College Algebra

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Contents

5 More on Functions

Outcome/Performance Criteria:

5. Create new functions from existing functions, understand and find inverse functions.the effect of certain compositions on the graphs of functions. Understand the effect of certain compositions on the graphs of functions.

   (a) Find the sum, difference, product or quotient of two functions.

   (b) Determine the domain of a sum, difference, product or quotient of two functions.

   (c) Find the partial fraction decomposition of a rational function.

   (d) Compute and simplify the composition of two functions.

   (e) Determine functions whose composition is a given function.

   (f) Use composition to determine whether two functions are inverses of each other.

   (g) Compute the inverse of a function.

   (h) Given the graph of a function, draw or identify the graph of its inverse.

   (i) Determine whether a function is one-to-one based on its graph.

   (j) When necessary, restrict the domain of a function so that it has an inverse. Give the inverse and its domain.

   (k) Identify the graphs of $y = x^2$, $y = x^3$, $y = \sqrt{x}$, $y = |x|$ and $y = \frac{1}{x}$.

   (l) Given the graph of a function, sketch or identify various transformations of the function.

5.1 Combinations of Functions

Performance Criteria:

5. (a) Find the sum, difference, product or quotient of two functions.
   (b) Determine the domain of a sum, difference, product or quotient of two functions.
   (c) Find the partial fraction decomposition of a rational function.

Given two functions f and g we can use them to create a new function that we'll call $f+g$. The first thing we need when working with a new function is some sort of description of how it works, often given as a formula. So, for any number x, we need to know how to find $(f+g)(x)$. The idea is simple: we find $f(x)$ and $g(x)$, and add the two together. That is, $(f+g)(x) = f(x)+g(x)$. If we have equations for f and g, we will usually combine them to get an equation for $f+g$, as shown in the next example.

◇ **Example 5.1(a):** For $f(x) = x-1$ and $g(x) = 3x^2+2x-5$, find the equation for $(f+g)(x)$.

Solution: $(f+g)(x) = f(x) - g(x) = (x-1)+(3x^2+2x-5) = 3x^2+3x-6$

Note that we are simply finding an equation for a new function, so the result should just be an expression with the unknown in it. There is no need to factor it unless asked to, and *we should definitely NOT set it equal to zero* and solve. It should be clear that we can define functions $f-g$, fg and $\frac{f}{g}$ in a similar manner. Their definitions are

$$(f-g)(x) = f(x)-g(x), \qquad (fg)(x) = f(x)g(x), \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Technically there are two more functions here, $g-f$ and $\frac{g}{f}$, since subtraction and division are not commutative. That is, changing the order of subtraction or division changes the result.

◇ **Example 5.1(b):** Again let $f(x) = x-1$ and $g(x) = 3x^2+2x-5$. Find the equations for $(f-g)(x)$, $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$.

Multiplying and Dividing Examples

Solution: $(f-g)(x) = (x-1)-(3x^2+2x-5) = -3x^2-x+4$

$$(fg)(x) = (x-1)(3x^2+2x-5) = 3x^3+2x^2-5x-3x^2-2x+5 = 3x^3-x^2-7x+5$$

$$\left(\frac{f}{g}\right)(x) = \frac{x-1}{3x^2+2x-5} = \frac{x-1}{(3x+5)(x-1)} = \frac{1}{3x+5}$$

Note that some care needs to be taken to distribute minus signs when computing $f-g$ or $g-f$.

Domains of Combinations of Functions

Suppose that we wish to construct one of the functions $f+g$, $f-g$, $g-f$, or fg from two functions f and g. It should be clear that such a function shouldn't be valid for values of x for which one or the other (or both) of f and g are not valid. Another way of saying this is to say that a value x is in the domain of one of those functions as long as it is the domain of *BOTH* f and g. The domain of f is a set of numbers, and the domain of g is another set of numbers. The domain of $f+g$, $f-g$, $g-f$, or fg is then the set of numbers that contains all numbers in the domains of both f and g. Such a new set constructed out of two known sets has a name:

Intersection of Two Sets

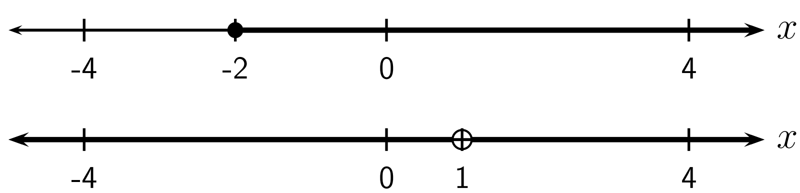
For two sets A and B, the **intersection** of A and B is the set containing all numbers that are in *both* A and B. We denote this new set by $A \cap B$.

