (a) Find $h(12)$.

   (b) Find all values of x for which $h(x) = 12$.

2. Let $f(x) = 2x^2 - 3x$.

   (a) Find $f(3)$ and $f(-5)$.

   (b) Find all values of x such that $f(x) = 9$.

   (c) Find all values of x such that $f(x) = 1$.

3. Let $g(x) = x - \sqrt{x+3}$.

   (a) Find all values of x such that $g(x) = 3$.

   (b) Find $g(-2)$.

   (c) Find all values of x such that $g(x) = 1.3$, rounding to the hundredth's place.

   (d) List three values of x for which $g(x)$ *CANNOT* be computed, and tell why.

   (e) List three values of x for which $g(x)$ *CAN* be computed.

   (f) With a bit of thought we can determine that $g(x)$ can be computed for all values of x greater than some number, and it cannot be computed for all values of x less than that number. What is the number?

   (g) Can $g(x)$ be found for the number you got for (f)?

4. Let $f(x) = \dfrac{3x+4}{x-2}$.

   (a) Find $f(6)$.

   [(b)] Find all x such that $f(x) = 6$.

   (c) Are there any values for which $g(x)$ cannot be found?

5. Let the function h be given by $h(x) = \sqrt{25 - x^2}$.

   (a) Find $h(-4)$.

   (b) Find $h(2)$, using your calculator. Your answer will be in decimal form and you will have to round it somewhere - round to the hundredth's place.

   (c) Why can't we find $h(7)$?

   (d) For what values of x *CAN* we find $h(x)$? Be sure to consider negative values as well as positive. We will refer to the set of values of x for which we can find $h(x)$ as the **domain** of h. More on this later...

   (e) Find all values of x such that $h(x) = 4$.

6. In this exercise we see a kind of function that is so simple that it can be confusing! Consider the function given by $h(x) = 12$. Find $h(0)$, $h(-43)$, $h(3\frac{4}{7})$. Your answers should all be the same! This is an example of a **constant function**. It is called this because the output has a constant value of 12, regardless of what the input is.

7. Remember that when dealing with a function, the notation $f(x)$ or $g(x)$ is basically just a y value. Keeping this in mind, find the x- and y-intercepts of each of the following functions from previous exercises.

   (a) $h(x) = 2x - 7$

   (b) $f(x) = 2x^2 - 3x$

   (c) $g(x) = x - \sqrt{x+3}$

   (d) $f(x) = \dfrac{3x+4}{x-2}$

   (e) $h(x) = \sqrt{25 - x^2}$

   (f) $h(x) = 12$

8. Let $f(x) = x^2 - 3x$, $g(x) = \sqrt{x+1}$. Find and simplify each of the following.

   (a) $f(x+1)$ (b) $g(2x)$ (c) $f(-2x)$ (d) $g(x-4)$

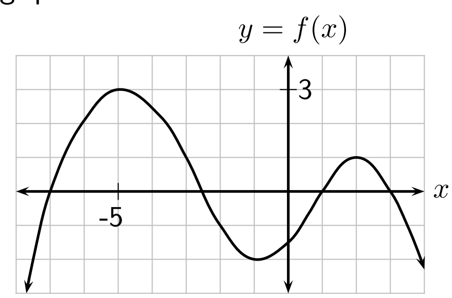
9. Consider the function $g(x) = x^2 + 5x$.

   (a) Find and simplify $g(a-3)$ by substituting $a-3$ for x and simplifying. *From this point on you will always be expected to simplify, even if not specifically asked to do so.*

   (b) Find and simplify $g(x+h)$.

   (c) Use your result from (b) to find and simplify $\dfrac{g(x+h) - g(x)}{(x+h) - x}$. *This sort of expression is very important in mathematics and its applications!* We'll see what it is about later.

10. Use the function shown to the right to answer the following questions.



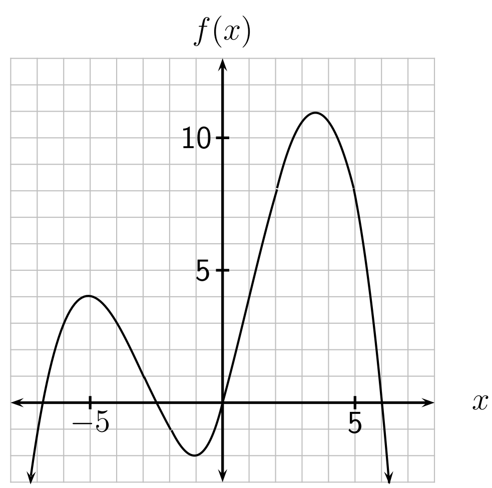
   (a) Give all of the x- and y-intercepts for the function f. Distinguish between the two!

   (b) Estimate the values of $f(-3)$, $f(-2)$ and $f(3)$.

   (c) Estimate *all* values of x for which $f(x) = 2$.

   (d) Estimate *all* values of x for which $f(x) = 3$.

11. Consider again the graph of the function f from Example 2.1(e), shown below and to the right. Use the graph to do each of the following:



   (a) Find $f(-3)$, $f(-1)$, $f(1)$.

   (b) Find x such that $f(x) = 1$. (There are four values of x for which this happens - give them all.)

   (c) Find $f(0)$.

   (d) Find all x such that $f(x) = 4$.

   (e) Find all x such that $f(x) = 10$.

12. We sometimes also want to know what input values can cause two different functions to have the same output values. Consider again $f(x) = 3x - 8$, along with $g(x) = x^2 - 3x$.

    (a) Find $f(4)$ and $g(4)$. They should be equal.

    (b) The question here is "How do we know that $x = 4$ will make the two functions equal?" (That is, make them have the same output.) Well, we want a value of x that makes $f(x) = g(x)$, so we set $3x - 8 = x^2 - 3x$ and solve. Do this.

    (c) You should have obtained $x = 4$ and $x =$ _____ in part (b). Substitute the other value that you found into both functions and verify that it does give the same output for both.

13. Find all values of t for which $r(t) = \sqrt{2t+5}$ and $s(t) = t + 1$ are equal. *Be sure to check your answer(s) in the original functions.*

14. Find all values of y for which $f(y) = \dfrac{y+3}{y^2 - y}$ and $g(y) = \dfrac{8}{y^2 - 1}$ are equal.

15. Consider the function $f(x) = 3x^4 - 5x^2$.

    (a) Find $f(-2)$ and $f(2)$, then find $f(-5)$ and $f(5)$.

    (b) Find *and simplify* $f(-x)$. What do you notice, in light of your answers to (a)? A function that behaves in this way is called an **even function**.

16. Now let $g(x) = x^3 + 5x$.

    (a) The function could be written as $g(x) = x^3 + 5x^r$ for what value of r? What kind of function do you suppose that this one is?

    (b) Find $f(-1)$ and $f(1)$, then $f(-4)$ and $f(4)$.

    (c) What can we say in general about $f(-x)$ for this function?

17. In light of the previous two exercises, what can you say about each of the functions

    (a) $h(x) = 12$ (See Exercise 6.)

    (b) $f(x) = x^2 - 5x + 2$

18. Find and simplify $f(x + h)$ for $f(x) = x^2 - 5x + 2$.

19. Find and simplify $g(x + h)$ for $g(x) = 3x - x^2$.

2.2 Domain and Range of a Function

Performance Criteria:

2. (c) Describe a set using set builder notation or interval notation.
   (d) Give the domain and range of a function from the equation of the function.
   (e) Give the domain and range of a function from its graph.

Consider the function $f(x) = \sqrt{25 - x^2}$. We can see that

$$f(0) = \sqrt{25 - 0^2} = 5, \qquad f(-5) = \sqrt{25 - (-5)^2} = 0, \qquad f(4) = \sqrt{25 - 4^2} = 3.$$

We can also see that $f(6) = \sqrt{25 - 6^2} = \sqrt{-11}$ is not possible to find unless we allow complex numbers, which we are not doing now. Because the value of x is squared in this function, $f(-6)$ results in the same situation, and is also not possible. In fact, we can only find $f(x)$ for values of x between -5 and 5, including both of those numbers. We call the collection of those numbers for which $f(x)$ can be found the **domain** of f. In this section we first look at how to describe such sets concisely, and then we look at the domains of some other functions.

Sets of Real Numbers

A **set** is a collection of objects, called **elements**; in our case the elements will always be numbers. Sets will be important at times in our study of functions, and it is helpful to have clear, concise ways of describing sets. When we have a set of just a few numbers, we can describe the set by listing its elements (generally listed from smallest to largest) and enclosing them with the symbols $\{$ and $\}$, used to indicate a set. For example, the set consisting of the numbers -1, 3 and 10 would be denoted by $\{-1, 3, 10\}$.

It is occasionally useful to consider sets with just a few elements, and even sets with just one element, like $\{5\}$. More often, however, we will consider sets that cannot be listed, like the numbers between -5 and 5, including both. When we say "all the numbers" we mean all the **real numbers**, which means every number that is not a complex number. The real numbers consist of all whole numbers and fractions, as well as more unusual numbers like $\sqrt{2}$ and π (and negatives of all these things as well). All the real numbers taken together is a set, which we denote with the symbol $\mathbb{R}$.

To describe a set like the set of all real numbers between -8 and 3, for example, we will usually use one of two methods. The first we'll describe is called **set builder notation**. The set of all real numbers between -8 and 3, including -8 and not including 3 is described in set builder notation as

$$\{x \in \mathbb{R} \mid -8 \le x < 3\} \qquad \text{or} \qquad \{x \mid -8 \le x < 3\}$$

The symbol $\in$ means "in the set" and the vertical bar represents the words "such that." The first statement above is then read as "the set of real numbers x such that x is greater than or equal to -8 and less than 3." Since we assume we are considering only real numbers unless told otherwise, we can eliminate the $\in \mathbb{R}$, which means "x is an element of the set of real numbers." For a set like all numbers greater than or equal to -2 the set builder notation would be $\{x \mid x \ge -2\}$

◇ **Example 2.2(a):** Give the set builder notation for each of the following sets:

(a) The set of all numbers between -5 and 3, not including -5 but including 3.

(b) The set of all numbers between 1 and 3, including both.

(c) The set of all numbers less than -2.

Solution:

(a) $\{x \mid -5 < x \leq 3\}$ (b) $\{x \mid 1 \leq x \leq 3\}$ (c) $\{x \mid x < -2\}$

Interval Notation for Sets of Real Numbers

Let's go back to the set of all real numbers between -8 and 3. Another method for describing sets like those that we have been looking at is called **interval notation**. This consists of writing the smaller of the two numbers bounding our set, followed by the larger: $-8, 3$. We then enclose these numbers with square brackets [ ] or parentheses ( ), depending on whether the numbers themselves are actually included in the set or not, respectively. For example, if -8 was to be included but 3 was not, we would write $[-8, 3)$. If we meant only the numbers between -8 and 3, but not including either, we would write $(-8, 3)$. If both -8 and 3 were to be included we'd write $[-8, 3]$.

For a set like all real numbers greater than -2, we need to have the concept of infinity, symbolized as ∞. And along with infinity comes negative infinity! Think of them this way: Neither of them is a number, so they cannot be included in sets. Infinity should be thought of as a "place" that is bigger than all real numbers, and negative infinity is a place that is smaller than all real numbers. Negative infinity is symbolized as $-\infty$. We use the concepts of positive and negative infinity like this: The set of all real numbers greater than or equal to -2 is written as $[-2, \infty)$. The set of all real numbers less than -2 is written as $(-\infty, -2)$. *Note that neither infinity or negative infinity can ever be included in a set, since neither is a number. Thus they are always enclosed by parentheses.*

◇ **Example 2.2(b):** Give the interval notation for each of the following sets:

(a) The set of all numbers between -5 and 3, not including -5 but including 3.

(b) The set of all numbers between 1 and 3, including both.

(c) The set of all numbers less than or equal to 4.

Solution:

(a) $(-5, 3]$ (b) $[1, 3]$ (c) $(-\infty, 4]$

Describing Other Sets of Numbers

Suppose that instead of all the numbers between -100 and 100 we needed to consider all the numbers less than or equal to -100 along with all those greater than or equal to 100. In other words, we want the set consisting of $(-\infty,-100]$ and $[100,\infty)$, combined. We call such a combination the **union** of the two sets, symbolized by $(-\infty,-100]\cup[100,\infty)$. Taken as one set, this is the set of all numbers that are less than or equal to -100 *OR* greater than or equal to 100. The same set can be described with set builder notation as $\{x \mid x \leq -100 \text{ or } x \geq 100\}$. Note that we don't use the word *and* because there are no numbers that are less than -100 and at the same time greater than 100. Another way of using set builder notation to describe this set would be $\{x \mid x \leq -100\} \cup \{x \mid x \geq 100\}$

◇ **Example 2.2(c):** Describe the set of all numbers less than -2 or greater than 2 using (a) set builder notation, and (b) interval notation.

Solution: The set is described with set builder notation by either of $\{x \mid x < -2 \text{ or } x > 2\}$ or $\{x \mid x < -2\} \cup \{x \mid x > 2\}$, and with interval notation by $(-\infty,-2)\cup(2,\infty)$.

Suppose that we are asked to describe the set of all real numbers not equal to seven. Although it is possible to use interval notation to describe this set, it is actually much easier to do using set builder notation: $\{x \mid x \neq 7\}$. If we wanted to describe the set of all real numbers not equal to 7 or 2, we would write $\{x \mid x \neq 2 \text{ and } x \neq 7\}$. Notice that we must use the word *and* in order to be logically correct here. Often we write instead $\{x \mid x \neq 2,\ 7\}$, which means the same thing.

Domains of Functions

Recall the function $f(x) = \sqrt{25-x^2}$ that we began this section with. We can now use set builder notation or interval notation to say that its domain is $\{x \mid -5 \leq x \leq 5\}$ or $[-5,5]$. To tell that this set is the domain of the function f we use the notation Dom$(f) = [-5,5]$, spoken as "the domain of f is the set ..." Consider the two functions $g(x) = x^2+5x$ and $h(x) = \dfrac{3}{x-7}$. It should be clear to you that we can compute $g(x)$ for *any* choice of x, so Dom$(g) = \mathbb{R}$. We run into trouble when we try to compute $h(7)$. We *can* compute $h(x)$ for any other value of x, however, so Dom$(h) = \{x \mid x \neq 7\}$.

