College Algebra

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Contents

3 Quadratic Functions, Circles, Average Rate of Change

Outcome/Performance Criteria:

3. Understand and apply quadratic functions. Find distances and midpoints, graph circles. Solve systems of two non-linear equations, find average rates of change and difference quotients.

    (a) Find the vertex and intercepts of a quadratic function; graph a quadratic function given in standard form or factored form.

    (b) Change a quadratic function from standard form to factored form, and vice-versa. Change from vertex form to standard form.

    (c) Graph a parabola whose equation is given in $y = a(x-h)^2+k$ form, relative to $y = x^2$.

    (d) Given the vertex of a parabola and one other point on the parabola, give the equation of the parabola.

    (e) Put the equation of a parabola in standard form $y = a(x-h)^2 + k$ by completing the square.

    (f) Solve a linear inequality; solve a quadratic inequality.

    (g) Solve a problem using a given quadratic model; create a quadratic model for a given situation.

    (h) Find the distance between two points in the plane; find the midpoint of a segment between two points in a plane.

    (i) Graph a circle, given its equation in standard form. Given the center and radius of a circle, give its equation in standard form.

    (j) Put the equation of a circle in standard form

    $$(x-h)^2 + (y-k)^2 = r^2$$

    by completing the square.

    (k) Solve a system of two non-linear equations.

    (l) Find the average rate of change of a function over an interval, from either the equation of the function or the graph of the function. Include units when appropriate.

    (m) Find and simplify a difference quotient.

3.1 Quadratic Functions

Performance Criteria:

3. (a) Find the vertex and intercepts of a quadratic function; graph a quadratic function given in standard form or factored form.

   (b) Change a quadratic function from standard form to factored form, and vice-versa. Change from vertex form to standard form.

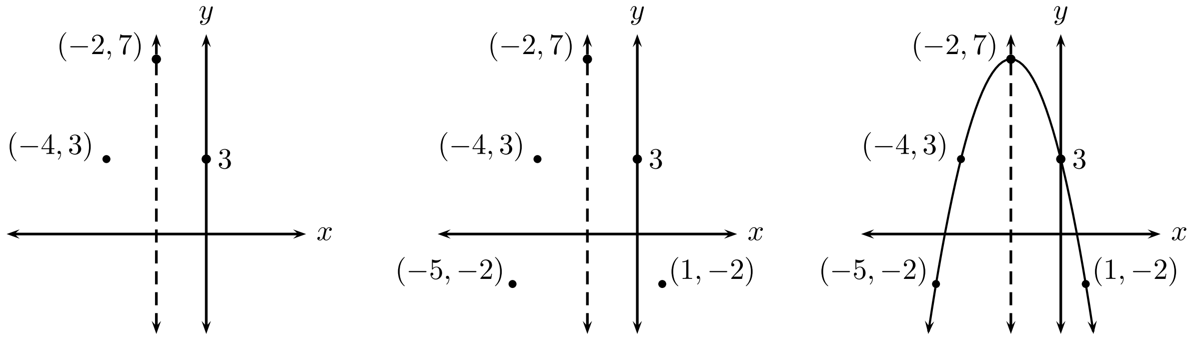
A **quadratic function** is a function that can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. You have seen a number of these functions already, of course, including several in "real world" situations $(h = -16t^2 + v_0t + h_0$, for example). We will refer to the value a, that x^2 is multiplied by, as the **lead coefficient**, and c will be called the **constant term**. We already saw and graphed such functions in Section 1.7, as equations of the form $y = ax^2 + bx + c$. Let's recall what we know about the graphs of such functions:

- The graph of $f(x) = ax^2 + bx + c$ is a parabola. The parabola opens up if $a > 0$ and down if $a < 0$.
- Every quadratic function has either an absolute maximum or an absolute minimum, depending on whether it opens down or up. The absolute maximum or minimum occurs at the x-coordinate of the vertex, and its value is the y-coordinate of the vertex.
- The vertex of the parabola has x-coordinate $x = -\dfrac{b}{2a}$ and the y-coordinate of the vertex is obtained by evaluating the function for that value of x. The vertical line passing through the vertex (so with equation $x = -\frac{b}{2a}$) is called the **axis of symmetry** of the parabola.
- The y-intercept of the parabola is $y = c$.
- Other than the vertex, every point on a parabola corresponds to another point on the other side of the parabola that has the same y-coordinate and is equidistant from the axis of symmetry.

Let's review Example 1.7(b) before continuing.

◇ **Example 1.7(b):** Graph the equation $y = -x^2 - 4x + 3$. Indicate clearly and accurately five points on the parabola.

Solution: First we note that the graph will be a parabola opening downward, with y-intercept 3. (This is obtained, *as always*, by letting $x = 0$.) The x-coordinate of the vertex is $x = -\dfrac{-4}{2(-1)} = -2$. Evaluating the equation for this value of x gives us $y = -(-2)^2 - 4(-2) + 5 = -4 + 8 + 3 = 7$, so the vertex is $(-2, 7)$. On the first graph at the top of the next page we plot the vertex, y-intercept, the axis of symmetry, and the point opposite the y-intercept, $(-4,3)$. To get two more points we can evaluate the equation for $x=1$ to get $y=-1^2-4(1)+3=-2$. We then plot the resulting point $(1,-2)$ and the point opposite it, as done on the second graph. The third graph then shows the graph of the function with our five points clearly and accurately plotted.



We will now be working with three different forms of quadratic functions:

- **Standard Form:** This is the form discussed above, that you are already familiar with.
- **Factored Form:** The factored form of a quadratic function is $f(x)=a(x-x_1)(x-x_2)$, where a is the same as the a of the standard form. x_1 and x_2 are just numbers.
- **Vertex Form:** The vertex form of a quadratic function is $f(x)=a(x-h)^2+k$, with a again being the same as for the other two forms. h and k are just numbers.

We'll explore all three forms just a bit in this section, and the next section will be devoted exclusively to the vertex form. First let's see how to change from factored or vertex form to standard form, which entails a bit of basic simplification.

◇ **Example 3.1(a):** Put $f(x)=-3(x+1)(x+4)$ and $h(x)=\frac{1}{3}(x+3)^2-2$ into standard form.

Solution: For quadratic functions like f that are in factored form, it usually easiest to multiply the two linear factors $x+1$ and $x+4$ first, then multiply in the constant -3:

$$f(x)=-3(x+1)(x+4)=-3(x^2+5x+4)=-3x^2-15x-12$$

The function h is given in vertex form. In this case we must follow the order of operations, first squaring $x+3$, then multiplying by $\frac{1}{3}$ and, finally, subtracting two:

$$h(x)=\tfrac{1}{3}(x+3)^2-2=\tfrac{1}{3}(x^2+6x+9)-2=(\tfrac{1}{3}x^2+2x+3)-2=\tfrac{1}{3}x^2+2x+1$$

The standard forms of the functions are $f(x)=-3x^2-15x-12$ and $h(x)=\frac{1}{3}x^2+2x+1$.

Next we see how to change from standard form to factored form.

◇ **Example 3.1(b):** Put $g(x) = -2x^2 + 8x + 10$ into factored form.

Solution: Here we first need to factor -2 out of the quadratic expression $-2x^2+8x+10$. To do this we must determine values e and f for which

$$-2(x^2 + ex + f) = -2x^2 + 8x + 10$$

We can see that e must be -4 and $f = -5$. We then have

$$g(x) = -2x^2 + 8x + 10 = -2(x^2 - 4x - 5) = -2(x + 1)(x - 5),$$

where the last expression was obtained by factoring $x^2 - 4x - 5$. Thus the factored form of $g(x) = -2x^2 + 8x + 10$ is $g(x) = -2(x + 1)(x - 5)$.

◇ **Example 3.1(c):** Put $h(x) = \frac{1}{4}x^2 + \frac{1}{2}x - 2$ into factored form.

Solution: Here we first need to factor $\frac{1}{4}$ out of $\frac{1}{4}x^2 + \frac{1}{2}x - 2$. This is most easily done by first getting a common denominator of four in each coefficient, then factoring the four out of the bottom of each coefficient as $\frac{1}{4}$:

$$h(x) = \tfrac{1}{4}x^2 + \tfrac{1}{2}x - 2 = \tfrac{1}{4}x^2 + \tfrac{2}{4}x - \tfrac{8}{4} = \tfrac{1}{4}(x^2 + 2x - 8)$$

We can then factor $x^2 + 2x - 8$ to get the factored form $h(x) = \frac{1}{4}(x + 4)(x - 2)$.

The factored form of a quadratic function is $f(x) = a(x - x_1)(x - x_2)$. In the above example, $a = \frac{1}{4}$, $x_1 = 4$ and $x_2 = -2$. We will now see that it is fairly easy to graph a quadratic function that is given in factored form.

◇ **Example 3.1(d):** Graph the function $h(x) = \frac{1}{4}(x + 4)(x - 2)$. Plot the intercepts, vertex and one more point.

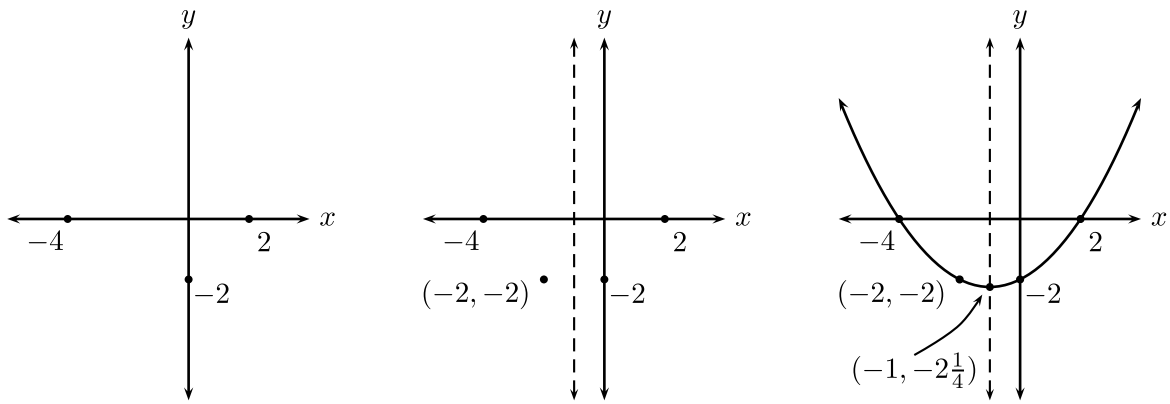
Solution: We first note that $x = -4$ and $x = 2$ are important values of x. Why? Well, it is easily seen that $h(-4) = h(2) = 0$ so, because $h(x)$ can be thought of as y, $y = 0$ when $x = -4$ or $x = 2$. This gives us the two x-intercepts, which are plotted on the axes to the left on the next page. Letting $x = 0$ gives us the y-intercept $y = g(0) = \frac{1}{4}(4)(-2) = -2$, also plotted to the left on the next page.

Using the fact that pairs of points on either side of a parabola are equidistant from the axis of symmetry, we can see that the axis of symmetry is the vertical line halfway between $x = -4$ and $x = 2$. The easiest way to find a value halfway between two numbers is to average them (you will see this idea gain soon!), so the axis of symmetry is $x = \frac{-4+2}{2} = -1$. This is shown on the axes in the center at the top of the next page, along with an additional point with the same y-coordinate as the y-intercept but equidistant from the axis of symmetry on the other side.

To finish up our graph we should at least find the vertex, and maybe a couple more points if asked. The x-coordinate of the vertex is -1, so its y-coordinate is

$$h(-1) = \tfrac{1}{4}(-1 + 4)(-1 - 2) = -\tfrac{9}{4} = -2\tfrac{1}{4}.$$

(*Note that we DO NOT have to put the function in standard form to evaluate it.*) The vertex is then at $\left(-1, -2\frac{1}{4}\right)$. This, and the other points we've found are plotted on the third set of axes below, and the graph of the function is also shown on that grid.



Section 3.1 Exercises   To Solutions

1. Put each of the following quadratic functions in standard form.

   (a) $f(x) = -\frac{1}{5}(x-2)(x+3)$ (b) $g(x) = 3(x-1)^2 + 2$

   (c) $y = -2(x-3)(x+3)$ (d) $h(x) = \frac{1}{2}(x+2)^2 - 5$

2. Put each of the following functions in factored form.

   (a) $f(x) = -\frac{1}{4}x^2 + x + \frac{5}{4}$ (b) $y = 2x^2 - 8x + 6$

   (c) $g(x) = \frac{1}{2}x^2 + 5x + \frac{9}{2}$ (d) $h(x) = -5x^2 + 5$

   (e) $y = 3x^2 - 15x + 18$ (f) $f(x) = \frac{1}{4}x^2 + \frac{3}{2}x - 4$

   (g) $y = -\frac{1}{3}x^2 + \frac{1}{3}x + 2$ (h) $g(x) = -x^2 + 5x + 14$

   Hint for part (h): Factor -1 out.