For the time being, there are only two things that can cause a problem with computing the value of a function:

- Having zero in the denominator of a fraction.
- Having a negative value under a square root (or any other *even* root).

◇ **Example 2.2(d):** Give the domain of the function $g(x) = \sqrt{x-3}$ using both set builder notation and interval notation.

Solution: To avoid having a negative under the square root, we must be sure that x is at least 3. Thus

$$\text{Dom}(g) = \{x \mid x \geq 3\} \qquad \text{or} \qquad \text{Dom}(g) = [3,\infty)$$

◇ **Example 2.2(e):** Give the domain of the function $f(x) = \dfrac{x-1}{x^2+x-20}$ using set builder notation or interval notation, whichever is more appropriate.

Solution: The denominator of f factors to give us $f(x) = \dfrac{x-1}{(x+5)(x-4)}$, so x cannot be either -5 or 4. Therefore $\text{Dom}(f) = \{x \mid x \neq -5,\ 4\}$. Note that there is no problem with x having the value 1, since that causes a zero in the numerator, which is not a problem. (It does mean that $f(1) = 0$.)

◇ **Example 2.2(f):** Give the domain of the function $h(x) = \dfrac{1}{\sqrt{x^2-16}}$ using interval notation.

Solution: To avoid having a negative under the square root, we must be sure that x is at least 4, but we also need to make sure we don't get a zero in the denominator, so x cannot be 4. We can see that x can also be less than -4 because then x^2 is greater than 16 and the square root will be defined. Thus $\text{Dom}(h) = (-\infty, -4) \cup (4, \infty)$.

Ranges of Functions

There is another concept that goes along with the domain of a function. Consider the function $f(x) = x^2 + 3$, whose domain is $\mathbb{R}$. Suppose that we were to find $f(x)$ for every real number - what values would we get? Well, the possible values of x^2 are all the positive real numbers and zero, or the set $[0, \infty)$. When we add three to all of those values, we will get every number greater than or equal to 3, or the set $[3, \infty)$. The set of all possible outputs of a function (when it is evaluated for every number in its domain) is called the **range** of the function, abbreviated Ran. So, for our example, Ran$(f) = [3, \infty)$. Ranges are generally more difficult to determine than domains.

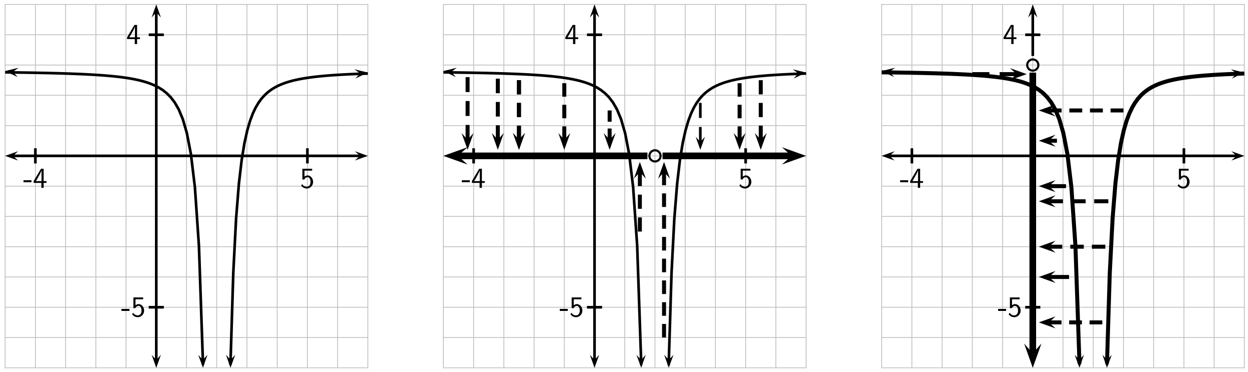
◇ **Example 2.2(g):** Find the range of $g(x) = \sqrt{x+5} - 2$.

Solution: We first note that $\text{Dom}(g) = [-5, \infty)$. When $x = -5$, $\sqrt{x+5} = 0$ and, as x gets larger, $\sqrt{x+5}$ gets larger as well, getting as large as we want. Thus the range of $\sqrt{x+5}$ is $[0, \infty)$. When we then subtract 2 from all those values we get $\text{Ran}(g) = [-2, \infty)$.

Domains and Ranges, Graphically

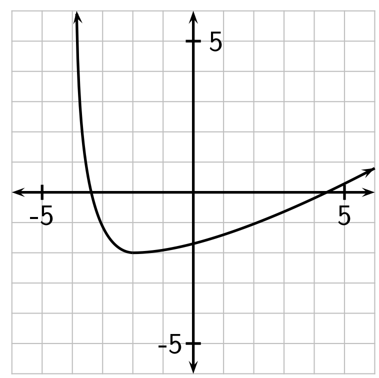
It is relatively simple to determine the domain and range of a function if we have its graph. As an example, let's consider the function $h(x) = \dfrac{3x^2-12x+11}{(x-2)^2}$, whose graph is shown below and to the left. To find the domain of the function we simply need to find all the points on the x-axis that correspond to a point on the curve. In the middle picture below I have tried to indicate how we "drop" or "raise" every point onto the x-axis. I have darkened the part of the

x-axis that is hit by points coming from the curve like this; it is the entire axis except the point $(2, 0)$. (It should be clear from the equation why this is!) Thus the domain is $\{x \mid x \neq 2\}$.



To determine the range we go horizontally to the y-axis from every point on the curve, as shown in the third picture above. The points "hit" are the ones on the y-axis that are shaded in that picture, consisting of the set $(-\infty, 3)$, so that is the range of the function.

◇ **Example 2.2(h):** Give the domain and range of the function whose graph is shown below and to the right. The left edge of the graph gets closer and closer to the vertical line $x = -4$ without ever touching it.



Solution: We can see that if we were to thicken the x-axis at all points above or below points on the curve, we would thicken the portion to the right of $x = -4$, but not -4 itself. the domain of the function is then $(-4, \infty)$.

If we instead thicken the y-axis at all points that are horizontal from some point on the curve, we will thicken the part from $y = -1$ up, including -1. Therefore the range of the function is $[-1, \infty)$.

Of course the set builder notation for the domain would be $\{x \mid x > -4\}$ and for the range it would be $\{y \mid y \geq -1\}$. (Note the use of the correct variable for each.)

Section 2.2 Exercises To Solutions

1. Give the set builder notation for each of the following sets.

   (a) All real numbers between $2\frac{1}{2}$ and 13, including both.

   (b) All real numbers between -12 and -2, including -2 but not -12.

   (c) All real numbers between -100 and 100, including neither.

(d) All real numbers greater than 3.

(e) All real numbers less than or equal to -5.

2. Give the interval notation for each of the sets described in Exercise 1.

3. Use interval notation and union to write each of the following sets.

   (a) The real numbers less than 3 or greater than or equal to 7.

   (b) The real numbers less than -4 or greater than 4.

   (c) The real numbers not equal to 7. (Note that this is the set of real numbers less than 7 or greater than 7.)

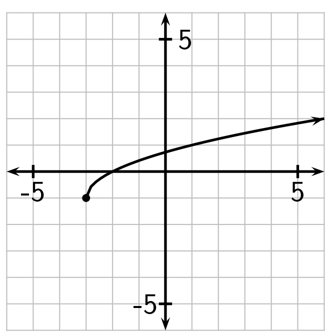
4. Use set builder notation to write each of the sets in the previous exercise.

5. Give the domain of each of the following functions using either interval notation or set builder notation.

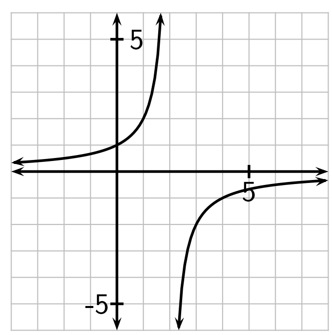
   (a) $f(x) = \sqrt{x+2}$ (b) $g(x) = \dfrac{4}{x-1}$ (c) $h(x) = \dfrac{x-2}{x^2+3x-10}$

6. For each graph, give the domain and range of the function whose graph is shown.

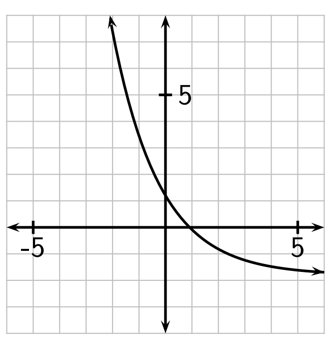
(a)



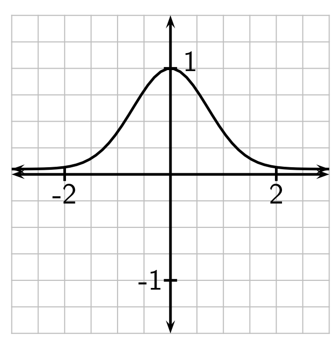
(b)



(c)



(d)



7. For each of the following,

   - attempt to determine the domain of the function algebraically
   - give the domain using either interval notation or set builder notation
   - Graph the function using technology, and ee if the graph seems to support your answer

- check your answer in the back of the book

(a) $y = \sqrt{x-4}$
(b) $f(x) = \sqrt{4-x}$
(c) $g(x) = \dfrac{5}{x+3}$
(d) $y = \sqrt{25-x^2}$
(e) $g(x) = \dfrac{x-1}{x+5}$
(f) $f(x) = \dfrac{7}{3x-2}$
(g) $y = \dfrac{5}{x^2-5x+4}$
(h) $f(x) = \dfrac{\sqrt{x-1}}{x-5}$
(i) $g(x) = x^2 - 5x + 4$

8. (a) Give the domain of $f(x) = \sqrt{x-3}$ using both set builder notation and interval notation.
   (b) Give the domain of $g(x) = \sqrt{3-x}$, noting how your answer compares to that of (a).
   (c) The domain of $h(x) = \dfrac{1}{\sqrt{3-x}}$ is slightly different than the domain of $g(x) = \sqrt{3-x}$. How is it different, and why? Give it.

9. (a) What is the domain of any function of the form $f(x) = ax^2 + bx + c$?
   (b) Suppose that the graph of a function is a parabola that has vertex $(-2, 5)$ and opens downward. Give the range of the function.
   (c) Determine the range of $y = x^2 + 4x - 5$ without graphing the function.

10. Use the following questions to help you determine the range of the function $y = \dfrac{x^2}{x^2+3}$.

- Can the function ever be negative?
- Can the function ever be positive?
- Can the function ever be zero?
- How does the size of the numerator compare with the size of the denominator? Can they ever be equal?
- Write down what you think the range of the function is. If the above questions are not enough for you to determine the range, substitute some values for x and find the corresponding values of y.
- Graph the function using technology and see if the graph supports your answer.

11. For each of the following given sets, determine a function whose domain is the set.

(a) $[-2, \infty)$
(b) $\{x \mid x \neq 5\}$
(c) $(-\infty, 5]$
(d) $\{x \mid x \neq -5, 1\}$
(e) $(-\infty, -3)$

2.3 Behaviors of Functions

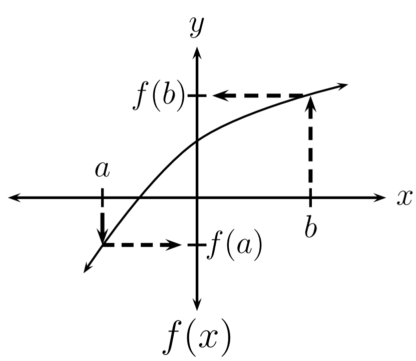
Performance Criteria:

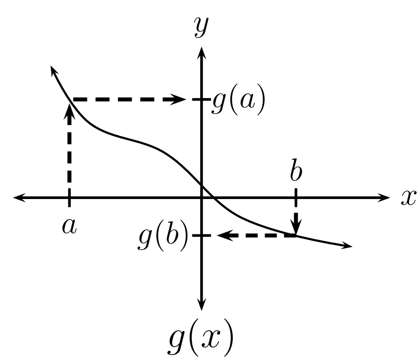
2. (f) Given the graph of a function,
   - give the intervals on which the function is increasing (decreasing)
   - give the intervals on which the function is positive (negative)
   - give the maxima and minima of the function and their locations, and determine whether each is relative or absolute
   - identify the function as even, odd, or neither

   (g) Given the definitions of even and odd functions, determine whether a function is even, odd or neither from its equation.

Increasing and Decreasing Functions

The graphs of two functions f and g are shown below. Note the following difference between the two functions: When we "input" two values a and b into f, if $a < b$ then $f(a) < f(b)$. With g, on the other hand, if $a < b$, then $g(a) > g(b)$. Put more simply, as x gets larger $f(x)$ gets larger, and $g(x)$ gets smaller as x gets larger. Graphically, as we move to the right on the graph of f we move up, and as we move to the right on the graph of g we move down.

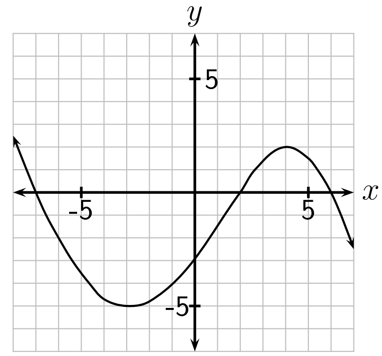




We call f an **increasing function** and g a **decreasing function**. To summarize:

- A function is **increasing** if increasing the input causes an increase in the output.
- A function is **decreasing** if increasing the input causes a *decrease* in the output.

Many functions that we will deal with are neither increasing or decreasing. Instead, they are increasing for some values of x, and decreasing for others. For example, the function h shown to the right at the top of the next page is increasing for values of x between -3 and 4. Using interval notation, we say the function is increasing on the interval $(-3, 4)$ or on the interval $[-3, 4]$. It doesn't really matter whether we include the endpoints of the interval or not, for technical reasons that I don't wish to go into here. Similarly, h is decreasing on the intervals $(-\infty, -3)$ and $(4, \infty)$ or $(-\infty, -3]$ and $[4, \infty)$. (We assume that the behavior we can't see, beyond the edges of the grid, continues in the manner indicated by the arrows on the ends of the graph.)

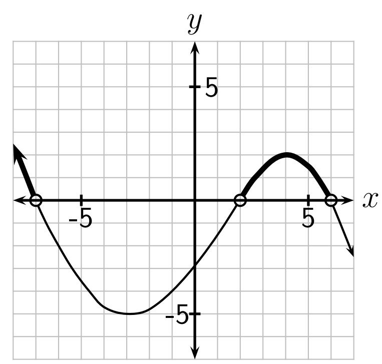


The function $h(x)$

Notice that when we talk about where a function is increasing or decreasing, we mean the x values, NOT the y values. For the idea of increasing and decreasing, and the others we are about to see, you might wish to think this way: The graph of the function is like a landscape, with hills and valleys. The x-axis is like a map, and every point on the landscape has a map location given by its x-coordinate. The y-coordinate of a point is its elevation. When we ask where a function is increasing, *we mean the "map locations,"* or the x-values.

Positivity and Negativity of a Function

We will often be interested in where a function is positive or negative. The function is the y values, so we want to know where y is positive on the graph of the function. The "where" part is given by x values, in the same way as done for increasing and decreasing. The graph of the function h is shown again, to the right, but this time the portions of the curve where y is positive are thickened. The"empty holes" where the function crosses the x-axis are to indicate that the y values are not positive at those points. To tell where the function is positive we give the x values corresponding to all the points on those thickened parts of the curve. Thus we say that the function is positive on the intervals $(-\infty, -7)$ and $(2, 6)$. Obviously the function is then negative on the intervals $(-7, 2)$ and $(6, \infty)$.



The function $h(x)$

Minima and Maxima of a Function

Looking at the graph of h again, we can see that the curve has a high point at $(4, 2)$. We call the value 2 a **maximum** of the function, and when talking about a maximum we want to identify two things:

i) *What the maximum value is*; this means the output, or y, value.

ii) *Where the maximum value occurs*; this means the x value for the point with the maximum y value.

In this case we would say that h has a maximum of 2 at $x = 4$. Note that the plural of maximum is *maxima*, and the plural of minimum is *minima*.

One can see that the point at $(4,2)$ is not actually the highest point on the graph, since every point on the part of the graph to the left of about -7.8 has a y value greater than two. However, the point at $(4,2)$ is higher than all the points on the curve that are near that point, so we say that the function has a **relative maximum** there. A relative maximum of a function occurs at a value of $x=a$ if $f(a)>f(x)$ for all values of x near a. (Take a minute and try to digest what this says!) On a graph, relative maxima occur at the tops of "hills" on the graph. Similarly, a relative minimum of a function occurs at a value of $x=b$ if $f(b)<f(x)$ for all values of x near b. The function h has a relative minimum of -5 at $x=-3$.

Any value $x=a$ such that $f(a)>f(x)$ for *ALL* other values of x is the location of what we call an **absolute maximum**. On a graph, this shows up as a "hill" that is higher than any other point on the graph, including points not actually shown on the picture. The function h has no absolute maximum (or absolute minimum). *Note that any absolute maximum is "automatically" a relative maximum as well.*

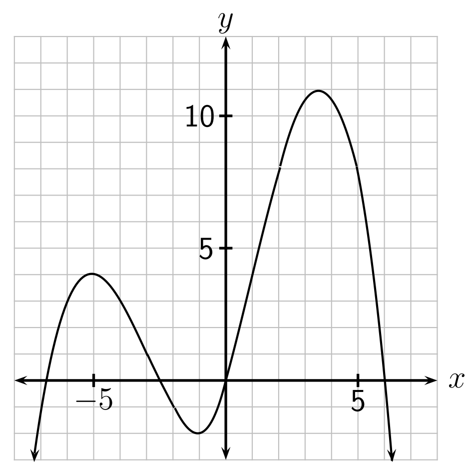
◇ **Example 2.3(a):** For the graph of the function f shown to the right, give

(a) the intervals on which the function is decreasing,

(b) the intervals on which the function is positive,

(c) all maxima and minima, including their locations and whether they are relative or absolute.

It will be necessary to approximate some values of x and y.



Solution: The function is decreasing on the intervals $(-5,-1)$ and $(3.5,\infty)$. It would also be correct to say instead that f is decreasing on $[-5,-1]$ and $[3.5,\infty)$. The function is positive on the intervals $(-6.8,-2.5)$ and $(0,6)$, with the first interval being approximate. Note that $x=0$ MUST be excluded because $f(0)$ is not positive.

The function has the following maxima and minima:

– an absolute maximum of 11 at about $x=3.5$

– a relative minimum of -2 at $x=-1$

– a relative maximum of 4 at $x=-5$

The function has no absolute minimum.

Even and Odd Functions

As you progress through these notes you will study various specific types of functions, like linear and quadratic functions, polynomial functions, rational functions, and exponential and logarithmic functions. Most of you will go on to take a course in trigonometry, in which you will study trigonometric functions. Overlapping all those kinds of functions, with the exception of exponential and logarithmic functions are two kinds of functions called even and odd functions. What I mean by "overlapping" is that linear, quadratic, polynomial, rational and trigonometric functions can also be even functions, odd functions, or they can be neither even nor odd.

Let's begin with the definitions of even and odd functions:

Even and Odd Functions

- $f(x)$ is said to be an **even function** if $f(-x) = f(x)$ for all x in the domain of f.
- $f(x)$ is said to be an **odd function** if $f(-x) = -f(x)$ for all x in the domain of f.

What is this saying? A function is even if putting in a value or its opposite (negative) both result in the same value. The classic example is $f(x) = x^2$; if we put in, say, 5 or -5 we get 25 out either way, and the same goes for all such pairs of values. More formally, to test whether a function is even, we evaluate it for $-x$ and see if the result is $f(x)$

◇ **Example 2.3(b):** Determine whether the function $g(x) = 3x^4 - x^2 + 6$ is even.

Another Example

Solution: Because $g(-x) = 3(-x)^4 - (-x)^2 + 6 = 3x^4 - x^2 + 6 = g(x)$, the function is even.

A function consisting whole number powers of x, each multiplied by a constant and all added or subtracted, is called a **polynomial function**; we will study such functions in more detail later. The function g is a polynomial function, and note that it can be written as $g(x) = 3x^4 - x^2 + 6x^0$, since $x^0 = 1$. Zero is considered an even number; note that all of the exponents in g are even, and g is an even function. As you might then guess, any polynomial function in which all of the exponents are odd, like $h(x) = x^3 - 7x$, is an odd function. This can be seen by the computation

$$h(-x) = (-x)^3 - 7(-x) = -x^3 + 7x = -(x^3 - 7x) = -h(x).$$

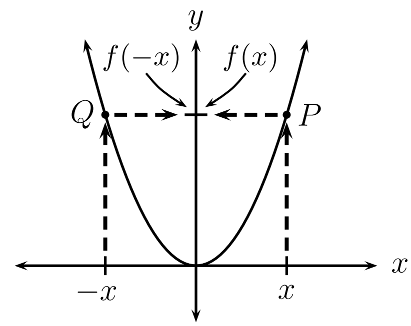
Not all functions are even or odd - in fact, "most" functions are neither even nor odd. An example of a function that is neither even or odd would be $f(x) = x^2 - 5x + 2$. Note that

$$f(-x) = (-x)^2 - 5(-x) + 2 = x^2 + 5x + 2 \neq f(x) \quad \text{and} \quad -f(x) = -x^2 + 5x - 2 \neq f(-x)$$

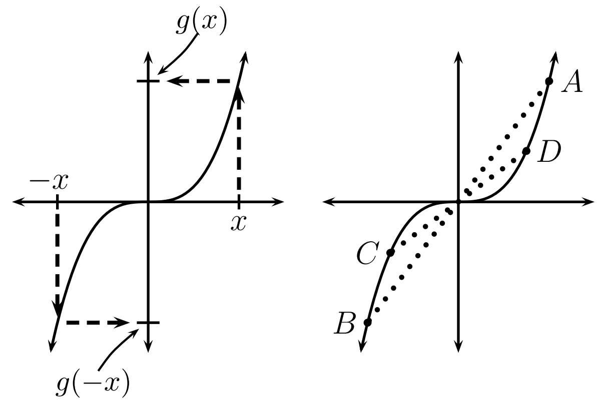
Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, f is neither even nor odd.

Graphs of Even and Odd Functions

Graphs of even and odd functions demonstrate special kinds of "balance" that we call **symmetry**. The graph of an even function is *symmetric with respect to the y axis*, which means that for every point on the graph there is a point on the opposite side of the y-axis and the same distance away. We illustrate this to the right using the function $f(x) = x^2$, an even function. Note that inputs of x and $-x$ both give us the same output; the points P and Q are symmetric with respect to the y-axis.



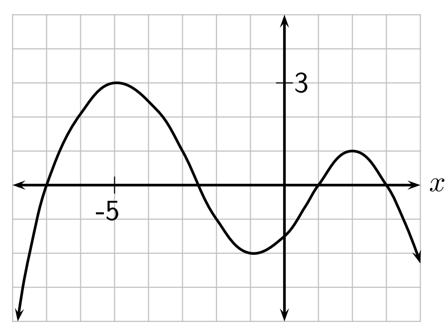
The two graphs to the right are for the function $g(x) = x^3$, an odd function. The first graph illustrates the definition of an odd function: If we input the opposite values x and $-x$ (seen as opposites on the x-axis), the corresponding outputs $g(x)$ and $g(-x)$ can be seen to be opposites on the y-axis. This is stated mathematically by $g(-x) = -g(x)$. The second graph illustrates the kind of symmetry that odd functions have, *symmetry with respect to the origin*. That is, for each point on the curve there is a corresponding point on the opposite side of the origin on a line through the origin, and the same distance from the origin. In the picture shown, the point A corresponds to B, and C corresponds to D.



Section 2.3 Exercises

To Solutions

1. For the function f, whose graph is shown below and to the right, answer the following questions. Both "tails" of the graph continue to spread outward and downward.



   (a) Give all of the x-intercepts and y-intercepts for the function f. Distinguish between the two!

   (b) Give the domain of the function.

   (c) Give the range of the function.

   (d) Give the intervals on which f is increasing.

   (e) Give the intervals on which f is negative.

   (f) Give all relative maxima and minima in the manner used in Example 2.3(a).

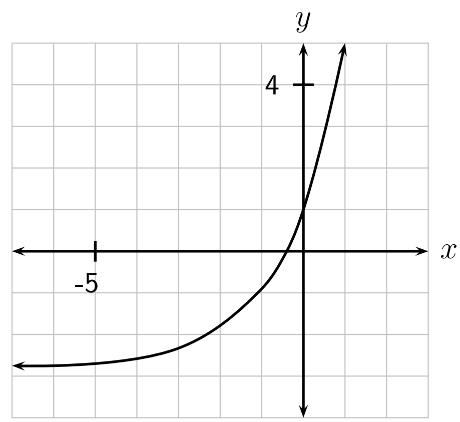
   (g) Does f have an absolute maximum? If so, what is it, and at what x value (or values) does it occur?

   (h) Does f have an absolute minimum? If so, what is it and at what x value (or values) does it occur?

(i) Estimate the value of $f(-4)$.

(j) Estimate all values of x where $f(x) = -1$.

2. For the function g, whose graph is shown below and to the right, answer the following questions. The right tail continues upward and rightward forever. The left tail continues leftward with y values getting closer and closer to, but never reaching, -3.



(a) Give the domain of the function.

(b) Give the range of the function.

(c) Estimate the values of $g(-2)$ and $g(-3)$.

(d) Estimate *all* values of x for which $g(x) = -1$.

(e) Estimate *all* values of x for which $g(x) = 5$.

(f) Give the intervals on which g is increasing.

(g) At what x values does the function g have relative maxima or minima? Answer with appropriate statements.

(h) Give the intervals on which the function is positive.

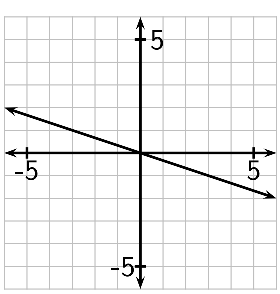
3. Essential to determining from the definition whether a function is even, odd or neither are the facts that

$$(-x)^n = x^n \text{ if } n \text{ is even, and } (-x)^n = -x^n \text{ if } n \text{ is odd.}$$

For each of the following functions, calculate $f(-x)$ by substituting $-x$ for x and using the above two facts. If the result is not $f(x)$ itself (indicating an even function), see if you can factor out a negative to get $-f(x)$, indicating an odd function. Conclude by telling whether the function is even, odd or neither.

(a) $f(x) = 3x + 1$ (b) $f(x) = x^2 - 5$ (c) $f(x) = 5x^3 - 7x + 1$
(d) $f(x) = 10$ (e) $f(x) = x^5 + 4x$

4. Each graph below is that of a function. For each, determine whether the function is odd, even, or neither.

(a)


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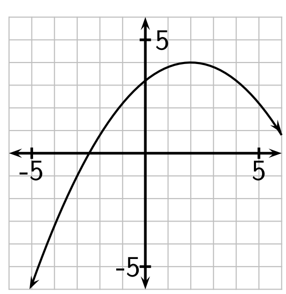
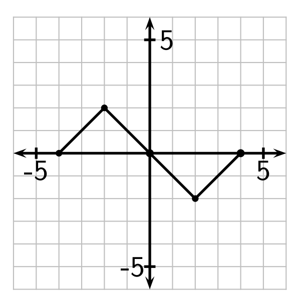
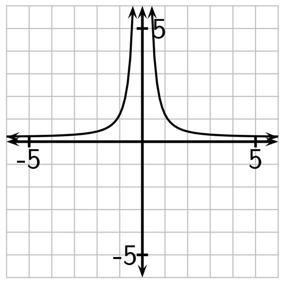
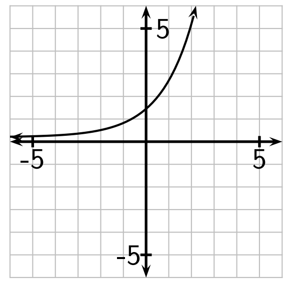
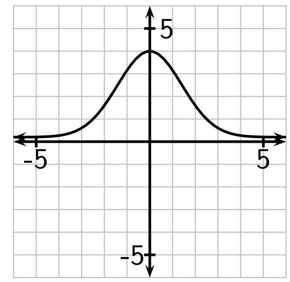


(c)


(d) 

(e) 

(f) 

5. Discuss the existence of relative and absolute maxima and/or minima for quadratic functions.

6. Consider the function $g(x) = -x^2 + 2x + 3$. Do all of the following without actually graphing the function. (You may wish to think about the appearance of the grph, though.)