3. A function $f(x) = ax^2 + bx + c$ is quadratic as long as a is not zero, but either of b or c (or both) *CAN* be zero. Put each of the following in factored form by first factoring out any common factor, then factoring what remains, if possible.

   (a) $y = 5x^2 - 5$ (b) $h(x) = x^2 - 4x$

4. Do the following for each quadratic function below.

   - Determine the x- and y- intercepts.
   - Determine the coordinates of the vertex in the manner described in Example 3.1(d).
   - Plot the intercepts and vertex, and then one more point that is the "partner" to the y-intercept, and sketch the graph of the function.

   Check your answers to the first two items in the back of the book, and check your graphs by graphing the function on your calculator or a computer/tablet/phone.

   (a) $f(x) = -\frac{1}{4}(x+1)(x-5)$
   (b) $y = 2(x-3)(x-1)$
   (c) $g(x) = \frac{1}{2}(x+1)(x+9)$
   (d) $h(x) = -5(x+1)(x-1)$
   (e) $y = -(x+2)(x-7)$
   (f) $f(x) = \frac{1}{4}(x+8)(x-2)$

5. Do the following for the function $y = -(x+1)^2 + 16$:

   (a) Put the equation of the function into standard form.
   (b) Put the equation of the function into factored form.
   (c) Give the coordinates of the vertex of the parabola.
   (d) Graph the parabola, checking your answer with your calculator or a computer/tablet/phone.

6. Repeat Exercise 5 for the function $f(x) = \frac{1}{3}(x-2)^2 - \frac{1}{3}$.

7. Find the equation in standard form of the parabola with x-intercepts -1 and 3 and y-intercept 1, using the following suggestions:

   - Use the x-intercepts to find the equation in factored form $y = a(x-x_1)(x-x_2)$ first - you will not know a at this point.
   - What is the x-coordinate for the y-intercept? Put the coordinates of the y-intercept in for x and y and solve to find a.
   - Multiply out to get the standard form of the equation of the parabola.
   - Check your answer by graphing it with your calculator or a computer/tablet/phone and seeing if it in fact has the given intercepts.

8. Find the equation, in standard form, of the parabola with one x-intercept of -2 and vertex $(1,-6)$. (**Hints:** What is the other x-intercept? Now do something like you did for the previous exercise.) Again, check your answer by graphing it on your calculator or a computer/tablet/phone.

9. Discuss the x-intercept situation for quadratic functions. (Do they always have at least one x-intercept? Can they have more than one?) Sketch some parabolas that are arranged in different ways (but always opening up or down) relative to the coordinate axes to help you with this.

10. Discuss the existence of relative and absolute maxima and/or minima for quadratic functions.

11. Consider the function $f(x) = (x-3)^2 - 4$.

    (a) Use the ideas of Section 2.6 to graph the function along with a dashed version of $y = x^2$.

    (b) Put the equation in standard form.

    (c) Use $x = -\dfrac{b}{2a}$ with the standard form of the equation to find the coordinates of the vertex of the parabola. If the results don't agree with your graph, figure out what is wrong and fix it.

    (d) Graph the standard form equation on your calculator or computer/tablet/phone and see if it agrees with your graph from (a). If it doesn't, figure out what is wrong and fix it.

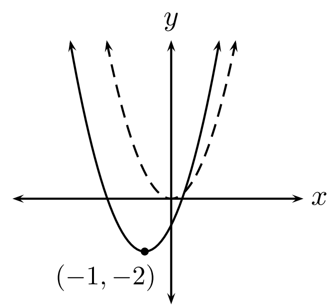
3.2 Quadratic Functions in Vertex Form

Outcome/Criteria:

3. (c) Graph a parabola with equation given in $y = a(x-h)^2 + k$ form, relative to $y = x^2$.
   (d) Given the vertex of a parabola and one other point on the parabola, give the equation of the parabola.
   (e) Put the equation of a parabola in the form $y = a(x-h)^2 + k$ by completing the square.

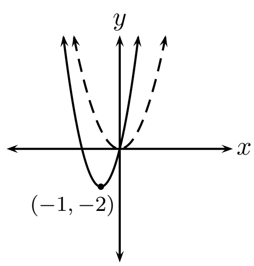
Graphing $y = a(x-h)^2 + k$

From Section 2.6 we know that the graph of the function $f(x) = (x+1)^2 - 2$ is the graph of $y = x^2$ shifted one unit left and two units down. This is shown to the right, where the dashed curve is the graph of $y = x^2$ and the solid curve is the graph of $f(x) = (x+1)^2 - 2$. Suppose now that we wish to graph an equation of the form $f(x) = a(x+1)^2 - 2$ for some value of a not equal to one. The question is "How does the value of a affect the graph?" The following example gives us a start on answering that question.

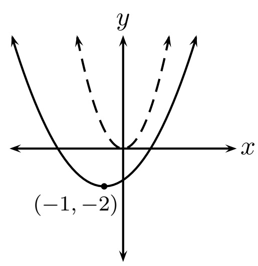


◇ **Example 3.2(a):** Graph $f(x) = a(x+1)^2 - 2$ for $a = 2, \frac{1}{3}, -\frac{1}{4}$ and -3.

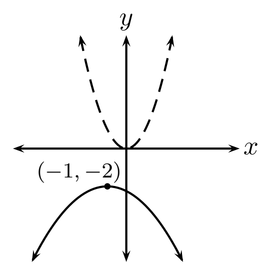
Solution: Below are the respective graphs as solid curves, with $y = x^2$ also plotted on each grid as a dashed line.



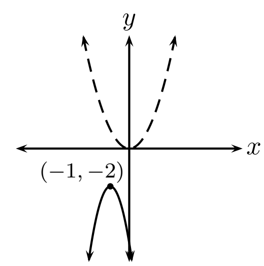
$f(x) = 2(x+1)^2 - 2$



$f(x) = \frac{1}{3}(x+1)^2 - 2$



$f(x) = -\frac{1}{4}(x+1)^2 - 2$



$f(x) = -3(x+1)^2 - 2$

From the above example we first observe that when a is positive, the graph of $f(x) = a(x+1)^2 - 2$ opens upward, and when a is negative the graph opens downward. Less obvios is the fact that when the absolute value of a is greater than one the graph of $f(x) = a(x+1)^2 - 2$ is narrower than the graph of $y = x^2$, and when the absolute value of a is less than one the graph of $f(x) = a(x+1)^2 - 2$ is wider than the graph of $y = x^2$. We can summarize all this, as well as some of what we already knew, in the following.

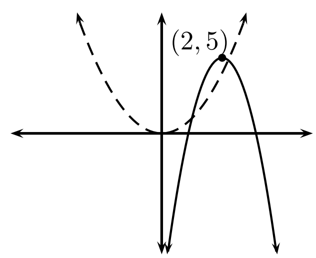
The Equation $y = a(x-h)^2 + k$

The graph of $y = a(x-h)^2+k$ is a parabola. The x-coordinate of the vertex of the parabola is the value of x that makes $x-h$ equal to zero, and the y-coordinate of the vertex is k. The number a is the same as in the form $y = ax^2+bx+c$ and has the following effects on the graph:

- If a is positive the parabola opens upward, and if a is negative the parabola opens downward.
- If the absolute value of a is less than one, the parabola is shrunk vertically to become "smashed flatter" than the parabola $y = x^2$.
- If the absolute value of a is greater than one, the parabola is stretched vertically, to become "narrower" than the parabola $y = x^2$.

◇ **Example 3.2(b):** Sketch the graph of $y = -4(x-2)^2+5$ on the same grid with a dashed graph of $y = x^2$. Label the vertex with its coordinates and make sure the shape of the graph compares correctly with that of $y = x^2$. Another Example

Solution: The value that makes the quantity $x-2$ zero is $x = 2$, so that is the x-coordinate of the vertex, and the y-coordinate is $y = 5$. Because $a = -4$, the parabola opens downward and is narrower than $y = x^2$. From this we can graph the parabola as shown to the right.



Finding the Equation of a Parabola

If we are given the coordinates of the vertex of a parabola and the coordinates of one other point on the parabola, it is quite straightforward to find the equation of the parabola using the equation $y = a(x-h)^2+k$. The procedure is a bit like what is done to find the equation of a line through two points, and we'll demonstrate it with an example.

◇ **Example 3.2(c):** Find the equation of the parabola with vertex $(-3, -4)$ with one x-intercept at $x = -7$.

Solution: Since the x-coordinate of the vertex is -3 , the factor $x-h$ must be $x+3$ so that $x = -3$ will make it zero. Also, k is the y-coordinate of the vertex, -4. From these two things we know the equation looks like $y = a(x+3)^2-4$. Now we need values of x and y for some other point on the parabola; if we substitute them in we can solve for a. Since -7 is an x-intercept, the y-coordinate that goes with it is zero, so we have the point $(-7, 0)$. Substituting it in we get

$$
\begin{aligned}
0 &= a(-7+3)^2 - 4 \\
4 &= (-4)^2 a \\
4 &= 16a \\
\tfrac{1}{4} &= a
\end{aligned}
$$

The equation of the parabola is then $y = \frac{1}{4}(x+3)^2 - 4$.

Changing From Standard Form to Vertex Form

What we have seen is that if we have the equation of a parabola in vertex form $y = a(x-h)^2 + k$ it is easy to get a general idea of what the graph of the parabola looks like. Suppose instead that we have a quadratic function in standard form $y = ax^2 + bx + c$. If we could put it in the vertex form $y = a(x-h)^2 + k$, then we would be able to graph it easily. The key to doing this is the following:

> Suppose we have $x^2 + cx$. If we add $d = (\frac{1}{2}c)^2$ to this quantity the result $x^2 + cx + d$ factors into two *EQUAL* factors.

The process of adding such a $d = (\frac{1}{2}c)^2$ is called **completing the square**. Of course we can't just add anything we want to an expression without changing its value. The key is that if we want a new term, we can add it as long as we subtract it as well. Here's how we use this idea to put $y = x^2 + bc + c$ into vertex form:

$y = x^2 - 6x - 3$	
$y = x^2 - 6x + \_\_\_ - 3$	pull constant term away from the x^2 and x terms
$y = x^2 - 6x + 9 - 9 - 3$	multiply 6 by $\frac{1}{2}$ and square to get 9, add and subtract
$y = (x^2 - 6x + 9) + (-9 - 3)$	group terms to form a quadratic trinomial with "leftover constants" and "leftover constants"
$y = (x-3)^2 - 12$	factor the trinomial, combine constants

And here's how to do it when the a of $y = ax^2 + bx + c$ is not one:

◇ **Example 3.2(d):** Put the equation $y = -2x^2 + 12x + 5$ into the form $y = a(x-h)^2 + k$ by completing the square.

$y = -2x^2 + 12x + 5$	
$y = -2(x^2 - 6x) + 5$	factor -2 out of the x^2 and x terms
$y = -2(x^2 - 6x + 9 - 9) + 5$	add and subtract 9 inside the parentheses
$y = -2(x^2 - 6x + 9) + 18 + 5$	bring the -9 out of the parentheses - it comes out as $+18$ because of the -2 factor outside the parentheses
$y = -2(x-3)^2 + 23$	factor the trinomial, combine the added constants

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Section 3.2 Exercises

To Solutions

1. For each of the following, sketch the graph of $y = x^2$ and the given function on the same coordinate grid. The purpose of sketching $y = x^2$ is to show how "open" the graph of the given function is, *relative to* $y = x^2$.

   (a) $f(x) = 3(x-1)^2 - 7$
   (b) $g(x) = (x+4)^2 - 2$
   (c) $h(x) = -\frac{1}{3}(x+2)^2 + 5$
   (d) $y = -(x-5)^2 + 3$

2. For each of the following, find the equation of the parabola meeting the given conditions. Give your answer in the vertex form $y = a(x-h)^2 + k$.

   (a) Vertex $V(-5,-7)$ and through the point $(-3,-25)$.
   (b) Vertex $V(6,5)$ and y-intercept 29.
   (c) Vertex $V(4,-3)$ and through the point $(-5,6)$.
   (d) Vertex $V(-1,-4)$ and y-intercept 1.

3. For each of the following, use completing the square to write the equation of the function in the vertex form $f(x) = a(x-h)^2 + k$. Check your answers by graphing the original equation and your vertex form equation together on your calculator or other device - they should of course give the same graph if you did it correctly! Hint for (c): Divide (or multiply) both sides by -1.