   (a) What sort of minima or maxima do we know the function has to have?

   (b) Give the value of any maximum or minimum, where it occurs and what kind of maximum or minimum it is.

   (c) Where is the function decreasing?

   (d) Where is the function positive?

   (e) What is the domain of the function?

   (f) What is the range of the function?

   (g) Use technology to graph the function, and use the graph to check your answers to (a)-(d).

7. Suppose that the function h is odd.

   (a) If $h(3) = -7$, what is $h(-3)$?   (b) If $h(5) = 12$, then $h($ _____ $) =$ _____

8. (a) Suppose that $f(4) = 7$ and $f(-4) = 7$ for some function f. Knowing only that, can we say for sure whether or not f is even? Explain.

   (b) For another function g we know that $g(-2) = 5$ and $g(2) = -7$. Can we say for sure whether or not g is odd? Explain.

9. Most of you will take trigonometry, where you will work with two functions called the **sine** and **cosine** functions. When we write $y = \sin x$, the letters sin are like the letter f for a function $f(x)$. So sine is like the box shown on the first page of Section 2.1 illustrating a function as a "machine." With the sine function we can find out what the machine does to a number by simply computing the sine of the number using the sin button on our calculators. The same holds for the cosine function.

   (a) Use your calculator to find $\sin x$ for several positive and negative pairs of the same number. Does the sine function appear to be even, odd, or neither?

(b) Use technology to graph $y = \sin x$. Does the graph support your answer to (a)?

(c) Repeat part (a) for the cosine function.

(d) Use technology to graph $y = \cos x$. Does the graph support your answer to (c)?

10. Sketch the graph of a function that is

- increasing on the intervals $(-\infty, -1)$ and $(2, 3)$ and decreasing everywhere else
- positive on $(-2, 1)$, zero at $x = -2$ and $x = 1$, and negative everywhere else.

11. Suppose that all of the conditions listed in Exercise 10 hold except that the function is also zero at $x = 3$. What sorts of minima and maxima (relative? absolute?) does the function have, and where do they occur? (One location is exact, the other two can only be determined to be in certain intervals - give the intervals in those cases.

12. Use your calculator to graph the function $f(x) = x^4 - 2x^3 - 5x^2 + 6x$. Then write a short paragraph about the function, addressing all of the concepts that have been discussed in this section.

2.4 Mathematical Models: Linear Functions

Outcome/Criteria:

2. (h) Use a linear model to solve problems; create a linear model for a given situation.
   (i) Interpret the slope and intercept of a linear model.

Using a Linear Model to Solve Problems

An insurance company collects data on amounts of damage (in dollars) sustained by houses that have caught on fire in a small rural community. Based on their data they determine that the expected amount D of fire damage (in dollars) is related to the distance d (in miles) of the house from the fire station. (Note here the importance of distinguishing between upper case variables and lower case variables!) The equation that seems to model the situation well is

$$D = 28000 + 9000d$$

This tells us that the damage D is a function of the distance d of the house from the fire station. Given a value for either of these variables, we can find the value of the other.

◇ **Example 2.4(a):** Determine the expected amount of damage from a house fire that is 3.2 miles from the fire station, 4.2 miles from the fire station and 5.2 miles from the fire station.

Solution: For convenience, let's rewrite the equation using function notation, and in the slope-intercept form: $D(d) = 9000d + 28000$. Using this we have

$$D(3.2) = 9000(3.2) + 28000 = 56800, \qquad D(4.2) = 65800, \qquad D(5.2) = 74800$$

The damages for distances of 3.2, 4.2 and 5.2 miles from the fire station are $\$56,800$, $\$65,800$ and $\$74,800$.

Note that in the above example, for each additional mile away from the fire station, the amount of damage increased by $9000.

◇ **Example 2.4(b):** If a house fire caused $47,000 damage, how far would you expect that the fire might have been from the fire station? *Round to the nearest tenth of a mile.*

Solution: Here we are given a value for D and asked to find a value of d. We do this by substituting the given value of D into the equation and solving for d:

$$\begin{aligned} 47000 &= 9000d + 28000 \\ 19000 &= 9000d \\ 2.1 &= d \end{aligned}$$

We would expect the house that caught fire to be about 2.1 miles from the fire station.

◇ **Example 2.4(c):** How much damage might you expect if your house was right next door to the fire station?

Solution: A house that is right next door to the fire station is essentially a distance of zero miles away. We would then expect the damage to be

$$D(0) = 9000(0) + 28000 = 28000.$$

The damage to the house would be $\$28,000$.

Interpreting the Slope and Intercept of a Linear Model

There are a few important things we want to glean from the above examples.

- When we are given a mathematical relationship between two variables, if we know one we can find the other.
- Recall that slope is rise over run. In this case rise is damage, measured in units of dollars, and the run is distance, measured in miles. Therefore the slope is measured in $\frac{\text{dollars}}{\text{miles}}$, or dollars per mile. The slope of 9000 dollars per mile tells us that for each additional mile farther from the fire station that a house fire is, the amount of damage is expected to increase by \$9000.
- The amount of damage expected for a house fire that is essentially right at the station is \$28,000, which is the D-intercept for the equation.

In general we have the following.

Interpreting Slopes and Intercepts of Lines

When an "output" variable depends linearly on another "input" variable,

- the slope has units of the output variable units over the input variable units, and it represents the amount of increase (or decrease, if it is negative) in the output variable for each one unit increase in the input variable,
- the output variable intercept ("y"-intercept) is the value of the output variable when the value of the input variable is zero, and its units are the units of the output variable. *The intercept is not always meaningful.*

The first of these two points illustrates what was pointed out after Example 2.4(a). As the distance increased by one mile from 3.2 miles to 4.2 miles the damage increased by $65800 - 56800 = 9000$ dollars, and when the distance increased again by a mile from 4.2 miles to 5.2 miles the damage again increased by $\$9000$.

When dealing with functions we often call the "input" variable (which is *ALWAYS* graphed on on the horizontal axis) the **independent variable**, and the "output" variable (which is always graphed on the vertical axis) is the **dependent variable**. Using this language we can reword the items in the previous box as follows.

Slope and Intercept in Applications

For a linear model $y = mx + b$,

- the slope m tells us the amount of increase in the dependent variable for every one unit increase in the independent variable
- the vertical axis intercept tells us the value of the dependent variable when the independent variable is zero

Section 2.4 Exercises

To Solutions

1. The amount of fuel consumed by a car is a function of how fast the car is traveling, amongst other things. For a particular model of car this relationship can be modeled mathematically by the equation $M = 18.3 - 0.02s$, where M is the mileage (in miles per gallon) and s is the speed (in miles per hour). *This equation is only valid when the car is traveling on the "open road" (no stopping and starting) and at speeds between 30 mph and 100 mph.*

   (a) Find the mileage at a speed of 45 miles per hour and at 50 miles per hour. Did the mileage increase, or did it decrease, as the speed was increased from 45 mph to 50 mph? By how much?

   (b) Repeat (a) for 90 and 95 mph.

   (c) Give the slope of the line. Multiply the slope by 5. What do you notice?

   (d) Find the speed of the car when the mileage is 17.5 mpg.

   (e) Give the slope of the line, with units, and interpret its meaning in the context of the situation.

   (f) Although we can find the m-intercept and nothing about it seems unusual, why should we not interpret its value?

2. The weight w (in grams) of a certain kind of lizard is related to the length l (in centimeters) of the lizard by the equation $w = 22l - 84$. This equation is based on statistical analysis of a bunch of lizards between 12 and 30 cm long.

   (a) Find the weight of a lizard that is 3 cm long. Why is this not reasonable? What is the problem here?

   (b) What is the w-intercept, and why does it have no meaning here?

   (c) What is the slope, with units, and what does it represent?

3. A salesperson earns \$800 per month, plus a 3% commission on all sales. Let P represent the salesperson's gross pay for a month, and let S be the amount of sales they make in a month. (Both are of course in dollars. Remember that to compute 3% of a quantity we multiply by the quantity by 0.03, the decimal equivalent of 3%.)

   (a) Find the pay for the salesperson when they have sales of \$50,000, and when they have sales of \$100,000.

(b) Find the equation for pay as a function of sales, given that this is a linear relationship.

(c) What is the slope of the line, and what does it represent?

(d) What is the P-intercept of the line, and what does it represent?

4. The equation $F = \frac{9}{5}C + 32$ gives the Fahrenheit temperature F corresponding to a given Celsius temperature C. This equation describes a line, with C playing the role of x and F playing the role of y.

   (a) What is the F-intercept of the line, and what does it tell us?

   (b) What is the slope of the line, and what does it tell us?

5. We again consider the manufacture of Widgets by the Acme Company. The costs for one week of producing Widgets is given by the equation $C = 7x + 5000$, where C is the costs, in dollars, and x is the number of Widgets produced in a week. This equation is clearly linear.

   (a) What is the C-intercept of the line, and what does it represent?

   (b) What is the slope of the line, and what does it represent?

   (c) If they make 1,491 Widgets in one week, what is their total cost? What is the cost for each individual Widget made that week? (The answer to this second question should *NOT* be the same as your answer to (b).)

6. The cost y (in dollars) of renting a car for one day and driving x miles is given by the equation $y = 0.24x + 30$. Of course this is the equation of a line. Explain what the slope and y-intercept of the line represent, *in terms of renting the car*.

7. A baby weighs 8 pounds at birth, and four years later the child's weight is 32 pounds. Assume that the childhood weight W (in pounds) is linearly related to age t (in years).

   (a) Give an equation for the weight W in terms of t. *Test it for the two ages that you know the child's weight, to be sure it is correct!*

   (b) What is the slope of the line, with units, and what does it represent?

   (c) What is the W-intercept of the line, with units, and what does it represent?

   (d) Approximately how much did the child weigh at age 3?

8. As the price of an item goes up, the number of units sold goes down, and this relationship is usually linear. If 5000 Widgets are sold at a price of $\$7.50$ and 3500 Widgets are sold at a price of $\$10.00$, determine an equation for the number x of Widgets sold as a linear function of price p. Note that your equation should have the form $x = mp + b$ - test it for the given values to make sure it works!

9. The temperature T, in degrees Fahrenheit, is linearly related to the number x of times that a cricket chirps in a minute. At $60°$F a cricket chirps 90 times a minute, and at $70°$F a cricket chirps 135 times a minute. Find an equation of the form $T = mx + b$ for temperature as a function of the number of chirps. Give the slope and intercept in exact form (fractions, no decimals).

2.5 Other Mathematical Models

Performance Criteria:

2. (j) Solve a problem using a function whose equation is given.
   (k) Determine the equation of a function modeling a situation.
   (l) Give the feasible domain of a function modeling a situation.

Introduction

In this section we will look at what functions have to do with anyone's concept of reality! First we consider this: When we look at a function in the form $y = x^2 - 5x + 2$, we say that "y is a function of x", meaning that y depends on x.

Often we will use letters other than x and y when using functions for "real life" quantities. Consider, for example, the function $h = -16t^2 + 144t$. This equation gives the height h (in feet) of a rock, thrown upward with a starting velocity of 144 feet per second, at any time t seconds after it is thrown. (The $-16t^2$ is the downward effect of gravity. Note that without gravity the equation would simply be $h(t) = 144t$, or "distance equals rate times time.") Here it is logical to use t and h instead of x and y, so we can see from the letter what quantity it represents.

We'll basically be working with two situations:

- Those for which the equation of the function will be provided. The equation will have to be given because it arises from concepts that you are probably not familiar with at this point of your education.
- Those for which you can (and will) create the equation for the function yourself.

It is relatively simple to work with functions whose equations are given, and you have already done this in parts of Chapter 1 and in the previous section. Usually you are given the input and asked for the output,or vice-versa. Since you are already familiar with that, I have not provided examples, but there are some exercises on this.

Finding Equations of Functions

Let's look at finding the equation of a function. As demonstrated in the next example, a good way to do this is to use the given information to find several input-output pairs *BEFORE* trying to find the equation for the function, then checking your equation with some of the input values for which you have already found output values. This process is similar to what we did to find equations for solving problems by guessing and checking. Let's look at an example.

◇ **Example 2.5(a):** A rectangle starts out with a width of 5 inches and a length of 8 inches. At time zero the width starts growing at a constant rate of 2 inches per minute, and the length begins growing by 3 inches per minute. Give an equation for the area A as a function of how many minutes t it has been growing.

Solution: Suppose that the rectangle had been growing for 6 minutes. Then the width would be $5 + 2(6) = 17$ inches and the length would be $8 + 3(6) = 26$ inches. The area would then be $(17)(26) = 442$ square inches. If the rectangle had been growing for 10 minutes, the width would be $5+2(10) = 25$ inches and the length would be $8+3(10) = 38$ inches. The area would then be $(25)(38) = 950$ square inches.

We now suppose that the rectangle had been growing for t minutes, with t unknown. Based on what we have just seen, we know that to get the width of the rectangle we would take the initial width of 5 inches and add on $2t$, which represents how much the width had grown. So the width would be $5 + 2t$. Similarly, the length would be $8 + 3t$. The area is then the length times the width, so we now have

$$A = (5 + 2t)(8 + 3t) = 40 + 15t + 16t + 6t^2 = 6t^2 + 31t + 40.$$

We can show that A is a function of t by writing $A(t) = 6t^2 + 31t + 40$, and we can check this against the results we used to get the formula. For example,

$$A(10) = 6(10)^2 + 31(10) + 40 = 600 + 310 + 40 = 950,$$

which agrees with what we got before we had an equation for the function.