   (a) $f(x) = x^2 - 6x + 14$
   (b) $g(x) = x^2 + 12x + 35$
   (c) $h(x) = -x^2 - 8x - 13$
   (d) $y = 3x^2 + 12x + 50$

4. Write the quadratic function $y = 3x^2 - 6x + 5$ in vertex form.

5. Write the quadratic function $h(x) = -\frac{1}{4}x^2 + \frac{5}{2}x - \frac{9}{4}$ in vertex form.

6. Sketch the graphs of $y = x^2$ and $f(x) = -\frac{3}{4}(x-1)^2 + 6$ without using your calculator, check with your calculator or other device.

7. Find the equation of the parabola with vertex $V(4,1)$ and y-intercept -7. Give your answer in $y = a(x-h)^2 + k$ form.

8. Write the quadratic function $y = -3x^2 + 6x - 10$ in vertex form.

9. Find the equation of the quadratic function whose graph has vertex $V(3,2)$ and passes through $(-1,10)$. Give your answer in vertex form.

10. Write the quadratic function $y = 5x^2 + 20x + 17$ in vertex form.

3.3 Linear and Quadratic Inequalities

Outcome/Criteria:

3. (f) Solve a linear inequality; solve a quadratic inequality.

Let's recall the notation $a > b$, which says that the number a is greater than the number b. Of course that is equivalent to $b < a$, b is less than a. So, for example, we could write $5 > 3$ or $3 < 5$, both mean the same thing. Note that it is not correct to say $5 > 5$, since five is not greater than itself! On the other hand, when we write $a \geq b$ it means a is greater than *or equal to* b, so we could write $5 \geq 5$ and it would be a true statement. Of course $a \geq b$ is equivalent to $b \leq a$.

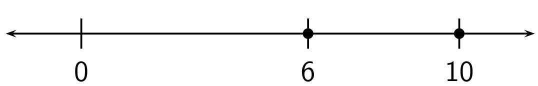
In this section we will consider algebraic inequalities; that is, we will be looking at inequalities that contain an unknown value. An example would be

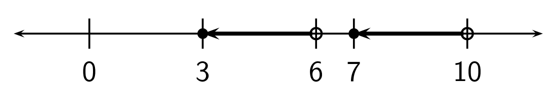
$$3x + 5 \leq 17,$$

and our goal is to *solve* the inequality. This means to find all values of x that make this true. This particular inequality is a **linear inequality**; before solving it we'll take a quick look at how linear inequalities behave.

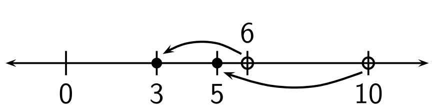
Solving Linear Inequalities

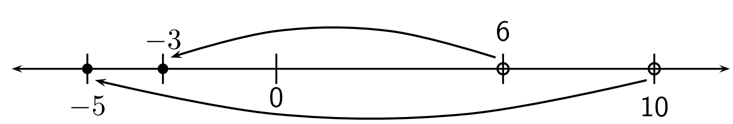
Consider the *true* inequality $6 < 10$; if we put dots at each of these values on the number line the larger number, ten, is farther to the right, as shown to the left below. If we subtract, say, three from both sides of the inequality we get $3 < 7$. The effect of this is to shift each number to the left by five units, as seen in the diagram below and to the right, so their relative positions don't change.





Suppose now that we divide both sides of the same inequality $6 < 10$ by two. We then get $3 < 5$, which is clearly true, and shown on the diagram below and to the left. If on the other hand, we divide both sides by -2 we get $-3 < -5$. This is not true! The diagram below and to the right shows what is going on here; when we divide both sides by a negative we reverse the positions of the two values on the number line, so to speak.





The same can be seen to be true if we multiply both sides by a negative. This leads us to the following principle.

Solving Linear Inequalities

To solve a linear inequality we use the same procedure as for solving a linear equation, *except that we reverse the direction of the inequality whenever we multiply or divide BY a negative.*

◇ **Example 3.3(a):** Solve the inequality $3x + 5 \leq 17$.

Solution: We'll solve this just like we would solve $3x + 5 = 17$. At some point we will divide by 3, but since we are not dividing by a negative, we won't reverse the inequality.

$$\begin{aligned} 3x + 5 &\leq 17 \\ 3x &\leq 12 \\ x &\leq 4 \end{aligned}$$

This says that the original inequality is true for any value of x that is less than or equal to 4. Note that zero is less than 4 and makes the original inequality true; this is a check to make sure at least that the inequality in our answer goes in the right direction.

◇ **Example 3.3(b):** Solve $-3x + 7 > 19$.

Solution: Here we can see that at some point we will divide by -3, and *at that point* we'll reverse the inequality.

$$\begin{aligned} -3x + 7 &> 19 \\ -3x &> 12 \\ x &< -4 \end{aligned}$$

This says that the original inequality is true for any value of x that is less than -4. Note that -5 is less than -4 and makes the original inequality true, so the inequality in our answer goes in the right direction.

Solving Quadratic Inequalities

Suppose that we wish to know where the function $f(x) = -x^2+x+6$ is positive. Remember that what this means is we are looking for the x values for which the corresponding values of $f(x)$ are greater than zero; that is, we want $f(x) > 0$. But $f(x)$ is $-x^2 + x + 6$, so we want to solve the inequality

$$-x^2 + x + 6 > 0.$$

Let's do this by first multiplying both sides by -1 (remembering that this causes the direction of the inequality to change!) and then factoring the left side:

$$\begin{aligned} -x^2 + x + 6 &> 0 \\ x^2 - x - 6 &< 0 \\ (x - 3)(x + 2) &< 0 \end{aligned}$$

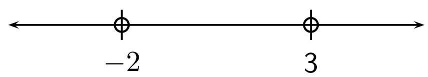
Now if $(x-3)(x+2)=0$, it must be the case that either $x-3=0$ or $x+2=0$. Unfortunately, solving the inequality is not so simple¡ but the method we will use in not that hard, either:

- Get zero on one side of the inequality, and a quadratic expression on the other side. In some way, get the coefficient a of x^2 to be positive.
- Factor the quadratic expression to find the values that make it zero.
- Plot the values that make the factors zero on a number line, as open circles if the inequality is *strict* ($<$ or $>$), or as solid dots if not ($\leq$ or $\geq$).
- The two (or occasionally one) points divide the number line into three (two) intervals. Test a value in each interval to see if the inequality holds there; if it does, shade that interval.
- Describe the shaded interval(s) using interval notation.

Let's illustrate by finishing solving the above inequality and with another example.

◇ **Example 3.3(c):** Solve the inequality $-x^2+x+6>0$. Another Example

Solution: We begin by multiplying both sides by $-$ to get the coefficient of x^2 to be positive, and we then factor the quadratic expression. These steps are shown above. The values that make the quadratic expression zero are 3 and -2, so we plot those on a number line. Because the inequality is not true for those particular values of x, we plot them as open circles:



Note that the two points -2 and 3 divide the number line into the three intervals $(-\infty,-2)$, $(-2,3)$ and $(3,\infty)$. For each interval, the inequality will be either true or false throughout the entire interval. We will test one value in each interval - in the first interval lets test $x=-3$, in the second interval $x=0$ and in the third interval $x=4$. (Note that the inequality we are testing is $(x-3)(x+2)<0$, which is equivalent to the original inequality.) All we need to test for is whether the inequality is true, so we only care about are whether things are positive or negative:

- For $x=-3$, $(x-3)(x+2)=(-3-3)(-3+2)=(-)(-)=+$, so the inequality $(x-3)(x+2)<0$ is false in the interval $(-\infty,-2)$.
- For $x=0$, $(x-3)(x+2)=(-3)(+2)=(-)(+)=-$, so $(x-3)(x+2)<0$ is true in the interval $(-2,3)$.
- For $x=4$, $(x-3)(x+2)=(4-3)(4+2)=(+)(+)=-$, so $(x-3)(x+2)<0$ is false in the interval $(3,\infty)$.

Based on the above, we can shade the portion of our number line where the inequality is true:

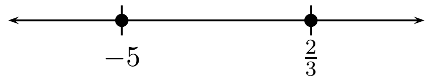
The solution to the inequality is then the interval $(-2,3)$.

◇ **Example 3.3(d):** Solve the inequality $3x^2 + 13x \geq 10$.

Solution: First we get zero on one side and factor the quadratic inequality:

$$\begin{aligned} 3x^2 + 13x &\geq 10 \\ 3x^2 + 13x - 10 &\geq 0 \\ (3x-2)(x+5) &\geq 0 \end{aligned}$$

The values that make $(3x-2)(x+5)$ zero are $x = -5$ and $x = \frac{2}{3}$. (The second of these is found by setting $3x - 2$ equal to zero and solving.) We plot those values on a number line, as *solid* dots because the inequality is greater than *or equal to*:



We then test, for example, $x = -6$, $x = 0$ and $x = 1$. When $x = -6$,

$$(3x-2)(x+5) = (-18-2)(-6+5) = (-)(-) = +,$$

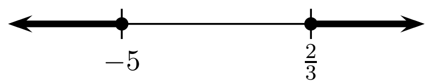
so the inequality $(3x-2)(x+5) \geq 0$ *IS* true for that value. For $x = 0$ we have

$$(3x-2)(x+5) = (-2)(+5) = (-)(+) = -,$$

so the inequality is not true for that value and, for $x = 1$,

$$(3x-2)(x+5) = (3-2)(1+5) = (+)(+) = +$$

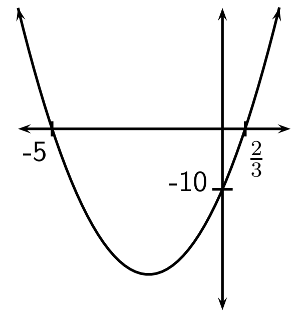
so the inequality is true for $x = 1$. Based on all this we can shade our number line:



This shows that the solution to the inequality is $(-\infty, -5] \cup [\frac{2}{3}, \infty)$.

NOTE: You may have guessed from the above that when we are solving a quadratic inequality for a number line divided into three parts, we can actually test a value in just one of the three intervals. The sign of the quadratic expression then changes as we move from one interval to any *adjacent* interval. So for the example we just did, we could have tested $x = 1$ and found that $(3x-2)(x+5)$ is positive in the interval $(\frac{2}{3}, \infty)$. The expression is then negative in the next interval to the left, $(-5, \frac{2}{3})$ and positive again in $(-\infty, -5)$. (I did not use $[$ and $]$ here because we are talking about where the expression is positive, not where the inequality is true.) We will see inequalities later where this approach does not necessarily work, so it is perhaps best to test every interval. Doing so can also alert you to any mistakes you might make in determining the sign within an interval.

To the right is the graph of $f(x) = 3x^2 + 13x - 10$. Compare the solution set from the last example with the graph of the function to see how the solution relates to the graph. The shaded number line shows us the intervals where the function is positive or negative - remember when looking at the graph that "the function" means the y values.



We can also use our knowledge of what the graph of a quadratic function looks like to solve an inequality, as demonstrated in the following example.

◇ **Example 3.3(e):** Solve the quadratic inequality $5 > x^2 + 4x$.

Solution: The given inequality is equivalent to $0 > x^2 + 4x - 5$. Setting $y = x^2 + 4x - 5$ and factoring gives us $y = (x + 5)(x - 1)$. Therefore the graph of y is a parabola, opening upward and with x-intercepts -5 and 1. y is then less than zero between -5 and 1, not including either. The solution to the inequality is then $-5 < x < 1$ or $(-5, 1)$.

Section 3.3 Exercises To Solutions

1. Solve each of the following linear inequalities.

   (a) $5x + 4 \geq 18$ (b) $7 - 2x \leq 13$

   (c) $4(x + 3) \geq x - 3(x - 2)$ (d) $4x + 2 > x - 7$

   (e) $5x - 2(x - 4) > 35$ (f) $5y - 2 \leq 9y + 2$

2. Solve each of the following inequalities. (**Hint:** Get zero on one side first, if it isn't already.) Give your answers as inequalities, OR using interval notation.

   (a) $x^2 + 8 < 9x$ (b) $2x^2 \leq x + 10$

   (c) $\frac{2}{3}x^2 + \frac{7}{3}x \leq 5$ (d) $5x^2 + x > 0$

   (e) $2x + x^2 < 15$ (f) $x^2 + 7x + 6 \geq 0$

3. Try solving each of the following inequalities using a method similar to that shown in this section.

   (a) $(x - 3)(x + 1)(x + 4) < 0$ (b) $(x - 5)(x + 2)(x - 2) \geq 0$

   (c) $(x - 3)^2(x + 2) > 0$ (d) $(x - 2)^3 \leq 0$

4. For each of the following, factor to solve the inequality.

(a) $x^3 + 3x^2 + 2x \geq 0$

(b) $10x^3 < 29x^2 + 21x$

5. (a) Determine **algebraically** where $x^2 - 4x + 4 > 0$. Then check your answer by graphing $f(x) = x^2 - 4x + 4$ and seeing where the function is positive.