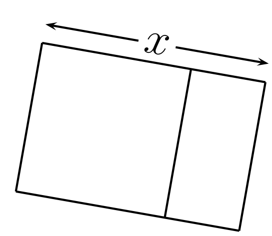
Feasible Domains of Functions

So far we have considered the domain of a function from a mathematical point of view. When using a function to describe a real-world phenomenon, however, we should recognize that not all values of the input variable are reasonable for the situation. Consider the equation $h(t) = -16t^2 + 144t$ describing the height of a projectile above the ground as a function of time after it was launched/thrown. Since the projectile is launched at time zero, it makes no sense to let t be negative. Furthermore, the rock will eventually hit the ground and stop, so the equation is no good for times after that. We'll call the times for which the equation *IS* valid the **feasible domain** of the function. To find when the rock hits the ground we substitute zero for the height and solve:

$$\begin{aligned} 0 &= -16t^2 + 144t \\ 0 &= -16t(t - 9) \\ t = 0 \quad &\text{or} \quad t = 9 \end{aligned}$$

The solution $t = 0$ is telling us that the projectile starts on the ground at time zero, and the solution $t = 9$ is giving the time that the projectile comes back down and hits the ground. The feasible domain of this function is then $[0, 9]$ or $\{t \mid 0 \leq t \leq 9\}$. Note that the feasible domain is far more restrictive than the mathematical domain, which allows all real numbers for t.

◇ **Example 2.5(b):** A farmer is going to create a rectangular field with two "compartments", as shown to the right. He has 2400 feet of fence with which to do this. Letting x represent the dimension shown on the diagram, write an equation for the total area A of the field as a function of x, simplifying your equation as much as possible. Give the feasible domain for the function.



Solution: Lets find the area for a value of x to help us determine an equation for the area as a function of x. Suppose that $x = 500$. Then the opposite side of the field is 500 also, and we have then used $2(500) = 1000$ feet of our fence. That leaves $2400 - 2(500) = 1400$ feet for the remaining *three* sections of fence. That means each of them has length $\frac{1400}{3}$ feet. Since the area of the field is length times width, the area is then

$$A = 500\left(\frac{1400}{3}\right) = 500\left(\frac{2400 - 2(500)}{3}\right)$$

It is not necessary that we actually calculate the area, as we can see now how it is obtained. At this point we can substitute x everywhere for 500 to get

$$A = x\left(\frac{2400 - 2x}{3}\right) = x(800 - \tfrac{2}{3}x) = 800x - \tfrac{2}{3}x^2$$

We now have the equation for the area as a function of x: $A = 800x - \frac{2}{3}x^2$. Now what is the feasible domain of the function? Clearly x must be greater than zero; it can't be negative, and if it is zero there will be no field! Is there an upper limit on x? Well, x certainly can't be larger than 2400, since that is how much fence we have, but we can do better than that. Notice that there are actually two pieces of fence with length x, and *together* they have to be less than 2400. (Again they can't total exactly 2400 because then there would be nothing left for the other three pieces of fence.) Therefore x itself must be less than half of 2400, or less than 1200. In conclusion, the feasible domain is $\{x \mid 0 < x < 1200\}$. This can instead be given using interval notation, as $(0, 1200)$.

Section 2.5 Exercises

To Solutions

1. The height h (in feet) of a rock thrown upward with a starting velocity of 144 feet per second at any time t seconds after it is thrown is given by the equation the function $h = -16t^2 + 144t$. (The $-16t^2$ is the downward effect of gravity. Note that without gravity the equation would simply be $h = 144t$, or "distance equals rate times time.")

   (a) How high will the rock be after 3 seconds?

   (b) How long will it take for the rock to hit the ground? (**Hint:** What is the height of the rock when it hits the ground?)

   (c) Later we will see that the rock will take the same amount of time to reach its high point as it does to come back down. Use this fact to find out *the maximum height the rock will reach*.

   (d) Find the time at which the rock is at a height of 224 feet. You should get two answers - how do you explain this?

2. Since the height of the rock can be found for any time by using the equation, we see that h is a function of t and we could write $h(t) = -16t^2 + 144t$.

   (a) *In words*, tell what $h(4)$ represents, including units. *This DOESN'T mean find* $h(4)$.

   (b) Find a value (or values) of t such that $h(t) = 150$. Give your answer(s) in decimal form, rounded to the nearest hundredth.

   (c) Write a sentence summarizing what you found out in part (b).

   (d) Why does it make no sense to compute $h(11)$? (**Hint:** See your answer to 1(c).)

3. The length L of the skid mark left by a car is proportional to the square of the car's speed s at the start of the skid. This means that there is an equation $L = ks^2$ relating L and s, where k is some constant (fixed number).

   (a) We need to find a value for k. We know that a car traveling 60 miles per hour leaves a 172 foot skid mark. Insert those values and solve for k. Round to the thousandth's place.

   (b) What are the units of k? Note that they must be such that they multiply times the units of s^2 to give the units for L.

   (c) Since k is a constant, it's value doesn't change from what you obtained in (a). Insert the value you obtained to get an equation relating L and s. Use your equation to find out the length of the skid mark that will be left by a car traveling at 50 miles per hour.

   (d) Use your equation to find out how fast the car would be going to leave a skid mark with a length of 100 feet. **Round to the tenth's place**.

4. The intensity I of light at a point is inversely proportional to the square of the point's distance d from the light. This means that I and d are related by an equation of the form $I = \dfrac{k}{d^2}$ for some constant k. The intensity of a light is 180 candlepower at a distance of 12 feet.

   (a) Assuming that a greater candlepower indicates brighter light, should the intensity at 18 feet be less or more than 180 candle power?

   (b) Use a method like that of the previous exercise to determine the intensity of the light at a distance of 18 feet. Does your answer agree with what you speculated in (a)?

5. Clearly, as the price of an item goes up, fewer will be sold, so *the number sold is a function of the price*. An economist for the Acme Company determines that the number w of widgets that can be sold at a price p is given by the equation $w = 20,000 - 100p$.

(a) Find the number of widgets that can be sold at a price of \$50, and the number that can be sold at \$100. **Show the arithmetic of the units.** Does the number sold increase or decrease, and by how many, when the price is increased from \$50 to \$100?

(b) Find the amount of increase or decrease when the price is raised from \$100 to \$150. Tell whether it is an increase or decrease when you give your answer.

(c) At what price would 7,000 widgets be sold?

(d) What is the feasible domain of this function? (**Hint:** Neither the price nor the number of Widgets can be negative.)

6. The amount of money brought in by sales is called the **revenue**. If 47 items that are priced at \$10 are sold, then the revenue generated is \$470. Consider the widgets from the previous exercise. (You will need to use the function from the previous exercise for most of this exercise!)

   (a) What will the revenue be if the price of a widget is \$80? Show/explain how you got your answer.

   (b) What will the revenue be if the price is \$110? \$140?

   (c) Note that the price increase from \$80 to \$110 is \$30. What is the corresponding increase in revenue? What is the increase in revenue as the price increases another \$30, from \$110 to \$140?

   (d) You should see by now that the revenue depends on the price set for a widget; that is, the *revenue is a function of the price.* Write an equation for this function, and simplify your equation.

   (e) Use your equation to find the revenue if the price of a widget is \$80, and check your answer with what you got in part (a). If they don't agree, then your equation is probably incorrect. (It could be that you got part (a) wrong, but that is less likely!)

   (f) What should the price be in order have a revenue of \$100,000? Round your answer to the nearest ten cents.

7. Biologists have determined that when x fish of a particular type spawn, the number y of their offspring that survive to maturity is modeled reasonably well by the equation

$$y = \frac{3700x}{x + 400}$$

.

   (a) How many survivors will there be if there are 2500 spawners?

   (b) Your answer to (a) should be rounded to the nearest whole number. Why?

(c) How many spawners are needed to get 3000 survivors?

(d) How many spawners must there be so that the number of survivors equal the number of spawners?

8. If a person takes a 30 milligram dose of a particular medicine, the amount of medicine left in the person's body at t hours after the medicine was taken is given by the equation $A(t) = 30(0.7)^t$. This is an example of an **exponential function**; we'll study them in more detail in Chapter 6. You may find it less confusing to work with the equation in the form $A = 30(0.7)^t$. Find the amount of medicine left in a person's body 3 hours after taking the medicine. Round to the tenth's place and give units with your answer. Remember that you must apply the exponent to the 0.7 *before* you multiply by 30, because of the order of operations.

9. Two people, who are 224 feet apart, start walking toward each other at the same time. One is walking at a rate of 1.5 feet per second and the other at 2 feet per second. Let time zero be when they start walking toward each other; clearly their distance apart is a function of the time elapsed since time zero.

   (a) Find out how far apart the people are at times $t = 0, 1, 3$ and 8 seconds. You may wish to record your results in a small table of two columns, the first labeled t and the second labeled as d.

   (b) Write a brief explanation, *in words*, of how to find the distance d for any time t.

   (c) Write an equation for the distance d as a function of the time t. *Give units with any constants in your equation.* Test it by finding $d(0)$, $d(3)$ and $d(8)$, *showing the arithmetic of the units as well as the numbers*, and making sure the results agree with what you got in (a).

   (d) Use your equation from (c) to determine how far apart the people will be after 20 seconds. *try showing how the units of the constants are worked with in this case.*

   (e) When will the people be 130 feet apart? When will they meet?

   (f) What is the mathematical domain of the function? What is the feasible domain?

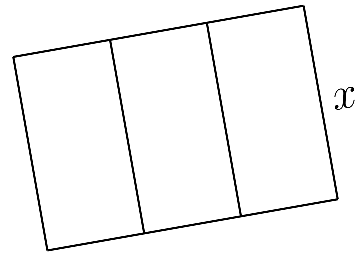
10. Recall that for a right triangle with legs of lengths a and b and hypotenuse with length c the Pythagorean Theorem tells us that $a^2 + b^2 = c^2$. Suppose that the legs of a right triangle start out with lengths 3 feet and 5 feet, and they both increase at 4 feet per hour.

   (a) Find the perimeter for several times.

   (b) Find, *but don't simplify*, a formula for the perimeter P of the triangle as a function of time t.

   (c) Simplify your formula from part (b).

11. A farmer is going to create a rectangular field with three "compartments", as shown to the right. He has 1000 feet of fence with which to do this. Letting x represent the dimension shown on the diagram, write an equation for the total area A of the field as a function of x. Simplify your equation as much as possible. **HINT:** To find your equation, experiment with a few specific numerical values of x, like you did (with t) in the previous exercise.



12. What is the mathematical domain of the function from the previous exercise? What is the feasible domain?

13. Billy and Bobby have a lawn service (called Billybob's Lawn Care). Billy, the faster of the two, can mow a three lawns every two hours by himself, and Bobby can only mow two lawns in two hours by himself.

    (a) How many lawns can each boy mow in one hour by themselves?

    (b) Find an equation for the number L of lawns that the two can mow in t hours *when working together*.

    (c) How long does it take the two of them together to mow one lawn? Give your answer in decimal form, remembering that it is in hours. Then give the answer in minutes.

    (d) What is the mathematical domain of the function? What is the feasible domain?

14. A 10 foot ladder is leaned against the side of a building. Note that the base of the ladder can be put at different distances from the wall, but changing the distance between the base of the ladder and the wall will cause the height of the top of the ladder to change.

    (a) There are three lengths or distances of interest here. What are they? Are any of them variables and, if so, which ones? Are any of them constants and, if so, which ones?

    (b) As the distance of the base of the ladder from the wall decreases, what happens to the top of the ladder?

    (c) Clearly the height to the top of the ladder is related to the distance of the base of the ladder from the wall. Write an equation relating these two variables, using h for the height and d for the distance of the base of the ladder from the wall.

    (d) Use your equation from (c) to determine the height of the ladder when the base is 6 feet from the wall. Then use your equation to determine the height of the ladder when the base is 2 feet from the wall.

    (e) You second answer to (d) should not be a whole number. Give it as a decimal, rounded to the nearest tenth of a foot. Then give it in simplified square root form.

    (f) Give the advantages and disadvantages of each of the two forms that you gave for your answer to (e).

(g) Give your answer in inches, rounded to the nearest whole inch. Then give it in feet and inches.

15. Consider the situation from the previous exercise.

    (a) Note that you can solve your equation from (c) to get $h = \pm\sqrt{100 - d^2}$, where h is the height of the top of the ladder and d is the distance from the bottom of the ladder to the base of the wall. Why can we disregard the negative sign?

    (b) Find the feasible domain for the function $h = \sqrt{100 - d^2}$.

16. The width of a rectangle is one-third its length.

    (a) Find the area of the rectangle as a function of the length of the rectangle. This means you should have an equation of the form A equals some mathematical expression containing l.

    (b) Find the area as a function of the width.

    (c) Find the perimeter P as a function of the length, then as a function of the width. (Remember that perimeter is the distance around the figure.)

17. The Acme Company has a factory that produces Widgets. The company spends \$5000 per week operating their factory (mortgage payments, heating, etc.). Additionally, it takes \$7 in parts, energy and labor to produce a Widget.

    (a) What are Acme's total costs for a week in which 8,000 Widgets are produced? What are the costs for a week in which 12,000 Widgets are produced?

    (b) Clearly the costs for a week are a function of the number of Widgets produced. Let C represent the costs for a week and let x represent the number of Widgets produced that week. Write an equation for the costs as a function of the number of Widgets produced.

18. Suppose that the price of gasoline is \$2.70 per gallon, and you are going to put some number g gallons of gas in your tank.

    (a) Write an equation for the cost C (in dollars) to you as a function g. gallons of gas in your car.

    (b) What does the feasible domain of this function depend on? Be as specific as possible!