   (b) Determine where $-x^2 - 6x - 11 \geq 0$. Check this answer by graphing the function $g(x) = -x^2 - 6x - 11$.

6. Consider the equation $\frac{1}{2}(x+1)^2 = 2$. One way to solve it would be to expand the left side (by "FOILing" it out and multiplying by $\frac{1}{2}$), then getting zero on one side and factoring. In this exercise you will solve it in a more efficient manner.

   - Begin by eliminating the fraction on the left, in the same way that we always do this.
   - Take the square root of both sides, remembering that you need to put a $\pm$ on the right.
   - You should be able to finish it from here! Check your answers (there are two) in the original equation.

7. (a) Determine the interval or intervals on which $\frac{1}{3}(x-2)^2 < 3$.

   (b) Graph $y = \frac{1}{3}(x-2)^2$ and $y = 3$ together on the same grid, using technology. Your answer to (a) should be the x values for which the graph of $y = \frac{1}{3}(x-2)^2$ is below the graph of $y = 3$.

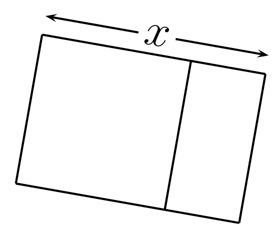
3.4 Applications of Quadratic Functions

Outcome/Criteria:

3. (g) Solve a problem using a given quadratic model; create a quadratic model for a given situation.

Quadratic functions arise naturally in a number of applications. As usual, we often wish to know the output for a given input, or what inputs result in a desired output. However, there are other things we may want to know as well, like the maximum or minimum value of a function, or when the function takes values larger or smaller than some given value. Let's begin by revisiting the following example:

◇ **Example 2.5(b):** A farmer is going to create a rectangular field with two "compartments", as shown to the right. He has 2400 feet of fence with which to do this. Letting x represent the dimension shown on the diagram, write an equation for the total area A of the field as a function of x, simplifying your equation as much as possible. Give the feasible domain for the function.



We determined that the equation for the area as a function of x is $A = 800x - \frac{2}{3}x^2$ for $0 < x < 1200$.

◇ **Example 3.4(a):** Determine the maximum area of the field, and the value of x for which it occurs.

Solution: Rearranging the function to get $A = -\frac{2}{3}x^2 + 800x$, we can see that the graph would be a parabola opening downward, so the maximum will be the A-coordinate of the vertex. The x-coordinate of the vertex is given by

$$x = -\frac{800}{2(-\frac{2}{3})} = \frac{800}{\frac{4}{3}} = (800)(\tfrac{3}{4}) = 600 \text{ feet}$$

Substituting this into the function we get

$$A = -\tfrac{2}{3}(600)^2 + 800(600) = 240,000 \text{ square feet}$$

The maximum area is $240,000$ square feet, which is attained when $x = 600$ feet.

◇ **Example 3.4(b):** Determine the values of x for which the area is $100,000$ square feet or more.

Solution: Here we need to solve the inequality

$$-\frac{2}{3}x^2 + 800x \geq 100,000.$$

We first change the inequality to equal and multiply both sides by three to clear the fraction. This gives us $-2x^2 + 2400x = 300,000$. We can then move the $300,000$ to the left side and divide by -2 to get $x^2 - 1200x + 150,000 = 0$. It is possible that this factors, but let's use the quadratic formula to solve for x:

$$x = \frac{-(-1200) \pm \sqrt{(-1200)^2 - 4(1)(150,000)}}{2(1)} = \frac{1200 \pm \sqrt{840,000}}{2} = 141.7, 1058.3$$

As stated in the previous example, the graph of the function is a parabola opening down, so the function has value greater than $100,000$ between the two values that we found. Thus the area of the field is greater than or equal to $100,000$ square feet when x is between 141.7 and 1058.3 feet.

Section 3.4 Exercises

To Solutions

1. The height h (in feet) of an object that is thrown vertically upward with a starting velocity of 48 feet per second is a function of time t (in seconds) after the moment that it was thrown. The function is given by the equation $h(t) = 48t - 16t^2$.

   (a) How high will the rock be after 1.2 seconds? Round your answer to the nearest tenth of a foot.

   (b) When will the rock be at a height of 32 feet? You should be able to do this by factoring, and you will get two answers; explain why this is, physically.

   (c) Find when the rock will be at a height of 24 feet. Give your answer rounded to the nearest tenth of a second.

   (d) When does the rock hit the ground?

   (e) What is the mathematical domain of the function? What is the feasible domain of the function?

   (f) What is the maximum height that the rock reaches, and when does it reach that height?

   (g) For what time periods is the height less than 15 feet? Round to the nearest hundredth of a second.

2. In statistics, there is a situation where the expression $x(1-x)$ is of interest, for $0 \leq x \leq 1$. What is the maximum value of the function $f(x) = x(1-x)$ on the interval $[0, 1]$? Explain how you know that the value you obtained is in fact a maximum value, not a minimum.

3. Use the quadratic formula to solve $s = s_0 + v_0 t - \frac{1}{2}gt^2$ for t. (**Hint:** Move everything to the left side and apply the quadratic formula. Keep in mind that t is the variable, and all other letters are constants.)

4. The Acme Company also makes and sells Geegaws, in addition to Widgets and Gizmos. Their weekly profit P (in dollars) is given by the equation $P = -800 + 27x - 0.1x^2$, where x is the number of Geegaws sold that week. Note that it will be possible for P to be negative; of course negative profit is really loss!

    (a) What is the P-intercept, *with units*, of the function? What does it mean, "in reality"?

    (b) Find the numbers of Geegaws that Acme can make and sell in order to make a true profit; that is, we want P to be positive. Round to the nearest whole Geegaw.

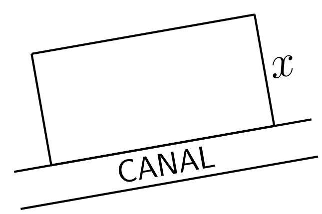
    (c) Find the maximum profit they can get. How do we know that it is a maximum, and not a minimum?

5. In a previous exercise you found the revenue equation $R = 20000p - 100p^2$ for sales of Widgets, where p is the price of a widget and R is the revenue obtained at that price.

    (a) Find the price that gives the maximum revenue, and determine the revenue that will be obtained at that price.

    (b) For what prices will the revenue be \$500,000 or less?

6. Another farmer is going to create a rectangular field (without compartments) against a straight canal, by putting fence along the three straight sides of the field that are away from the canal. (See picture to the right.) He has 1000 feet of fence with which to do this.



    (a) Write an equation (in simplified form) for the area A of the field, in terms of x.

    (b) Give the feasible domain of the function.

    (c) Determine the maximum area of the field.

    (d) Determine all possible values of x for which the area is at least 90,000 square feet.

3.5 Distance, Midpoint and Circles

Performance Criteria:

3. (h) Find the distance between two points in the plane; find the midpoint of a segment between two points in a plane.

   (i) Graph a circle, given its equation in standard form. Given the center and radius of a circle, give its equation in standard form.

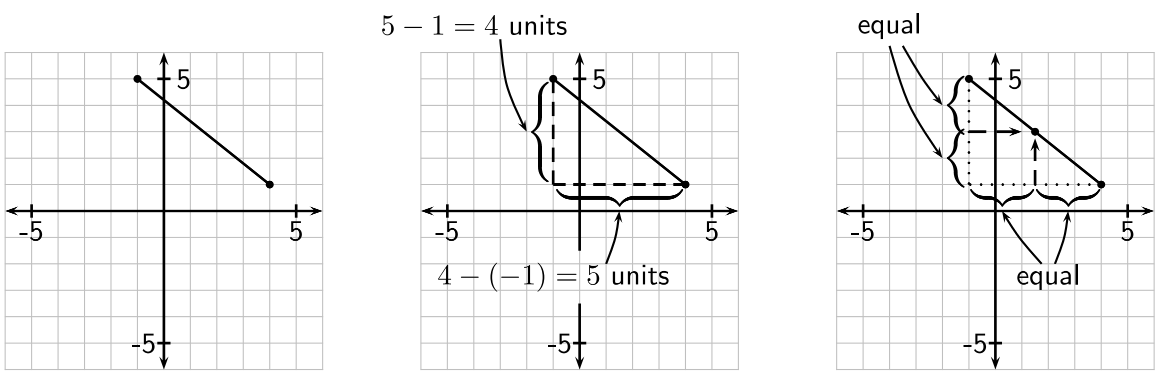
   (j) Put the equation of a circle in standard form

$$(x-h)^2+(y-k)^2=r^2$$

by completing the square.

Length and Midpoints of a Segment

In this section we will consider some geometric ideas that are not central to the main ideas of this course, but which are occasionally of interest. Consider a **segment** in the xy-plane like the one from $(-1,5)$ to $(4,1)$, shown to the below and to the left. (A segment is a section of a line that has ends in both directions.) There are two things we'd like to know about this segment: its length, and the coordinates of its midpoint.



To find the length of the segment we can think of a right triangle whose legs are shown by the dashed segments in the middle picture above, and apply the Pythagorean theorem $a^2+b^2=c^2$. In this case we'll let a be the length of the horizontal leg; we can see that its length is five units, which can be obtained by subtracting the x-coordinates: $4-(-1)=5$. b is then the length of the vertical leg, given by $5-1=4$. Substituting into the quadratic formula gives $5^2+4^2=c^2$, so $c=\sqrt{5^2+4^2}=\sqrt{41}$. (We don't need the usual $\pm$ with the root because c must be positive in this situation, because it is a length.)

To find the midpoint we use an idea that should be intuitively clear: The x-coordinate of the midpoint should be halfway between the x-coordinates of the endpoints, and the y-coordinate of the midpoint should be halfway between the y-coordinates of the endpoints. This is shown in the picture above and to the right. To find a value halfway between two other values *we average the two values*. The x-coordinate of the midpoint is then the average of -1 and 4, which is

$x = \dfrac{-1+4}{2} = 1\frac{1}{2}$, and the y-coordinate of the midpoint is $y = \dfrac{1+5}{2} = 3$. The coordinates of the midpoint are then $\left(1\frac{1}{2}, 3\right)$.

We can summarize the general procedure for finding the length and midpoint of a segment as follows:

Length and Midpoint of a Segment

- To find the length of a segment between two points P and Q, subtract the x-coordinates to find the horizontal distance (or negative of the distance, it doesn't matter) between the points. Then subtract the y-coordinates to find the vertical distance. Substitute those two distances as a and b of the Pythagorean formula $a^2 + b^2 = c^2$; the *positive* value of c is the length of the segment.

- To find the coordinates of the midpoint of a segment, average the x-coordinates of the endpoints to get the x-coordinate of the midpoint. Average the y-coordinates of the endpoints to get the y-coordinate of the midpoint.

- The above two things can be done symbolically as follows. For a segment with endpoints $P(x_1, y_1)$ and $Q(x_2, y_2)$, the length l and midpoint M of the segment can be found by

$$l = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \qquad \text{and} \qquad M = \left(\frac{x_1 + x_2}{2}\, , \, \frac{y_1 + y_2}{2}\right)$$

◇ **Example 3.5(a):** Find the length and midpoint of the segment with endpoints $(-5, -3)$ and $(7, 10)$. Give the length in exact form *and* as a decimal rounded to the nearest tenth of a unit. Midpoint Example

Solution: The length is

$$l = \sqrt{(-5-7)^2 + (-3-10)^2} = \sqrt{(-12)^2 + (-13)^2} = \sqrt{144 + 169} = \sqrt{313} = 17.7$$

and the midpoint is

$$M = \left(\frac{-5+7}{2}\, , \frac{-3+10}{2}\right) = \left(\frac{2}{2}\, , \frac{7}{2}\right) = (1, 3\tfrac{1}{2})$$

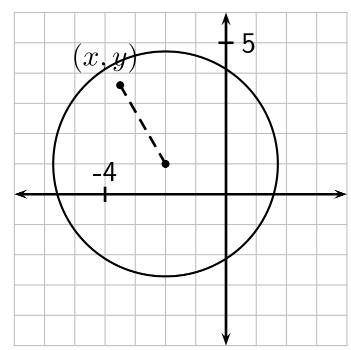
◇ **Example 3.5(b):** One endpoint of a segment is $(5, -4)$ and the midpoint is $(7, 3)$. Find the coordinates of the other endpoint.

Solution: Note that we know one of (x_1, y_1) or (x_2, y_2) but not both, and we also know the midpoint. If we take $(5, -4)$ to be (x_2, y_2) we have that

$$(7, 3) = \left(\frac{x_1 + 5}{2}, \frac{y_1 + (-4)}{2}\right)$$

This gives us the two equations $7 = \frac{x_1 + 5}{2}$ and $3 = \frac{y_1 + (-4)}{2}$. Solving these for x_1 and y_1 gives us $x_1 = 9$ and $y_1 = 10$, so the missing endpoint is then $(9, 10)$.

Circles in the Plane



Consider the circle shown to the right. It is easily seen that the center of the circle is the point $(-2, 1)$ and the radius is three units. The point (x, y) is an arbitrary point on the circle; regardless of where it is on the circle, its distance from the center is three units. We can apply the Pythagorean theorem or the formula for the length of a segment to the dashed segment to get

$$[x - (-2)]^2 + (y - 1)^2 = 3^2 \qquad (1)$$

This equation must be true for any point (x, y) on the circle, and any ordered pair (x, y) that makes the equation true *MUST* be on the circle. Therefore we say that (1) is the equation of the circle. In general the equation of the circle with center (h, k) and radius r is $(x-h)^2+(y-k)^2 = r^2$. We usually do a little simplification; we would simplify equation (1) to $(x+2)^2+(y-1)^2 = 9$.

Equation of a Circle

The equation of the circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

◇ **Example 3.5(c):** Give the equation of the circle with center $(4, -7)$ and radius 6.

Solution: By the above, the equation would be $(x - 4)^2 + (y - (-7))^2 = 6^2$, which can be cleaned up a little to get $(x - 4)^2 + (y + 7)^2 = 36$. We don't usually multiply out the two terms $(x - 4)^2$ and $(y + 7)^2$.

This last example illustrates where formality is sometimes a bit confusing. Where we see a minus sign $(y - k)$ in the equation $(x - h)^2 + (y - k)^2 = r^2$, in this last example we have a plus sign,

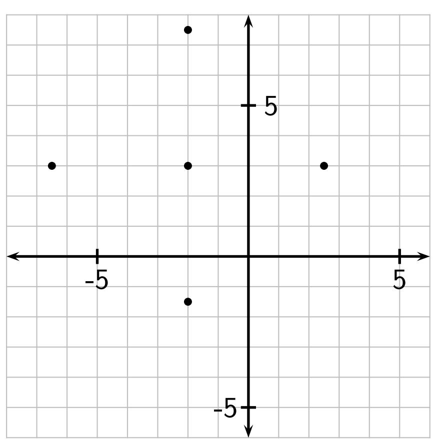
$(y+7)$. Here is a nice way to think about this concept: Recall that when solving the equation $x^2 - 2x - 15 = 0$ we factor to get $(x-5)(x+3) = 0$. The solutions are then the values that make the two factors $(x-5)$ and $(x+3)$ equal to zero, 5 and -3. Note also that for the equation $(x-4)^2 + (y+7)^2 = 36$ from Example 3.5(c), the values of x and y that make $(x-4)$ and $(y+7)$ equal to zero are the coordinates of the center of the circle. These two things again illustrate the important general principle we first saw in Section 1.7:

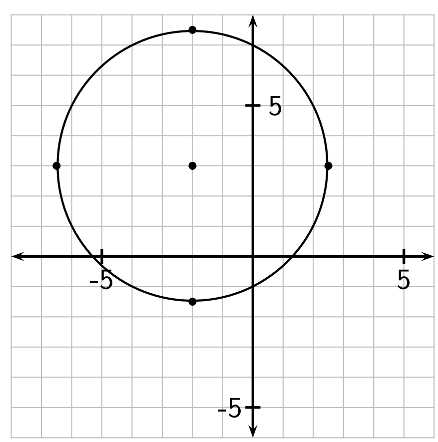
> *When we have a quantity $(x \pm a)$, the value of x (or any other variable in that position) that makes the entire quantity zero is likely an important value of that variable.*

Suppose we know that the equation $(x-4)^2+(y+7)^2 = 36$ is the equation of a circle, and we recognize that the values $x = 4$ and $y = -7$ that make $(x-4)$ and $(y+7)$ both zero are probably important. For a circle, there is only one point that is more important than any others, the center. Therefore $(4, -7)$ must be the center of the circle.

◇ **Example 3.5(d):** Sketch the graph of the circle with equation $(x+2)^2+(y-3)^2 = 20$. Correctly locate the center and the four points that are horizontally and vertically aligned with the center. (The locations of the four points will have to be somewhat approximate.)

Solution: The center of the circle is $(-2, 3)$ since $x = -2$ and $y = 3$ make $(x+2)$ and $(y-3)$ equal to zero. Also, $r^2 = 20$ so $r = \sqrt{20} \approx 4.5$. (We ignore the negative square root because the radius is a positive number.) To graph the circle we first plot the center, then plot four points that are 4.5 units to the right and left of and above and below the center. Those points are shown on the graph below and to the left. We then sketch in the circle through those four points and centered at $(-2, 3)$. This is shown below and to the right.





Putting the Equation of a Circle in Standard Form

Consider the equation $x^2 + y^2 + 2x - 10y + 22 = 0$. It turns out this is the equation of a circle, but it is not in the form that we have been working with, so we can't easily determine the center and radius. What we would like to do is put the equation into the standard form $(x-h)^2+(y-k)^2=r^2$. To do this we use the **completing the square** procedure that we saw in Section 3.2. Remember the key idea:

> Suppose we have x^2+cx. If we add $d=(\frac{1}{2}c)^2$ to this quantity the result x^2+cx+d factors into two *EQUAL* factors.

When using this idea with circles there will one slight change; Rather than adding and subtracting the same value on one side of an equation (which is, of course, equivalent to adding zero), we'll add the desired value $d=(\frac{1}{2}c)^2$ to both sides of the equation. That will keep the equation "in balance."

◇ **Example 3.5(e):** $x^2+y^2+2x-10y+22=0$ is the equation of a circle. Put it into standard form, then give the center and radius of the circle. Another Example

Solution:

$x^2+y^2+2x-10y+22$	$= 0$	the original equation
$x^2+y^2+2x-10y$	$= -22$	subtract 22 from both sides
x^2+2x+y^2-10y	$= -22$	rearrange terms on the left side to get the x terms together and the y terms together
$x^2+2x+1+y^2-10y+25$	$= -22+1+25$	add $[\frac{1}{2}(2)]^2=1$ and $[\frac{1}{2}(10)]^2=25$ to both sides
$(x+1)(x+1)+(y-5)(y-5)$	$= 4$	factor the left side and simplify the right
$(x+1)^2+(y-5)^2$	$= 4$	equation in standard form

The circle has center $(-1,5)$ and radius 2.

Section 3.5 Exercises

To Solutions

1. (a) Find the distance from $(-3,1)$ to $(5,7)$.

   (b) Find the midpoint of the segment from $(-3,1)$ to $(5,7)$.

2. (a) Find the distance from $(-1,8)$ to $(0,1)$. Give your answer in simplified square root form *AND* as a decimal, rounded to the nearest hundredth.

   (b) Find the midpoint of the segment from $(-1,8)$ to $(0,1)$.

3. (a) Find the distance from $(1.3, -3.6)$ to $(5.9, 2.1)$. Give your answer as a decimal, rounded to the nearest hundredth.

   (b) Find the midpoint of the segment from $(1.3, -3.6)$ to $(5.9, 2.1)$.

4. Find the other endpoint of the segment with endpoint $(-3, -7)$ and midpoint $(-1, 2\frac{1}{2})$.

5. Find the other endpoint of the segment with endpoint $(5, -2)$ and midpoint $(0, 0)$. Try to do this with no calculation; visualize a graph of the segment.

6. Determine the equation of the circle with the given center and radius.

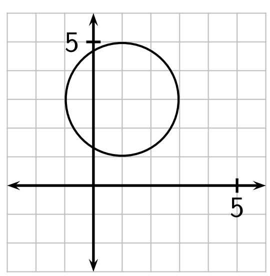
   (a) $(2, 3)$, $r = 5$ (b) $(2, -3)$, $r = 5$ (c) $(0, -5)$, $r = 6$.

7. Give the center and radius for each of the circles whose equations are given. When the radius is not a whole number, give the radius in both exact form and as a decimal, rounded to the the nearest tenth.

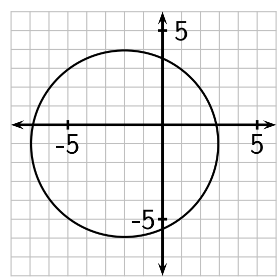
   (a) $(x + 3)^2 + (y - 5)^2 = 4$ (b) $(x - 4)^2 + y^2 = 28$ (c) $x^2 + y^2 = 7$

8. Give the equation of each circle shown below.

   (a)



   (b)



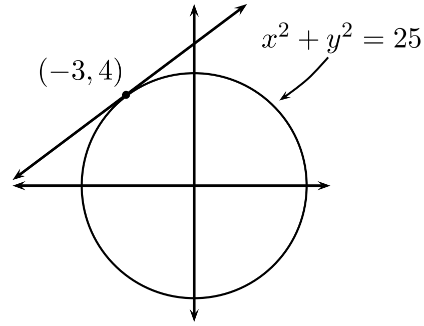
9. Put each equation into standard form by completing the square, then give the center and radius of each circle.

   (a) $x^2 + y^2 = -2x + 6y + 6$ (b) $x^2 - 4x + y^2 + 6y = 12$

   (c) $x^2 + y^2 - 6x + 2y = 18$ (d) $x^2 + y^2 + 10x + 15 = 0$

10. Determine an equation for the circle that has a diameter with end points $A(7, 2)$ and $B(-1, 6)$. (The diameter is a segment whose endpoints are on the circle and that passes through the center of the circle.)

11. A line that touches a circle in only one point is said to be **tangent** to the circle. The picture to the right shows the graph of the circle $x^2 + y^2 = 25$ and the line that is tangent to the circle at $(-3, 4)$. Find the equation of the tangent line. (**Hint:** The tangent line is perpendicular to the segment that goes from the center of the circle to the point $(-3, 4)$.)



12. Equations of circles are not functions, so we don't generally talk about a maximum or minimum in the case of a circle. Determine the highest and lowest points (that is, the points with largest and smallest y values) on the circle with equation $(x - 2)^2 + (y + 3)^2 = 49$.

13. A line is called a **perpendicular bisector** of a segment if the line intersects the segment at its midpoint and the line is perpendicular to the segment. Determine the equation of the perpendicular bisector of the segment with endpoints $(1, -1)$ and $(5, 5)$.

14. Determine the equation of the circle for which the segment with endpoints $(1, -1)$ and $(5, 5)$ is a diameter. (A **diameter** is a segment passing through the center of the circle, and whose endpoints are on the circle.)

3.6 Systems of Two Non-Linear Equations

Performance Criteria:

3. (k) Solve a system of two non-linear equations.

Let's begin by considering the system of equations

$$\begin{aligned} x^2 + y &= 9 \\ x - y + 3 &= 0 \end{aligned}$$

The second equation is linear, but the first is not, its graph is a parabola. If you think of a line and a parabola floating around in the xy-plane, they might not cross at all, they might touch in just one point, or they might cross in two places. (Try drawing pictures of each of these situations.) In this section we will see how to solve systems of equations like this one, where one or both of the equations are non-linear. In some cases the equations may not even represent functions.

Of course solving a system of equations means that we are looking for (x, y) pairs that make *BOTH* equations true. There are multiple options for going about this in this case:

- Solve the first equation for y and substitute into the second equation.
- Solve the second equation for x or y and substitute into the first equation.
- Add the two equations.
- Solve both equations for y and set the results equal to each other.

The least desirable of these options would be to solve the second equation for x, because we would get $x = y - 3$, and substituting that into the first equation would give $(y-3)^2 + y = 9$. This would require squaring $y - 3$, which is not hard, but the other choices above would not require such a computation. Let's take a look at how a couple of those options would go.

◇ **Example 3.6(a):** Solve $\begin{aligned} x^2 + y &= 9 \\ x - y + 3 &= 0 \end{aligned}$ by solving the second equation for y and substituting into the first equation.

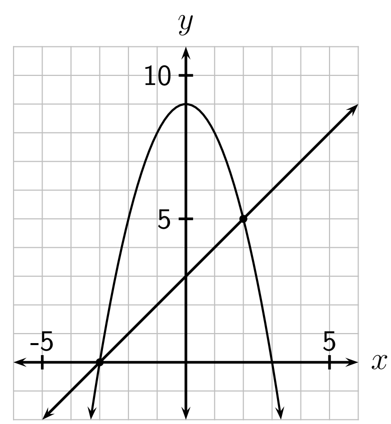
Solution: Solving the second equation for y gives $y = x + 3$. Substituting that into the first equation we get $x^2 + x + 3 = 9$. Subtracting nine from both sides and factoring leads to the solutions $x = -3$ and $x = 2$. We can substitute these values into either equation to find the corresponding values of y. Using the second equation, when $x = -3$ we get $y = 0$, and when $x = 2$, $y = 5$. The solutions to the system are then $(-3, 0)$ and $(2, 5)$.