2.6 A Return to Graphing

Performance Criteria:

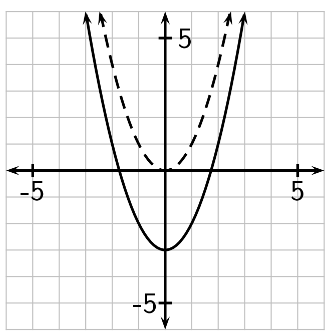
5. (m) Given the graph of a function, sketch or identify translations of the function.

   (n) Given the graph of a function, sketch or identify vertical reflections of the function.

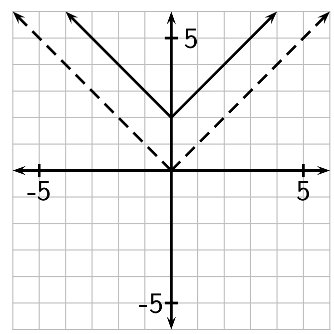
Let's begin with an example.

◇ **Example 2.6(a):** Sketch the graph of $y = x^2$, dashed. On the same grid, graph $f(x) = x^2 - 3$, solid. On a different grid, repeat for $y = |x|$ and $g(x) = |x| + 2$.

Solution: We'll skip the details of finding points to plot, which was covered thoroughly in Chapter 1. The graphs are shown below.



$y = x^2$ and $f(x) = x^2 - 3$



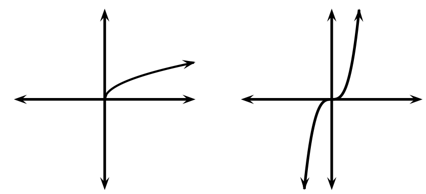
$y = |x|$ and $g(x) = |x| + 2$

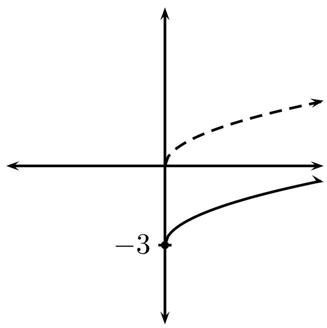
The point of this example, of course, was not simply to graph the given functions. We can observe the effect of adding or subtracting a constant on the graph of a function. We see that adding two to $y = |x|$ causes the graph to move up two units, and subtracting four from $y = x^2$ moves the graph down by four units. We call any movement that does not change the shape or orientation of a graph a **translation** of of the graph. So we are seeing that adding or subtracting a constant to a function translates (or "shifts") the graph up or down by the absolute value of the constant, with the shift being upward if the constant is positive and downward if the constant is negative.

Taking advantage of this fact, and others we'll see in this section, requires knowing the graphs of some common functions. At this point the reader should review the graphs of some basic equations (which are of course functions) in Section 1.7. Here is another example demonstrating what we discovered above.

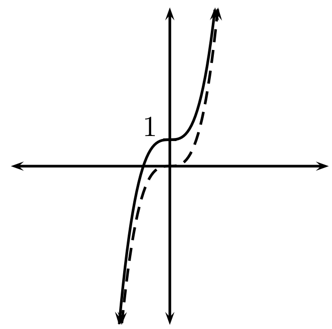
◇ **Example 2.6(b):** Sketch the graphs of $y = \sqrt{x} - 3$ and $y = x^3 + 1$.

Solution: The graphs of $y = \sqrt{x}$ and $y = x^3$ are shown to the right. To obtain the graph of $y = \sqrt{x} - 3$ we simply shift the graph of $y = \sqrt{x}$ down by three units, and we obtain the graph of $y = x^3 + 1$ we shift the graph of $y = x^3$ up by one unit. The resulting graphs are shown at the top of the next page.





$y = \sqrt{x}$ (dashed) and
$y = \sqrt{x} - 3$ (solid)

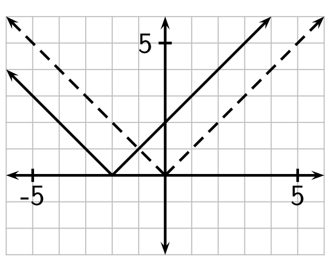


$y = x^3$ (dashed) and
$y = x^3 + 1$ (solid)

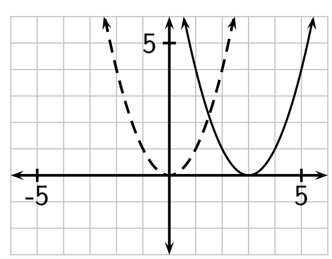
For the next example, compare the functions being graphed with those graphed in Example 2.6(a), and observe carefully the difference between the functions and the resulting graphs.

◇ **Example 2.6(c):** Sketch the graph of $y = |x|$, dashed. On the same grid, graph $g(x) = |x + 2|$, solid. On a different grid, repeat for $y = x^2$ and $h(x) = (x - 3)^2$.

Solution: We recall the idea that for the function g the value $x = -2$ is important because it makes $x + 2$ zero. If we evaluate g for some values of x on either side of -2 and plot the resulting points and graph, we get the solid graph below and to the left. Similarly, the critical value of x for the function h is $x = 3$ because it makes $x - 3$ zero. Evaluating for values near $x = 3$ and plotting results in the graph to the right below.

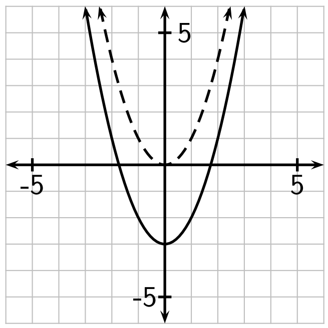


$y = |x|$ and $g(x) = |x + 2|$

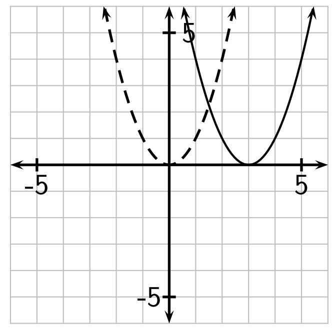


$y = x^2$ and $h(x) = (x - 3)^2$

What can we gather in general from these examples? Let's compare the second graph above with the one from Example 2.6(a):



$y = x^2$ and $f(x) = x^2 - 3$



$y = x^2$ and $h(x) = (x-3)^2$

In both cases we are modifying $y = x^2$ by subtracting three, but the difference is this: In $f(x) = x^2 - 3$ we are subtracting three *after* squaring, and in $h(x) = (x-3)^2$ we are subtracting three *before* squaring. In the first case we simply obtain y values for $y = x^2$ and subtract three from all of them, causing the graph to shift downward by three units. In the second case, evaluating $h(x) = (x-3)^2$ for $x = 3$ gives the same y value as evaluating $y = x^2$ at $x = 0$. In fact, evaluating $h(x) = (x-3)^2$ for any x value gives the same result as evaluating $y = x^2$ for a value of x that is three less. This may be a bit confusing, but the net result is that *subtracting three before squaring results in shifting the graph to the right by three*. Let's summarize all of this:

Translations (Shifts) of Graphs:

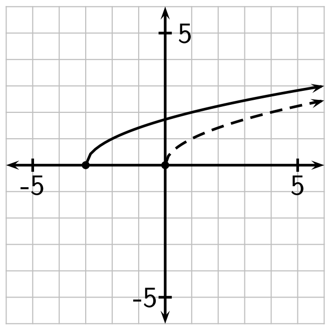
Let a and b be *positive* constants and let $f(x)$ be a function.

- The graph of $y = f(x+a)$ is the graph of $y = f(x)$ shifted a units to the left. That is, what happened at zero for $y = f(x)$ happens at $x = -a$ for $y = f(x+a)$.
- The graph of $y = f(x-a)$ is the graph of $y = f(x)$ shifted a units to the right. That is, what happened at zero for $y = f(x)$ happens at $x = a$ for $y = f(x-a)$.
- The graph of $y = f(x) + b$ is the graph of $y = f(x)$ shifted up by b units.
- The graph of $y = f(x) - b$ is the graph of $y = f(x)$ shifted down by b units.

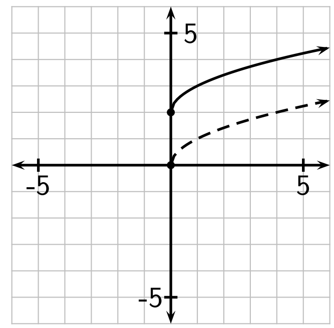
Note that when we add or subtract the value a *before* the function acts, the shift is in the x direction, and the direction of the shift is the opposite of what our intuition tells us should happen. When we add or subtract b *after* the function acts, the shift is in the y direction and the direction of the shift is up for positive and down for negative, as our intuition tells us it should be.

◇ **Example 2.6(d):** Sketch the graphs of $y = \sqrt{x+3}$ and $y = \sqrt{x} + 2$.

Solution: The graph of $y = \sqrt{x}$ is shown as the dashed curve on both grids below. For $y = \sqrt{x+3}$ we are adding three *before* taking the root, so the graph is shifted three units to the left, as shown on the left grid below. For $y = \sqrt{x} + 2$, two is added *after* taking the square root, so the graph is shifted up by two units. This is shown on the grid to the right below.



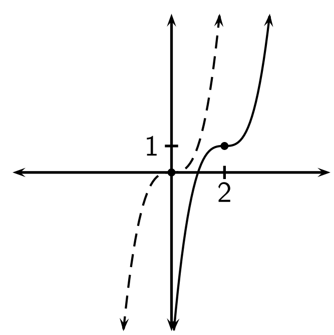
$y = \sqrt{x+3}$ (solid)



$y = \sqrt{x} + 2$ (solid)

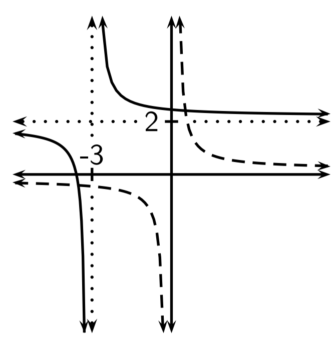
◇ **Example 2.6(e):** Sketch the graph of $y = (x-2)^3 + 1$ on the same grid with the graph of $y = x^3$.

Solution: We first note that this is a modification of $y = x^3$, whose graph is the dashed curve to the right. Note that we are modifying by first subtracting two *before* cubing and adding one *after* cubing. Thus the graph is moved two units to the right and one unit up. The final result is seen in the solid graph to the right.



◇ **Example 2.6(f):** Sketch the graph of $y = \dfrac{1}{x+3} + 2$ on the same grid with the graph of $y = \dfrac{1}{x}$.

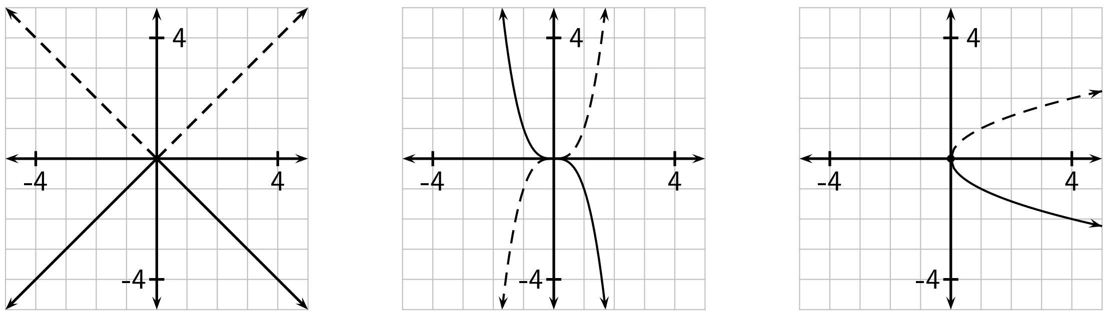
Solution: This is a modification of $y = \dfrac{1}{x}$, shown as the dashed curve to the right. In this case we are modifying by adding three before the "one over," and adding two afterward. The resulting graph is moved three units to the left and two units up. The final result is seen in the solid graph to the right.



The vertical and horizontal dotted lines in the above graph *ARE NOT* parts of the graph of the equation $y = \dfrac{1}{x+3} + 2$, but they help us see where the graph (the solid curve) is located. The vertical, line, whose equation is $x = -3$, is called a **vertical asymptote** of the graph, and the horizontal line $y = 2$ is called a **horizontal asymptote**.

Vertical Reflections (Flips) of Graphs

On the three grids below we see the graphs of $y = -|x|$, $y = -x^3$ and $y = -\sqrt{x}$ (solid), along with $y = |x|$, $y = x^3$ and $y = \sqrt{x}$ (dashed).

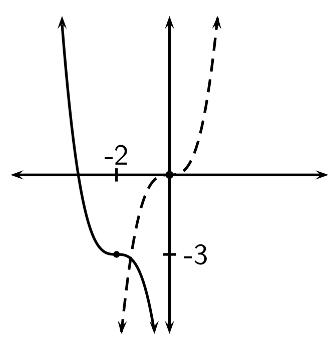


$y = |x|$ and $y = -|x|$ $\qquad$ $y = x^3$ and $y = -x^3$ $\qquad$ $y = \sqrt{x}$ and $y = -\sqrt{x}$

It may not be readily apparent at first glance what is happening here, but the effect of multiplying a function by negative one (which is what is really happening when we make the function negative) is to flip its graph vertically over the x-axis. We call this a **reflection** of the graph over the x-axis. We will often see such reflections combined with the previously seen translations.

◇ **Example 2.6(g):** Sketch the graph of $y = -(x+2)^3 - 3$ on the same grid with the graph of $y = x^3$.

Solution: The graph of $y = x^3$ is the dashed curve to the right. The effect of adding two before cubing is to shift the graph to the left by two units. The multiplication by negative one takes place *after* the cubing (order of operations!) and results in the graph being flipped vertically over the x-axis. Finally, three is subtracted at the end, shifting the graph down by three units. The final result is the solid graph to the right.



Section 2.6 Exercises

To Solutions

You will check some of your answers for this section by graphing your results on either a graphing calculator or using an online grapher. I would again suggest

www.desmos.com,

which allows you to graph two functions on the same graph, as we have been doing in this section.