NOTE: When stating your answer *you must make it clear which values of x and y go together*. The easiest way to do this is probably just giving the solutions as ordered pairs, as done in the above example.

◇ **Example 3.6(b):** Solve
$$\begin{aligned} x^2+y &= 9 \\ x-y+3 &= 0 \end{aligned}$$
by the addition method.

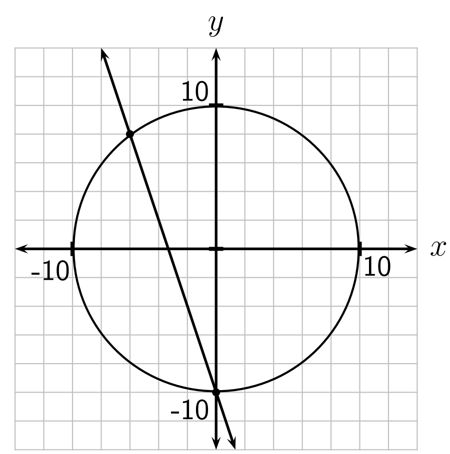
Solution: Adding the two equations gives $x^2+x+3=9$, the same equation we arrived at in the previous example. The remainder of the steps are exactly the same as in that example.

Recall that when a system of two *linear* equations in two unknowns has a solution, it is the (x,y) pair representing the point where the graphs of the two lines cross. In this case the solution is the point where the graphs of $x^2+y=9$ and $x-y+3=0$ cross. Note that these can both be solved for y to get $y=-x^2+9$ and $y=x+3$. Clearly the graph of the second is a line with slope one and y-intercept three. From what we learned in Section 2.6, the graph of $y=-x^2+9$ is the graph of $y=x^2$ reflected across the x-axis and shifted up nine units. Both of the equations are graphed together on the grid to the right, and we can see that the graphs cross at $(-3,0)$ and $(2,5)$, the solutions to the system.



◇ **Example 3.6(c):** Consider the two equations $3x+y=-10$ and $x^2+y^2=100$. Determine the solution(s) to the system of equations first by graphing both on the same grid, then by solving algebraically.

Solution: Solving the first equation for y gives $y=-3x-10$, a line with slope of -3 and y-intercept -10. The second equation is that of a circle with center at the origin and radius ten. Both of these are graphed together on the grid to the right, where we can see that the solutions to the system are $(0,-10)$ and $(-6,8)$. To solve the system algebraically we would solve the first equation for y, which we've already done, and substitute the result into the second equation:



$$\begin{aligned} x^2+(-3x-10)^2 &= 100 \\ x^2+9x^2+60x+100 &= 100 \\ 10x^2+60x &= 0 \\ 10x(x+6) &= 0 \end{aligned}$$

Therefore $x=0$ or $x=-6$. Substituting $x=0$ into $y=-3x-10$ results in $y=-10$, so $(0,-10)$ is a solution to the system. Substituting $y=-6$ into the same equation results in $x=8$, giving the solution $(-6,8)$.

Section 3.6 Exercises To Solutions

1. Solve the system $\begin{aligned} x^2+y &= 9 \\ x-y+3 &= 0 \end{aligned}$ a third way, by solving the second equation for x and substitution the result into the first equation. (Earlier we said this isn't quite as efficient as the methods shown in Examples 3.6(a) and (b), but it should give the same results.)

2. Consider now the non-linear system of equations $\begin{aligned} x^2+y &= 9 \\ 2x+y &= 10 \end{aligned}$.

   (a) Solve this system. How does your solution differ from that of Exercise 1 and the examples?

   (b) What would you expect to see if you graph the two equations for this system together? Graph them together to see if you are correct.

3. Do the following for the system $\begin{aligned} x+3y &= 5 \\ x^2+y^2 &= 25 \end{aligned}$.

   (a) Graph the two equations on the same grid. (Try doing this without using your calculator.) What does it appear that the solution(s) to the system is (are)?

   (b) Solve the system algebraically. If the results do not agree with your graph, try to figure out what is wrong.

4. For the system $\begin{aligned} x^2+y^2 &= 8 \\ y-x &= 4 \end{aligned}$,

   (a) determine the solution(s) to the system by graphing the two equations on one grid, then

   (b) solve the system algebraically. Make sure your two answers agree, or find your error!

5. There is a particular way of using the substitution method for solving a system of equations that often comes up. It arises when we are solving a system that is given with both equations already solved for y, like the system $\begin{aligned} y &= x^2-4x+1 \\ y &= 2x-7 \end{aligned}$. What we say is that we "set the two expressions for y equal," which is of course the same as substituting the value of y from one equation into y in the other equation. Do this to solve this system.

6. Consider the system $\begin{aligned} x+y^2 &= 3 \\ x^2+y^2 &= 5 \end{aligned}$.

   (a) Solve the system algebraically - you should get *four* solution pairs.

   (b) Graph the two equations on the same grid "by hand." Then check your graph and your answer to (a) by graphing the two equations with a device.

7. The Acme company manufactures Doo-dads. If x is the number of Doo-dads they make and sell in a week, their costs for the week (in dollars) are given by $C = 2450 + 7x$, and their revenues (also in dollars) are $R = 83x - 0.1x^2$. Find the numbers of Doo-dads that are made and sold when they break even (revenues equal costs).

3.7 Average Rates of Change, Secant Lines and Difference Quotients

Performance Criteria:

3. (l) Find the average rate of change of a function over an interval, from either the equation of the function or the graph of the function. Include units when appropriate.

   (m) Find and simplify a difference quotient.

Average Rate of Change

◇ **Example 3.7(a):** Suppose that you left Klamath Falls at 10:30 AM, driving north on on Highway 97. At 1:00 PM you got to Bend, 137 miles from Klamath Falls. How fast were you going on this trip?

Solution: Since the trip took two and a half hours, we need to divide the distance 137 miles by 2.5 hours, with the units included in the operation:

$$\text{speed} = \frac{137 \text{ miles}}{2.5 \text{ hours}} = 54.8 \text{ miles per hour}$$

Of course we all know that you weren't going 54.8 miles per hour the entire way from Klamath Falls to Bend; this speed is the *average* speed during your trip. When you were passing through all the small towns like Chemult and La Pine you likely slowed down to 30 or 35 mph, and between towns you may have exceeded the legal speed limit. 54.8 miles per hour is the speed you would have to go to make the trip in two and a half hours driving at a constant speed for the entire distance. The speed that you see any moment that you look at the speedometer of your car is called the *instantaneous* speed.

Speed is a quantity that we call a **rate of change**. It tells us the change in distance (in the above case, measured in miles) for a given change in time (measured above in hours). That is, for every change in time of one hour, an additional 54.8 miles of distance will be gained. In general, we consider rates of change when one quantity depends on another; we call the first quantity the **dependent variable** and the second quantity the **independent variable**. In the above example, the distance traveled depends on the time, so the distance is the dependent variable and the time is the independent variable. We find the average rate of change as follows:

$$\text{average rate of change} = \frac{\text{change in dependent variable}}{\text{change in independent variable}}$$

The changes in the variables are found by subtracting. The order of subtraction does not matter in theory, but we'll follow the fairly standard convention

$$\text{average rate of change} = \frac{\text{final value minus intial value for dependent variable}}{\text{final value minus intial value for independent variable}}$$

This should remind you of the process for finding a slope. In fact, we will see soon that every average rate of change can be interpreted as the slope of a line.

The example on the next page will illustrate what we have just talked about.

◇ **Example 3.7(b):** Again you were driving, this time from Klamath Falls to Medford. At the top of the pass on Highway 140, which is at about 5000 feet elevation, the outside thermometer of your car registered a temperature of 28°F. In Medford, at 1400 feet of elevation, the temperature was 47°. What was the average rate of change of temperature with respect to elevation?

Solution: The words "temperature with respect to elevation" implies that temperature is the dependent variable and elevation is the independent variable. From the top of the pass to Medford we have

$$\text{average rate of change} = \frac{47°\text{F} - 28°\text{F}}{1400 \text{ feet} - 5000 \text{ feet}} = \frac{19°\text{F}}{-3600 \text{ feet}} = -0.0053°\text{F per foot}$$

How do we interpret the negative sign with our answer? Usually, when interpreting a rate of change, its value is *the change in the dependent variable for each INCREASE in one unit of the independent variable*. So the above result tells us that the temperature (on that particular day, at that time and in that place) *decreases* by 0.0053 degrees Fahrenheit for each foot of elevation *gained*.

Average Rate of Change of a Function

You probably recognized that what we have been calling independent and dependent variables are the "inputs" and "outputs" of a function. This leads us to the following:

Average Rate of Change of a Function

For a function $f(x)$ and two values a and b of x with $a < b$, the **average rate of change of** f **with respect to** x over the interval $[a, b]$ is

$$\left.\frac{\Delta f}{\Delta x}\right|_{[a,b]} = \frac{f(b) - f(a)}{b - a}$$

There is no standard notation for average rate of change, but it is standard to use the capital Greek letter delta (Δ) for "change." The notation $\left.\frac{\Delta f}{\Delta x}\right|_{[a,b]}$ could be understood by any mathematician as change in f over change in x, over the interval $[a, b]$. Because it is over an interval it must necessarily be an average, and the fact that it is one change divided by another indicates that it is a *rate* of change.

◇ **Example 3.7(c):** For the function $h(t) = -16t^2 + 48t$, find the average rate of change of h with respect to t from $t = 0$ to $t = 2.5$.

Solution:
$$\left.\frac{\Delta h}{\Delta t}\right|_{[0,2.5]} = \frac{h(2.5) - h(0)}{2.5 - 0} = \frac{20 - 0}{2.5 - 0} = 8$$

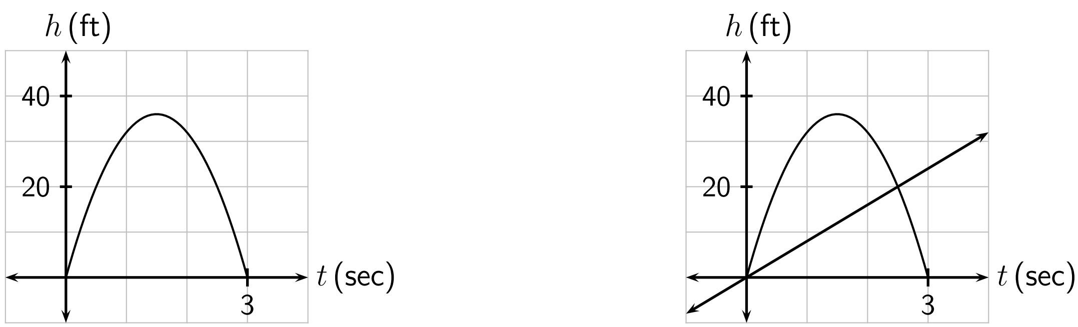
You might recognize $h(t) = -16t^2 + 48t$ as the height h of a projectile at any time t, with h in feet and t in seconds. The units for our answer are then feet per second, indicating that over the first 2.5 seconds of its flight, the height of the projectile increases at an average rate of 8 feet per second.

Secant Lines

The graph below and to the left is for the function $h(t) = -16t^2 + 48t$ of Example 3.7(c). To the right is the same graph, with a line drawn through the two points $(0, 0)$ and $(2.5, 20)$. Recall that the average rate of change in height with respect to time from $t = 0$ to $t = 2.5$ was determined by

$$\left.\frac{\Delta h}{\Delta t}\right|_{[0,2.5]} = \frac{h(2.5) - h(0)}{2.5 - 0} = \frac{20 - 0}{2.5 - 0} = 8 \text{ feet per second}$$

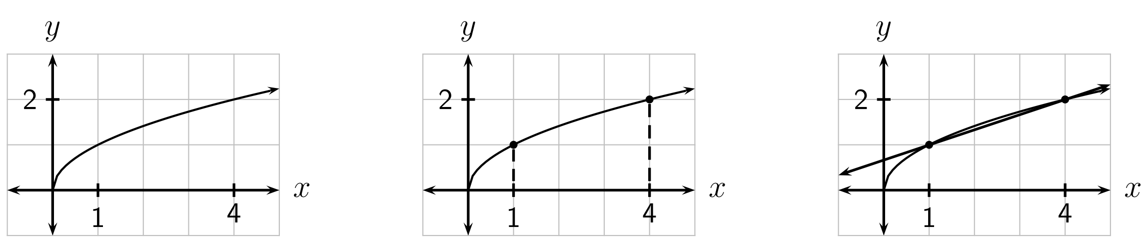
With a little thought, one can see that the average rate of change is simply the slope of the line in the picture to the right below.



A line drawn through two points on the graph of a function is called a **secant line**, and the slope of a secant line represents the rate of change of the function between the values of the independent variable where the line intersects the graph of the function.

◇ **Example 3.7(d):** Sketch the secant line through the points on the graph of the function $y = \sqrt{x}$ where $x = 1$ and $x = 4$. Then compute the average rate of change of the function between those two x values.

Solution: The graph below and to the left is that of the function. In the second graph we can see how we find the two points on the graph that correspond to $x = 1$ and $x = 4$. The last graph shows the secant line drawn in through those two points.



The average rate of change is $\frac{2-1}{4-1} = \frac{1}{3}$, the slope of the secant line.

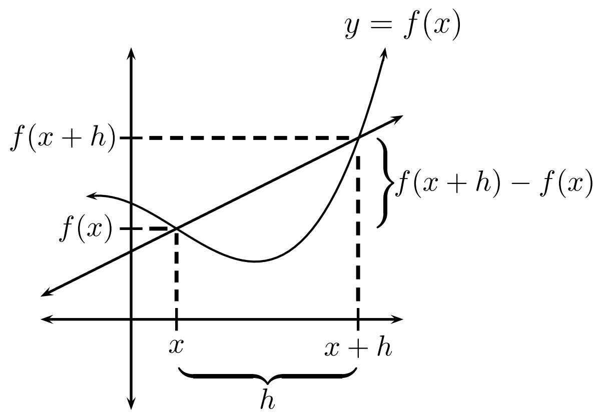
Difference Quotients

The following quantity is the basis of a large part of the subject of calculus:

> **Difference Quotient for a Function**
>
> For a function $f(x)$, the **difference quotient** for f is
>
> $$\frac{f(x+h)-f(x)}{h}$$

Let's rewrite the above expression to get a better understanding of what it means. With a little thought you should agree that

$$\frac{f(x+h)-f(x)}{h} = \frac{f(x+h)-f(x)}{(x+h)-a}$$



This looks a lot like a slope, and we can see from the picture to the right that it is. The quantity $f(x+h) - f(x)$ is a "rise" and $(x+h) - x = h$ is the corresponding "run." The difference quotient is then the slope of the tangent line through the two points $(x, f(x))$ and $(x+h, f(x+h))$.

◇ **Example 3.7(e):** For the function $f(x) = 3x - 5$, find and simplify the difference quotient. Another Example

$$\frac{f(x+h)-f(x)}{h} = \frac{[3(x+h)-5]-[3x-5]}{h} = \frac{3x+3h-5-3x+5}{h} = \frac{3h}{h} = 3$$

Note that in the above example, the h in the denominator eventually canceled with one in the numerator. *This will always happen if you carry out all computations correctly!*

◇ **Example 3.7(f):** Find and simplify the difference quotient $\dfrac{g(x+h)-g(x)}{h}$ for the function $g(x) = 3x - x^2$. Another Example Another Example

Solution: With more complicated difference quotients like this one, it is sometimes best to compute $g(x+h)$, then $g(x+h) - g(x)$, before computing the full difference quotient:

$$g(x+h) = 3(x+h) - (x+h)^2 = 3x + 3h - (x^2 + 2xh + h^2) = 3x + 3h - x^2 - 2xh - h^2$$

$$g(x+h) - g(x) = (3x + 3h - x^2 - 2xh - h^2) - (3x - x^2) = 3h - 2xh - h^2$$

$$\frac{g(x+h) - g(x)}{h} = \frac{3h - 2xh - h^2}{h} = \frac{h(3 - 2x - h)}{h} = 3 - 2x - h$$

Section 3.7 Exercises To Solutions

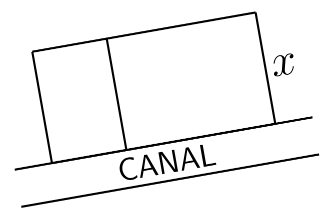
1. In Example 2.5(a) we examined the growth of a rectangle that started with a width of 5 inches and a length of 8 inches. At time zero the width started growing at a constant rate of 2 inches per minute, and the length began growing by 3 inches per minute. We found that the equation for the area A (in square inches) as a function of time t (in minutes) was $A = 6t^2 + 31t + 40$. Determine the average rate of increase in area with respect to time from time 5 minutes to time 15 minutes. Give units with your answer!

2. The height h (feet) of a projectile at time t (seconds) is given by $h = -16t^2 + 144t$.

   (a) Find the average rate of change in height, with respect to time, from 4 seconds to 7 seconds. Give your answer as a sentence that includes these two times, your answer, and whichever of the words *increasing* or *decreasing* that is appropriate.

   (b) Find the average rate of change in height, with respect to time, from time 2 seconds to time 7 seconds. Explain your result *in terms of the physical situation*.

3. A farmer is going to create a rectangular field with two compartments against a straight canal, as shown to the right, using 1000 feet of fence. No fence is needed along the side formed by the canal.



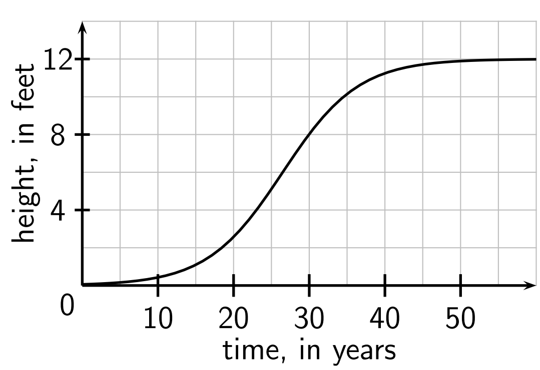
   (a) Find the total area of the field when x is 200 feet and again when x is 300 feet.

   (b) Find the average rate of change in area, with respect to x, between when x is 200 feet and when it is 300 feet. Give units with your answer.

4. The graph to the right shows the heights of a certain kind of tree as it grows. Use it to find the average rate of change of height with respect to time from

   (a) 10 years to 50 years

   (b) 25 years to 30 years



5. In a previous exercise you found the revenue equation $R = 20000p - 100p^2$ for sales of Widgets, where p is the price of a widget and R is the revenue obtained at that price.

   (a) Find the average change in revenue, with respect to price, from a price of \$50 to \$110. Include units with your answer.

   (b) Sketch a graph of the revenue function and draw in the secant line whose slope represents your answer to (a). Label the relevant values on the horizontal and vertical axes.

6. (a) Sketch the graph of the function $y = 2^x$. Then draw in the secant line whose slope represents the average rate of change of the function from $x = -2$ to $x = 1$.

   (b) Determine the average rate of change of the function from $x = -2$ to $x = 1$. Give your answer in (reduced) fraction form.

7. Find the average rate of change of $f(x) = x^3 - 5x + 1$ from $x = 1$ to $x = 4$.

8. Find the average rate of change of $g(x) = \dfrac{3}{x-2}$ from $x = 4$ to $x = 7$.

9. Consider the function $h(x) = \frac{2}{3}x - 1$.

   (a) Find the average rate of change from $x = -6$ to $x = 0$.

   (b) Find the average rate of change from $x = 1$ to $x = 8$.

   (c) You should notice two things about your answers to (a) and (b). What are they?

10. Find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ for each of the following.

    (a) $f(x) = x^2 - 3x + 5$

    (b) $f(x) = 3x^2 + 5x - 1$

    (c) $f(x) = 3x - 1$

    (d) $f(x) = x^2 + 7x$

    (e) $f(x) = 3x - x^2$

    (f) $f(x) = -\frac{3}{4}x + 2$

    (g) $f(x) = x^2 + 4x$

    (h) $f(x) = 7x^2 + 4$

11. For the function $f(x) = 3x^2$, find and simplify the difference quotient.

12. (a) Compute and simplify $(x+h)^3$

    (b) For the function $f(x) = x^3 - 5x + 1$, find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$.

    (c) In Exercise 7 you found the average rate of change of $f(x) = x^3 - 5x + 1$ from $x = 1$ to $x = 4$. That is equivalent to computing the difference quotient with $x = 1$ and $x + h = 4$. Determine the value of h , then substitute it and $x = 1$ into your answer to part (b) of this exercise. You should get the same thing as you did for Exercise 7!

13. For this exercise you will be working with the function $f(x) = \dfrac{1}{x}$.

(a) Find and simplify $f(x+h) - f(x)$ by carefully obtaining a common denominator and combining the two fractions.

(b) Using the fact that dividing by h is the same as multiplying by $\frac{1}{h}$, find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$.

3.8 Chapter 3 Exercises

To Solutions

1. Solve each of the following linear inequalities.

   (a) $8 - 5x \leq -2$

   (b) $6x - 3 < 63$

   (c) $3(x-2) + 7 < 2(x+5)$

   (d) $-4x + 3 < -2x - 9$

   (e) $8 - 5(x+1) \leq 4$

   (f) $7 - 4(3x+1) \geq 2x - 5$

2. Solve each of the following inequalities. (**Hint:** Get zero on one side first, if it isn't already.) Give your answers as inequalities, OR using interval notation.

   (a) $21 + 4x \geq x^2$

   (b) $y^2 + 3y \geq 18$

   (c) $8x^2 < 16x$

   (d) $\dfrac{1}{15}x^2 \leq \dfrac{1}{6}x - \dfrac{1}{10}$

3. Consider the function $f(x) = \frac{1}{4}(x+1)(x-3)$.

   (a) Sketch the graph of the function *without calculating anything*, just using the equation of the function as given. You should be able to locate exactly two points on the parabola.

   (b) Based on your graph, will the rate of change of the function from $x = -3$ to $x = 2$ be negative, or will it be positive? Add something to your graph to help you answer this question, and provide a few sentences of explanation.

4. **To do this exercise you need to have done Exercise 13 of the Chapter 1 Exercises.** In this section we found out how to determine the equation of a parabola from two points on the parabola, *where one of the points is the vertex*. If we don't know the vertex, we must have three points on the parabola in order to determine its equation. This exercise will lead us through the process for doing that. We will find the equation of the parabola containing the points $(-3,3)$, $(1,3)$, and $(3,-3)$.

   (a) Put the coordinates $(-3,3)$ into the standard form $y = ax^2 + bx + c$ of a parabola, and rearrange to get something like $2a + 5b - c = -4$.

   (b) Repeat (a) for the other two points on the parabola.

   (c) You now have a system of three equations in three unknowns. Solve the system in the manner described in Exercise 13 of the Chapter 1 Exercises.

   (d) You have now found a, b and c, so you can write the equation $y = ax^2 + bx + c$ of the parabola, so do so. Then graph the equation using technology and check to see that it does indeed pass through the given points.

5. Find the equation of the parabola containing the points $(0,8)$, $(1,3)$ and $(4,0)$. Check your answer by graphing with technology.

A Solutions to Exercises

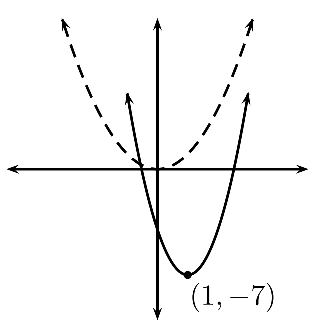
A.3 Chapter 3 Solutions

Section 3.1 Solutions Back to 3.1 Exercises

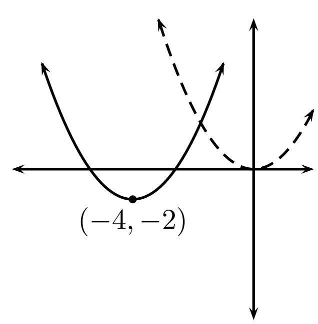
1. (a) $f(x) = -\frac{1}{5}x^2 - \frac{1}{5}x + \frac{6}{5}$ (b) $g(x) = 3x^2 - 6x + 5$
   (c) $y = -2x^2 + 18$ (d) $h(x) = \frac{1}{2}x^2 + 2x - 3$
2. (a) $f(x) = -\frac{1}{4}(x+1)(x-5)$ (b) $y = 2(x-3)(x-1)$
   (c) $g(x) = \frac{1}{2}(x+1)(x+9)$ (d) $h(x) = -5(x+1)(x-1)$
   (e) $y = 3(x-2)(x-3)$ (f) $f(x) = \frac{1}{4}(x+8)(x-2)$
   (g) $y = -\frac{1}{3}(x+2)(x-3)$ (h) $g(x) = -(x+2)(x-7)$
3. (a) $y = 5(x+1)(x-1)$ (b) $h(x) = x(x-4)$
4. (a) The x-intercepts are -1 and 5, the y-intercept is $\frac{5}{4}$ and the vertex is $(2, \frac{9}{4})$.
   (b) The x-intercepts are 1 and 3, the y-intercept is 6 and the vertex is $(2, -2)$.
   (c) The x-intercepts are -1 and -9, the y-intercept is $\frac{9}{2}$ and the vertex is $(-5, -8)$.
   (d) The x-intercepts are -1 and 1, the y-intercept is 5 and the vertex is $(0, 5)$.
   (e) The x-intercepts are -2 and 7, the y-intercept is 14 and the vertex is $(2\frac{1}{2}, 20\frac{1}{4})$.
   (f) The x-intercepts are -8 and 2, the y-intercept is -4 and the vertex is $(-3, -6\frac{1}{4})$.
5. (a) $f(x) = -x^2 - 2x + 15$ (b) $f(x) = -(x-3)(x+5)$ (c) $(-1, 16)$
6. (a) $y = \frac{1}{3}x^2 - \frac{4}{3}x + 1$ (b) $y = \frac{1}{3}(x-3)(x-1)$ (c) $(2, -\frac{1}{3})$
9. The graph of a quadratic function can have 0, 1 or 2 x-intercepts.
10. Every quadratic function has exactly one absolute maximum (if the graph of the function opens downward) or one absolute minimum (if the graph opens upward), but not both. The are no relative maxima or minima other than the absolute maximum or minimum.