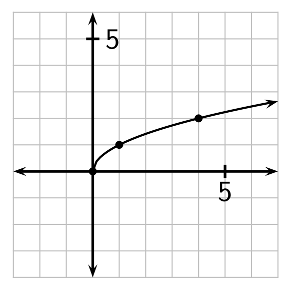
1. For each of the quadratic functions below,

   - sketch the graph of $y = x^2$ **accurately** on a coordinate grid,
   - **on the same grid**, sketch what you think the graph of the given function would look like,
   - graph $y = x^2$ and the given function together on your calculator or Desmos to check your answer

   (a) $f(x) = (x+1)^2$ (b) $y = x^2 - 4$

   (c) $g(x) = (x+2)^2 - 1$ (d) $h(x) = -(x-4)^2 + 2$

2. The graph of $y = \sqrt{x}$ is shown to the right, with the three points $(0,0)$, $(1,1)$ and $(4,2)$ indicated with dots. For each of the following, sketch what you think the graph of each function would look like, along with a dashed copy of the graph of $y = \sqrt{x}$. On the graph of the new function, indicate three points showing the new positions of the three previously mentioned points. Check your answers in the back, making sure to see that you have the correct points plotted.



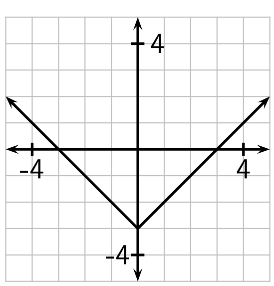
$y = \sqrt{x}$

   (a) $y = \sqrt{x-1}$ (b) $f(x) = \sqrt{x} - 3$

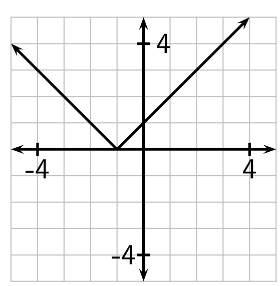
   (c) $g(x) = -\sqrt{x-2} + 4$

3. For each of the graphs shown, give a function of the form $y = \pm|x \pm a| \pm b$ whose graph would be the one shown. Check your answers by graphing them with some sort of technology and seeing if the results are as shown.

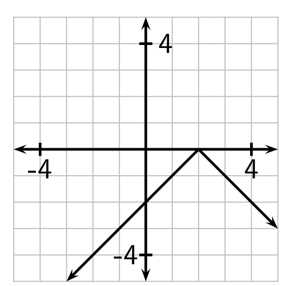
(a)



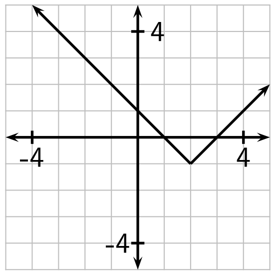
(b)



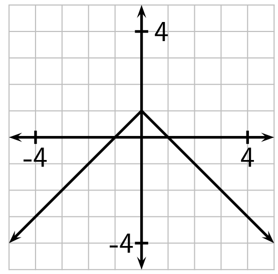
(c)



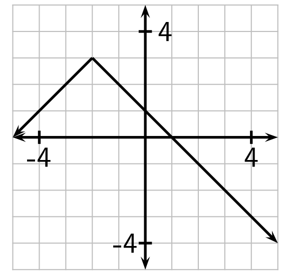
(d)



(e)



(f)



4. For each of the following, graph $y = x^3$ and the given function on the same graph. Check your answers in the solutions.

(a) $f(x) = -x^3$ (b) $y = x^3 + 4$ (c) $g(x) = (x-1)^3$

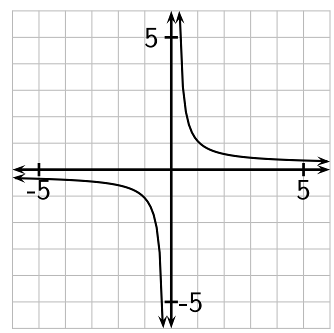
(d) $y = (x+3)^3 + 1$ (e) $h(x) = -(x+2)^3$

5. For each of the following, graph $y = \frac{1}{x}$ (its graph is shown below and to the right) and the given function on the same graph, with the graph of $y = \frac{1}{x}$ dashed.

(a) $y = \dfrac{1}{x+3}$ (b) $y = \dfrac{1}{x} + 3$

(c) $y = -\dfrac{1}{x}$ (d) $y = \dfrac{1}{x+2} + 1$

(e) $y = -\dfrac{1}{x-2}$



$y = \dfrac{1}{x}$

6. Give the domain and range of each function without graphing them (you may wish to think about what their graphs would look like, though).

(a) $y = (x+3)^2 - 2$ (b) $f(x) = \sqrt{x+3}$ (c) $g(x) = -x^2 + 5$

(d) $h(x) = \dfrac{1}{x-2}$ (e) $y = -|x+4| + 7$ (f) $y = \sqrt{x-2} + 1$

2.7 Chapter 2 Exercises

To Solutions

1. For each of the following,

   - attempt to determine the domain of the function algebraically
   - give the domain using either interval notation or set builder notation
   - Graph the function using technology, and ee if the graph seems to support your answer
   - check your answer in the back of the book

   (a) $f(x) = \dfrac{x+3}{x^2 - x}$

   (b) $h(x) = \sqrt{x+2}$

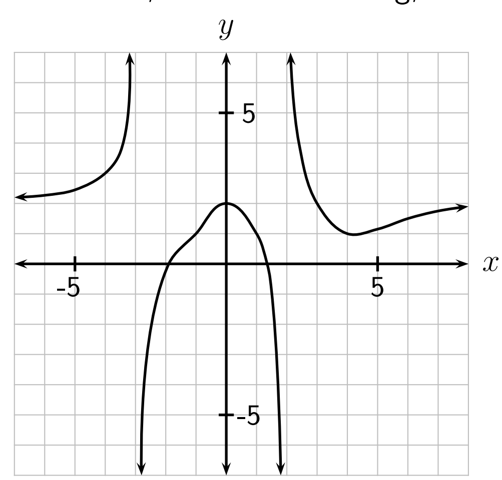
   (c) $g(x) = \dfrac{8}{x^2 - 1}$

   (d) $f(x) = \dfrac{x+2}{x^2 - 5x + 4}$

   (e) $g(x) = \sqrt{3x - 2}$

   (f) $y = \dfrac{x^2}{x^2 + 3}$

2. The graph of a function h is shown below and to the right. The left and right tails continue outward with y values getting closer and closer to, but never reaching, 2. The tails going up and down continue that way, getting ever closer to, but never touching, the vertical lines $x = -3$ and $x = 2$.



   (a) Give the domain of the function.

   (b) Give the range of the function.

   (c) Give the values of $h(-4)$ and $h(4)$.

   (d) Give *all* values of x for which $h(x) = 1$.

   (e) Give *all* values of x for which $h(x) = 2$.

   (f) Give the intervals on which h is decreasing.

   (g) Give the intervals where h is negative.

   (h) Does h have any relative maxima or minima? If so, give the values of x where they are, and give the function value at each of those x values.

   (i) Repeat (h) for absolute maxima and minima.

A Solutions to Exercises

A.2 Chapter 2 Solutions

Section 2.1 Solutions Back to 2.1 Exercises

1. (a) $h(12) = 17$ (b) $x = \frac{19}{2}$

2. (a) $f(3) = 9,\ f(-5) = 65$ (b) $x = -\frac{3}{2},\ 3$ (c) $x = \frac{3}{4} \pm \frac{\sqrt{17}}{4}$ or $x = -0.28,\ 1.78$

3. (a) $x = 1, 6$ (b) $g(-2) = -3$ (c) $x = -0.35,\ 3.95$

4. (a) $f(6) = \frac{11}{2}$ (b) $x = \frac{16}{3}$

5. (a) (a) $h(-4) = 3$ (b) $h(2) \approx 4.58$
   (c) $h(7)$ doesn't exist, because we can't take the square root of a negative number.
   (d) We must make sure that $25 - x^2 \geq 0$. This holds for all values of x between -5 and 5, including those two.

6. $h(0) = h(-43) = h(3\frac{4}{7}) = 12$

7. (a) x-intercept: $x = \frac{7}{2}$ y-intercept: $y = -7$
   (b) x-intercepts: $x = 0, \frac{3}{2}$ y-intercept: $y = 0$
   (c) x-intercept: $x = \frac{1}{2} + \frac{\sqrt{13}}{2},\ \frac{1}{2} - \frac{\sqrt{13}}{2}$ y-intercept: $y = -\sqrt{3}$
   (d) x-intercept: $x = -\frac{4}{3}$ y-intercept: $y = -2$
   (e) x-intercept: $x = -5, 5$ y-intercept: $y = 5$
   (f) x-intercept: there isn't one y-intercept: $y = 12$

8. (a) $f(x+1) = x^2 - x - 2$ (b) $g(2x) = \sqrt{2x+1}$
   (c) $f(-2x) = 4x^2 + 6x$ (d) $g(x-4) = \sqrt{x-3}$

9. (a) $g(a-3) = a^2 - a - 6$ (b) $g(x+h) = x^2 + 2xh + h^2 + 5x + 5h$
   (c) $\dfrac{g(x+h) - g(x)}{(x+h) - x} = 2x + h + 5$

10. (a) The x-intercepts are $x = -7,\ 1,\ 3,\ x \approx -2.5,$ and the y-intercept is $y \approx -1.5$
    (b) $f(-3) = 1,\ f(-2) = -1,\ f(3) = 0$ (c) $x = -6, -3.5$ (d) $x = -5$

11. (a) $f(-3) = 1,\ f(-1) = -2,\ f(1) = 4$ (b) $x \approx -6.6,\ -3,\ 0.3,\ 5.9$ (c) $f(0) = 0$
    (d) $x = -5,\ 1,\ 1.1,\ \approx 5.5$ (e) $x \approx 2.7,\ 4.3$

12. (a) $f(4) = g(4) = 4$ (c) $x = 2$

13. $t = 2$ (-2 doesn't check) 14. $y = 3$ (1 doesn't check)

18. $f(x+h) = x^2 + 2xh + h^2 - 5x - 5h + 2$ 19. $g(x+h) = 3x + 3h - x^2 - 2xh - h^2$

Section 2.2 Solutions

Back to 2.2 Exercises

1. (a) $\{x \mid 2\frac{1}{2} \leq x \leq 13\}$ (b) $\{x \mid -12 < x \leq -2\}$
   (c) $\{x \mid -100 < x < 100\}$ (d) $\{x \mid x > 3\}$ (e) $\{x \mid x \leq -5\}$
2. (a) $[2\frac{1}{2}, 13]$ (b) $(-12, -2]$ (c) $(-100, 100)$ (d) $(3, \infty)$ (e) $(-\infty, -5]$
3. (a) $(-\infty, 3) \cup [7, \infty)$ (b) $(-\infty, -4) \cup (4, \infty)$ (c) $(-\infty, 7) \cup (7, \infty)$
4. (a) $\{x \mid x < 3 \text{ or } x \geq 7\}$ (b) $\{x \mid x < -4 \text{ or } x > 4\}$ (c) $\{x \mid x \neq 7\}$
5. (a) $[-2, \infty)$ or $\{x \mid x \geq -2\}$ (b) $\{x \mid x \neq 1\}$ (c) $\{x \mid x \neq -5, 2, \}$
6. (a) The domain is $[-3, \infty)$ and the range is $[-1, \infty)$
   (b) The domain is $\{x \mid x \neq 2\}$ and the range is $\{y \mid y \neq 0\}$
   (c) The domain is all real numbers and the range is $(-2, \infty)$
   (d) The domain is all real numbers and the range is $(0, 1]$
7. (a) $[4, \infty)$ $\{x \mid x \geq 4\}$ (b) $(-\infty, 4]$ or $\{x \mid x \leq 4\}$ (c) $\{x \mid x \neq 3\}$
   (d) $[-5, 5]$ or $\{x \mid -5 \leq x \leq 5\}$ (e) $\{x \mid x \neq -5\}$ (f) $\{x \mid x \neq \frac{2}{3}\}$
   (g) $\{x \mid x \neq 1, 4\}$ (h) $[1, 5) \cup (5, \infty)$ or $\{x \mid 1 \leq x \text{ and } x \neq 5\}$ (i) $\mathbb{R}$ or all real numbers
8. (a) $\{x \mid x \geq 3\}$, $[3, \infty)$ (b) $\{x \mid x \leq 3\}$, $(-\infty, 3]$
   (c) The domain of h does not include 3, because $x = 3$ would cause a zero in the denominator. The domain of h is $\{x \mid x < 3\}$, $(-\infty, 3)$
9. (a) all real numbers (b) $(-\infty, 5]$ (c) $[-9, \infty)$

Section 2.3 Solutions

Back to 2.3 Exercises

1. (a) The x-intercepts are $x = -7, 1, 3,$ $x \approx -2.5$. The y-intercept is $y \approx -1.5$.
   (b) The domain of f is all real numbers.
   (c) The range of f is $(-\infty, 3]$
   (d) f is increasing on the intervals $(-\infty, -5)$ and $(-1, 2)$.
   (e) f is negative on $(-\infty, -7)$, $(-2.5, 1)$ and $(3, \infty)$.
   (f) There is a relative maximum of 3 at $x = -5$, a relative maximum of 1 at $x = 2$ and a relative minimum of -2 at $x = -1$.
   (g) f has an absolute maximum of 3 at $x = -5$.
   (h) f has no absolute minimum.
   (i) $f(-4) \approx 2.5$
   (j) $f(x) = -1$ at $x = -2$ and $x \approx 0.4$

2. (a) The domain of g is all real numbers. (b) The range of g is $(-3, \infty)$.
   (c) $g(-2) \approx -2$ and $g(-3) \approx -2.5$. (d) $g(x) = -1$ when $x = -1$.
   (e) $g(x) = 5$ when $x = 1$.
   (f) g is increasing on the interval $(-\infty, \infty)$.
   (g) g does not have any relative (or absolute) maxima or minima.
   (h) g is positive on about $(-0.4, \infty)$.