Section 3.2 Solutions Back to 3.2 Exercises

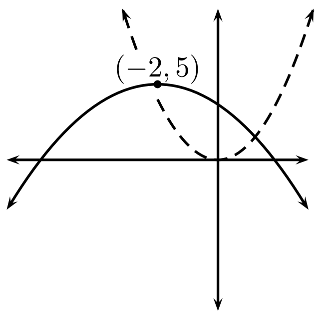
1. (a)



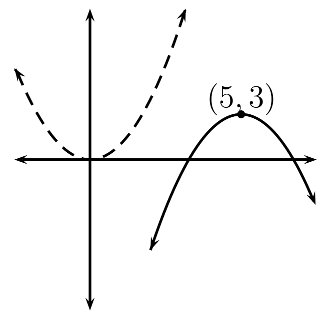
(b)



(c)



(d)



2. (a) $y = -\frac{9}{2}(x+5)^2 - 7$ (b) $y = \frac{2}{3}(x-6)^2 + 5$
   (c) $y = \frac{1}{9}(x-4)^2 - 3$ (d) $y = 5(x+1)^2 - 4$

4. $y = 3(x-1)^2 + 2$

5. $h(x) = -\frac{1}{4}(x-5)^2 + 4$

7. $y = -\frac{1}{2}(x-4)^2 + 1$

8. $y = -3(x-1)^2 - 7$

9. $y = \frac{1}{2}(x-3)^2 + 2$

10. $y = 5(x+2)^2 - 3$

Section 3.3 Solutions Back to 3.3 Exercises

1. (a) $x \geq \frac{14}{5}$ (b) $x \geq -3$ (c) $x \geq -1$
   (d) $x > -3$ (e) $x > 9$ (f) $y \geq -1$

2. (a) $1 < x < 8$, OR $(1, 8)$
   (b) $-2 \leq x \leq \frac{5}{2}$, OR $[-2, \frac{5}{2}]$
   (c) $-5 \leq x \leq \frac{3}{2}$, OR $[-5, \frac{3}{2}]$
   (d) $x < -\frac{1}{5}$ or $x > 0$, OR $(-\infty, -\frac{1}{5}) \cup (0, \infty)$
   (e) $-5 < x < 3$, OR $(-5, 3)$
   (f) $x \leq -6$ or $x \geq -1$, OR $(-\infty, -6] \cup [-1, \infty)$

3. (a) $x < -4$ or $-1 < x < 3$, OR $(-\infty, -4) \cup (-1, 3)$
   (b) $-2 \leq x \leq 2$ or $x \geq 5$, OR $[-2, -2] \cup [5, \infty)$
   (c) $-2 < x < 3$ or $x > 3$, OR $(-2, 3) \cup (3, \infty)$
   (d) $x \leq 2$, OR $(-\infty, 2]$

4. (a) $-2 \leq x \leq -1$ or $x \geq 0$, OR $[-2, -1] \cup [0, \infty)$
   (b) $x < -\frac{3}{5}$ or $0 < x < \frac{7}{2}$, OR $(-\infty, -\frac{3}{5}) \cup (0, \frac{7}{2})$

Section 3.4 Solutions

Back to 3.4 Exercises

1. (a) 34.6 feet
   (b) The rock will be at a height of 32 feet at 1 second (on the way up) and 2 seconds (on the way down).
   (c) 0.6 seconds, 2.4 seconds
   (d) The rock hits the ground at 3 seconds.
   (e) The mathematical domain is all real numbers. The feasible domain is [0,3]
   (f) The rock reaches a maximum height of 36 feet at 1.5 seconds.
   (g) The rock is at a height of less than 15 feet on the time intervals $[0, 0.35]$ and $[2.65, 3]$.

2. We can see that $f(x) = x - x^2$, so its graph is a parabola opening downward. From the form $f(x) = x(1 - x)$ it is easy to see that the function has x-intercepts zero and one, so the x-coordinate of the vertex is $x = \frac{1}{2}$. Because $f(\frac{1}{2}) = \frac{1}{4}$, the maximum value of the function is $\frac{1}{4}$ at $x = \frac{1}{2}$.

3. $t = \dfrac{v_0 \pm \sqrt{v_0^2 - 2gs + 2gs_0}}{g}$

4. (a) The P-intercept of the function is -800 dollars, which means that the company loses \$800 if zero Geegaws are sold.
   (b) Acme must make and sell between 34 and 236 Geegaws to make a true profit.
   (c) The maximum profit is \$1022.50 when 135 Geegaws are made and sold. We know it is a maximum because the graph of the equation is a parabola that opens downward.

5. (a) Maximum revenue is \$1,000,000 when the price is \$100 per Widget.
   (b) The price needs to be less than or equal to \$29.29. or greater than or equal to \$170.71.

6. (a) $A = x(1000 - 2x) = 1000x - 2x^2$
   (b) $0 < x < 500$ or $(0, 500)$
   (c) The maximum area of the field is 125,000 square feet when $x = 250$ feet.
   (d) x must be between 117.7 and 382.3 feet.

Section 3.5 Solutions

Back to 3.5 Exercises

1. (a) $d = 10$ (b) $M = (1, 4)$

2. (a) $d = 5\sqrt{2} = 7.07$ (b) $M = (-\frac{1}{2}, 4\frac{1}{2})$

3. (a) $d = 7.32$ (b) $(3.6, -0.75)$

4. $(1, 12)$

5. $(-5, 2)$

6. (a) $(x-2)^2 + (y-3)^2 = 25$ (b) $(x-2)^2 + (y+3)^2 = 25$ (c) $x^2 + (y+5)^2 = 36$

7. (a) The center is $(-3, 5)$ and the radius is 2.
   (b) The center is $(4, 0)$ and the radius is $2\sqrt{7} = 5.3$.

(c) The center is $(0,0)$ and the radius is $\sqrt{7} = 2.6$.

8. (a) $(x-1)^2 + (y-3)^2 = 4$ (b) $(x+2)^2 + (y+1)^2 = 25$

9. (a) $(x+1)^2 + (y-3)^2 = 16$, the center is $(-1,3)$ and the radius is 4
   (b) $(x-2)^2 + (y+3)^2 = 25$, the center is $(2,-3)$ and the radius is 5
   (c) $(x-3)^2 + (y+1)^2 = 28$, the center is $(3,-1)$ and the radius is $2\sqrt{7} = 5.3$
   (d) $(x+5)^2 + y^2 = 10$, the center is $(-5,0)$ and the radius is $\sqrt{10} = 3.2$

10. $(x-3)^2 + (y-4)^2 = 20$

11. $y = \frac{3}{4}x + \frac{25}{4}$ or $y = \frac{3}{4}x + 6\frac{1}{4}$

12. $(2,10)$, $(2,-5)$

13. $y = -\frac{2}{3}x + 4$

14. $(x-3)^2 + (y-2)^2 = 13$

Section 3.6 Solutions

Back to 3.6 Exercises

2. (a) The solution is $(1,8)$. If we were to graph the two equation we would see a line and a parabola intersecting in only one point.
   (b) See graph below.

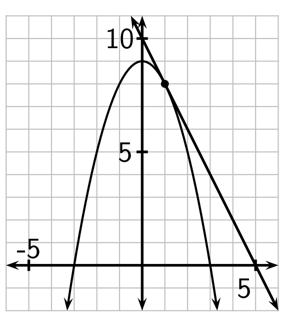
3. (a) See graph below.
   (b) The solutions are $(5,0)$ and $(-4,3)$.

4. (a) See graph below.
   (b) The solution is $(-2,2)$

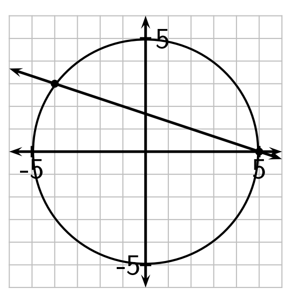
5. The solutions are $(2,-3)$ and $(4,1)$.

7. They will break even when they produce either 34 Doo-dads or 726 Doo-dads.

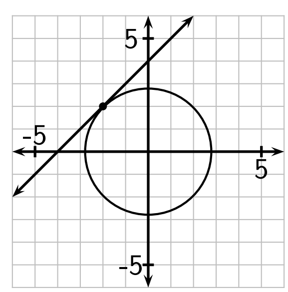
2(b)



3(a)



4(b)



Section 3.7 Solutions

Back to 3.7 Exercises

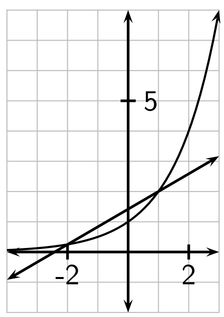
1. 151 square inches per minute

2. (a) From 4 seconds to 7 seconds the height decreased at an average rate of 32 feet per second.
   (b) The average rate of change is 0 feet per second. This is because at 2 seconds the projectile is at a height of 224 feet, on the way up, and at 7 seconds it is at 224 feet again, on the way down.

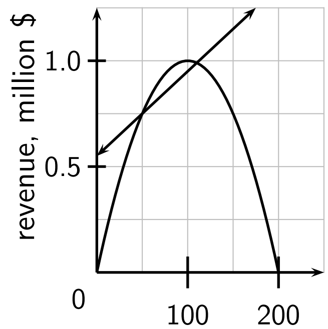
3. (a) When $x = 200$ feet, the area is 80,000 square feet, and when $x = 300$ feet, the area is 30,000 square feet.
   (b) -500 square feet per foot

4. (a) $\frac{11.5}{40} = 0.29$ feet per year (b) $\frac{3}{5} = 0.6$ feet per year

5. (a) \$4000 of revenue per dollar of price (b) See graph below.



Exercise 6(a)



Exercise 5(b)

6. (a) See graph above. (b) $\frac{7}{12}$

7. 16 8. $-\frac{3}{10} = -0.9$

9. (a) $\frac{2}{3}$ (b) $\frac{2}{3}$
   (c) The answers are the same, and they are both equal to the slope of the line.

10. (a) $2x + h - 3$ (b) $6x + 3h + 5$ (c) 3
    (d) $2x + h + 7$ (e) $-2x - h + 3$ (f) $-\frac{3}{4}$
    (g) $2x + h + 4$ (h) $14x + 7h$

11. $6x + 3h$

12. (a) $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$
    (b) $f(x+h) = x^3 + 3x^2h + 3xh^2 + h^3 - 5x - 5h + 1$
    $f(x+h) - f(x) = 3x^2h + 3xh^2 + h^3 - 5h$
    $$\frac{f(x+h) - f(x)}{h} = 3x^2 + 3xh + h^2 - 5$$
    (c) $3(1)^2 + 3(1)(3) + 3^2 - 5 = 16$

13. (a) $f(x+h) - f(x) = \dfrac{-h}{x(x+h)}$ (b) $\dfrac{f(x+h) - f(x)}{h} = \dfrac{-1}{x(x+h)}$

Section 3.8 Solutions Back to 3.8 Exercises

1. (a) $x \geq 2$ (b) $x < 11$ (c) $x < 9$
   (d) $x > 6$ (e) $x \geq -\frac{1}{5}$ (f) $x \leq \frac{4}{7}$

2. (a) $x \leq -3$ or $x \geq 7$, OR $(-\infty, -3] \cup [7, \infty)$

   (b) $x \leq -6$ or $x \geq 3$, OR $(-\infty, -6] \cup [3, \infty)$

   (c) $0 < x < 2$, OR $(0, 2)$

   (d) $1 \leq x \leq \frac{3}{2}$, OR $[1, \frac{3}{2}]$