3. (a) $f(-x) = -3x + 1$, the function is neither even nor odd
   (b) $f(-x) = x^2 - 5 = f(x)$, the function is even
   (c) $f(-x) = -5x^3 + 7x + 1$, the function is neither even nor odd
   (d) $f(-x) = 10 = f(x)$, the function is even
   (e) $f(x) = -x^5 - 4x = -f(x)$, the function is odd

4. (a) odd (b) neither (c) odd
   (d) even (e) neither (f) even

5. Every quadratic function has exactly one absolute maximum (if the graph of the function opens downward) or one absolute minimum (if the graph opens upward), but not both. The are no relative maxima or minima other than the absolute maximum or minimum.

Section 2.4 Solutions Back to 2.4 Exercises

1. (a) 17.4 mpg, 17.3 mpg, The mileage decreased by 0.1 mpg as the speed increased from 45 to 50 mph.
   (b) 16.5 mpg, 16.4 mpg, The mileage decreased by 0.1 mpg as the speed increased from 90 to 95 mph.
   (c) The slope of the line is -0.02, and when you multiply it by 5 you get -0.1. This the change in mileage (negative indicating a decrease) in mileage as the speed is increased by 5 mph.
   (d) 40 mph
   (e) The slope of the line is $-0.02\ \frac{\text{mpg}}{\text{mph}}$. This tells us that the car will get 0.02 miles per gallon less fuel economy for each mile per hour that the speed increases.
   (f) We should not interpret the value of the m-intercept because it should represent the mileage when the speed is zero miles per hour. We are told in the problem that the equation only applies for speeds between 30 and 100 miles per hour. Furthermore, when the speed is zero mph the mileage should be infinite, since no gas is being used!

2. (a) The weight of a lizard that is 3 cm long is -18 grams. This is not reasonable because a weight cannot be negative. The problem is that the equation is only valid for lizards between 12 and 30 cm in length.
   (b) The w-intercept is -84, which would be the weight of a lizard with a length of zero inches. It is not meaningful for the same reason as given in (a).
   (c) The slope is 22 grams per centimeter. This says that for each centimeter of length gained by a lizard, the weight gain will be 22 grams.

3. (a) \$2300, \$3800 (b) $P = 0.03S + 800$
   (c) the slope of the line is 0.03 dollars of pay per dollar of sales, the commission rate.
   (a) The P-intercept of the line is \$800, the monthly base salary.

4. (a) The F-intercept is 32°, which is the Fahrenheit temperature when the Celsius temperature is 0 degrees.
   (b) The slope of the line is $\frac{9}{5}$ degrees Fahrenheit per degree Celsius; it tells us that each degree increase in Celsius temperature is equivalent to $\frac{9}{5}$ degrees increase in Fahrenheit temperature.

5. (a) The C-intercept is \$5000. This represents costs that Acme has regardless of how many Widgets they make. These are usually referred to as *fixed costs*.
   (b) The slope of the line is \$7 per Widget. It says that each additional Widget produced adds \$7 to the total cost. Business people often refer to this as "marginal cost."
   (c) The total cost is \$15,437, the cost per Widget is \$10.35

6. The slope of \$0.24 per mile is the additional cost for each additional mile driven. The intercept of \$30 is the the base charge for one day, without mileage.

7. (a) $W = 6t + 8$
   (b) The slope is 6 pounds per year, and represents the additional weight the baby will gain each year after birth.
   (c) The W-intercept of 8 pounds is the weight at birth, which is time zero.
   (d) The child weighed 26 pounds at 3 years of age.

8. $x = -600p + 9500$ 9. $T = \frac{2}{9}x + 40$

Section 2.5 Solutions

Back to 2.5 Exercises

1. (a) 288 feet (b) 9 seconds (c) 324 feet
   (d) 2.26 seconds and 6.74 seconds. There are two times because the is at a height of 224 feet once on the way up, and again on the way down.

2. (a) $h(4)$ represents the height of the rock at four seconds after it was thrown.
   (b) $t = 1.20$ seconds, $t = 7.80$ seconds
   (c) The rock reaches a height of 150 feet at 1.20 seconds and 7.80 seconds after it was thrown.
   (d) Since the rock hits the ground at $t = 9$ seconds, it makes no sense to use the equation for $t = 11$ seconds.

3. (a) $k = 0.048$ (b) $\frac{\text{ft}}{\text{mph}^2}$ (c) $L = 120$ feet (d) $s = 45.6$ miles per hour

4. (a) At 18 feet the intensity should be *less* than 180 candlepower.
   (b) At 18 feet the intensity is 80 candlepower.

5. (a) 15000 can be sold at \$50 and 10000 can be sold at \$100; the number sold decreases by 5000.
   (b) The number sold decreases by 5000 again.
   (c) \$130 (d) The feasible domain is $[0, 200]$.

6. (a) At a price of \$80, $w = 20,000 - 8000 = 12,000$ widgets will be sold. The revenues will then be $(12,000)(80) = \$960,000$
   (b) \$990,000, \$840,000
   (c) As the price increases from \$80 to \$110, the revenue increases by \$30,000. As the price increases from \$110 to \$140, the revenue *decreases* by \$150,000.
   (d) $R = wp = (20,000 - 100p)p = 20000p - 100p^2$
   (f) \$5.13 or \$194.87

7. (a) 3190 (b) You can't have part of a (live) fish! (c) 1714 (d) 0 or 3300

8. There will be 10.3 milligrams left after 3 hours.

9. (c) $d = 224 - 3.5t$ (d) 154 feet (e) $t \approx 26.9$ seconds, $t = 64$ seconds

10. (b) $P = (3 + 4t) + (5 + 4t) + \sqrt{(3 + 4t)^2 + (5 + 4t)^2}$
    (c) $P = 8 + 8t + \sqrt{32t^2 + 64t + 34}$

11. $A = 500x - 2x^2$

12. The mathematical domain is all real numbers, but the feasible domain is $(0, 250)$ because if x is over 250 there will not be enough fence to make the four sections with length x.

13. (a) Billy can mow 1.5 lawns in an hour and Bobby can mow 1 lawn in an hour.
    (b) $L = 1.5t + 1t = 2.5t$
    (c) Together the two of them can mow one lawn in $\frac{2}{5}$ of an hour, which is $\frac{2}{5}(60) = 24$ minutes.
    (d) The mathematical domain of the function is all real numbers, but the feasible domain is $[0, \infty)$.

14. (a) The three lengths/distance of interest are the length of the ladder, the height of the top of the ladder, and the distance of the base of the ladder from the wall. The height of the ladder and the distance of the base from the wall are variables, and the length of the ladder is a constant.
    (b) As the distance of the base from the wall decreases, the height of the top of the ladder increases.
    (c) $d^2 + h^2 = 10^2$ or $d^2 + h^2 = 100$
    (d) The height of the ladder is 8 feet when the base is 6 feet from the wall. When the base of the ladder is 2 feet from the wall the height is 9.8 feet.
    (e) 9.8 feet, $4\sqrt{6}$ feet.

(f) The first value is nice because it is clear how much it is. The second number is hard for us to interpret, but it is exact. Someone could use it to get the answer to as many decimal places as they desired.

(g) 9.8 feet rounded to the nearest inch is 118 inches, or 9 feet 10 inches.

15. (a) We can disregard the negative sign because the height must be a positive number.

(b) The feasible domain is $(0, 10)$.

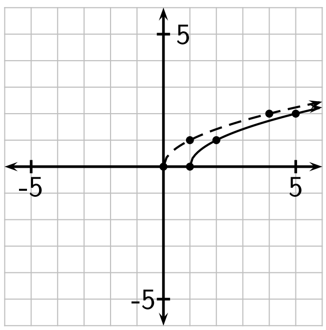
16. (a) $A = l(\frac{1}{3}l) = \frac{1}{3}l^2$ (b) $A = w(3w) = 3w^2$

(c) $P = 2l + 2(\frac{1}{3}l) = \frac{8}{3}l$, $P = 2(3w) + 2w = 8w$

17. (a) $5000 + 7(8000) = \$61,000$, $5000 + 7(12000) = \$89,000$

(b) $C = 5000 + 7x$

Section 2.6 Solutions Back to 2.6 Exercises

2. (a)


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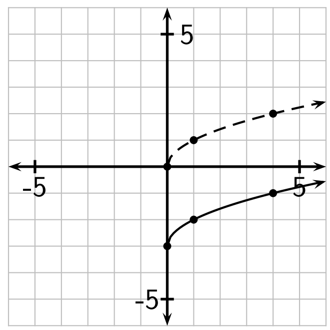

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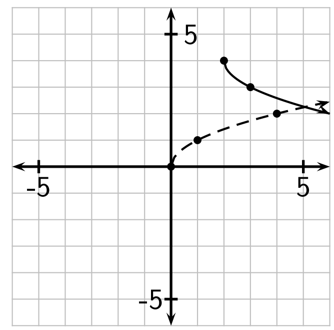

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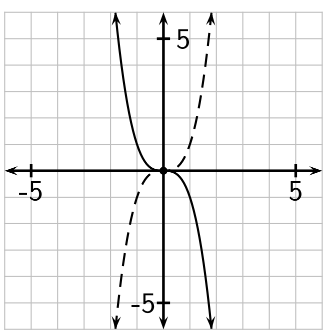

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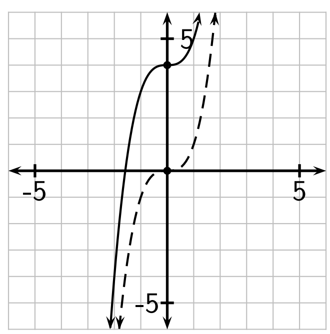

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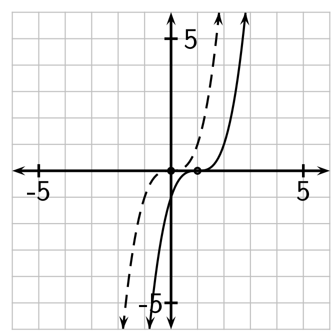

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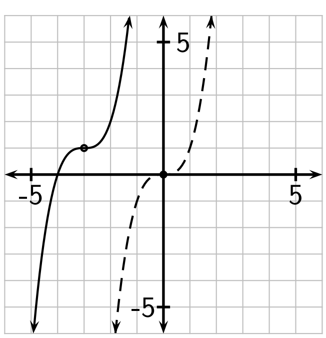

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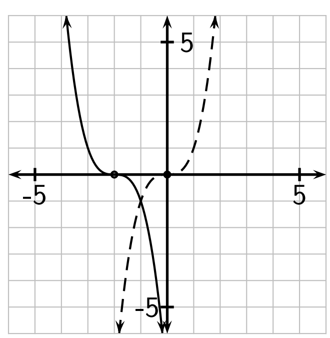
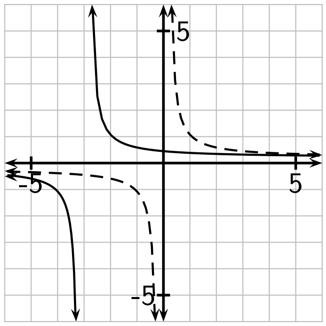
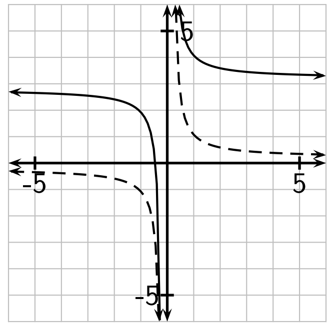


5. (a)



(b)



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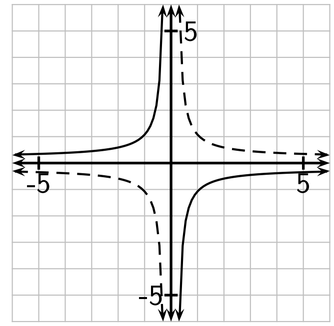

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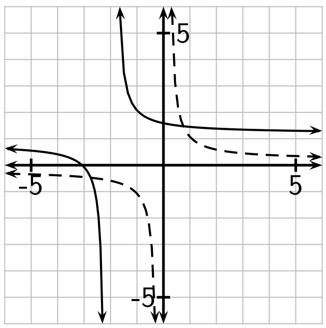

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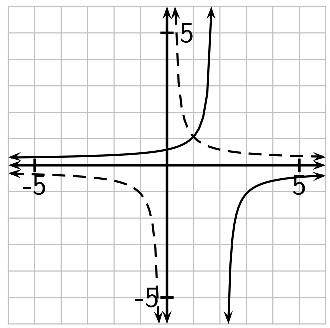

6. (a) Domain: $\mathbb{R}$ Range: $[-2, \infty)$
   (b) Dom$(f) = [-3, \infty)$ Ran$(f) = [0, \infty)$
   (c) Dom$(g) = \mathbb{R}$ Ran$(g) = (-\infty, 5]$
   (d) Dom$(h) = \{x \mid x \neq 2\}$ Ran$(h) = \{y \mid y \neq 0\}$
   (e) Domain: $\mathbb{R}$ Range: $(-\infty, 7]$
   (f) Domain: $[2, \infty)$ Range: $[1, \infty)$

Section 2.7 Solutions Back to 2.7 Exercises

1. (a) $\{x \mid x \neq 0, 1\}$ (b) $[-2, \infty)$ or $\{x \mid x \geq -2\}$ (c) $\{x \mid x \neq -1, 1\}$
   (d) $\{x \mid x \neq 1, 4\}$ (e) $[\frac{2}{3}, \infty)$ (f) $\mathbb{R}$ or all real numbers

2. (a) Dom$(h) = \{x \mid x \neq -3, 2\}$ (b) Ran(h) is all real numbers.
   (c) $h(-4) = 3$ and $h(4) \approx 1$ (d) $h(x) = 1$ for $x = -1, 1, 4$
   (e) $h(x) = 2$ when $x = 0$ and $x = 3$.
   (f) h is *decreasing* on the intervals $(0, 2)$ and $(2, 4)$.
   (g) h is negative on $(-3, -1.9)$ (or $(-3, 2)$) and $(1.3, 2)$
   (h) h has a relative maximum of 2 at $x = 0$ and a relative minimum of 1 at $x = 4$.
   (i) h has no absolute maxima or minima.