Using this new concept and its notation, along with the notation $\text{Dom}(f)$ for the domain of a function f, we have

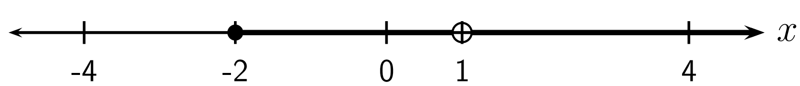
$$\text{Dom}(f+g) = \text{Dom}(f-g) = \text{Dom}(g-f) = \text{Dom}(fg) = \text{Dom}(f) \cap \text{Dom}(g).$$

◇ **Example 5.1(c):** Let $f(x) = \sqrt{x+2}$ and $g(x) = \dfrac{1}{x-1}$. Give the domain of $(g-f)(x)$.

Solution: We can see that $\text{Dom}(f) = [-2, \infty)$ and $\text{Dom}(g) = \{x \,|\, x \neq 1\}$. We can easily visualize the intersection of those two sets if we show them each on a number line. The top number line shows the domain of f, and the one below it shows the domain of g.



The domain of $g-f$ is then the intersection of those two sets, which means the points that are shaded on *both* number lines:



We can now give the domain of $g-f$, using interval notation, as $[-2, 1)$ together with $(1, \infty)$.

Creating the intersection of two sets is an operation on the two sets that yields a new set, in the same way that addition or multiplication are operations on two numbers that yield a new number. We now introduce another operation for combining two sets in a different way:

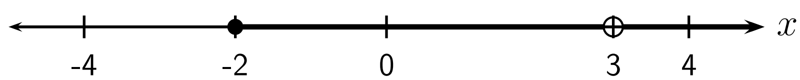
> **Union of Two Sets**
>
> For two sets A and B, the **union** of A and B is the set containing all numbers that are in *either* A or B (including those points that are in both). We denote this new set by $A \cup B$.

Using this idea and notation, the domain of $g - f$ from the Example 5.1(c) is $[-2, 1) \cup (1, \infty)$. Note that we are not saying that the domain of $g - f$ is the union of the domains of f and g! We obtain the domain of $g - f$ by taking the intersection of the domains of the two functions f and g, but the result of that intersection is most cleanly described as the union of two sets (that are *not necessarily the domains of f and g*).

The domain of $\frac{f}{g}$ (and, of course, $\frac{g}{f}$), is "almost" the intersections of the domains of f and g, but there is an additional consideration. Any value in both the domain of f and the domain of g will be in the domain of $\frac{f}{g}$, *as long as it doesn't make the value of g zero*. So we find the domain of $\frac{f}{g}$ by taking the intersection of the domains of f and g and removing any values of x for which $g(x) = 0$.

◇ **Example 5.1(d):** Let $f(x) = \sqrt{x+2}$ and $g(x) = x^2 - 9$. Give the domain of $(\frac{f}{g})(x)$.

Solution: We know that $\mathrm{Dom}(f) = [-2, \infty)$ and $\mathrm{Dom}(g)$ is all real numbers. The intersection of these two sets is just $[-2, \infty)$. However, we see that $g(x) = x^2 - 9 = (x+3)(x-3)$ is zero when $x = -3$ and when $x = 3$. Therefore we need to remove the point $x = 3$ from the interval $[-2, \infty)$, giving us a set that looks like this:



So $\mathrm{Dom}(\frac{f}{g}) = [-2, 3) \cup (3, \infty)$.

Partial Fraction Decomposition

There are times when we wish to take a function of the form $f(x) = \dfrac{Ax + B}{(x - x_1)(x - x_2)}$ and find two functions of the form

$$g(x) = \frac{C}{x - x_1} \qquad \text{and} \qquad h(x) = \frac{D}{x - x_2}$$

such that f is the sum of g and h. We illustrate the method for doing this in the next example.

◇ **Example 5.1(e):** Let $f(x) = \dfrac{x+17}{(x-1)(x+5)}$. Find two functions g and h of the form given above such that $f(x) = (g+h)(x)$. Another Example

Solution: First we let $g(x) = \dfrac{C}{x-1}$ and $h(x) = \dfrac{D}{x+5}$, and add them together:

$$g(x)+h(x) = \frac{C}{x-1} + \frac{D}{x+5} = \frac{C(x+5)}{(x-1)(x+5)} + \frac{D(x-1)}{(x-1)(x+5)} = \frac{Cx+5C+Dx-D}{(x-1)(x+5)}$$

Now if $f(x)$ is to equal $g(x)+h(x)$, the last fraction on the line above must equal $\dfrac{x+17}{(x-1)(x+5)}$. Note that both fractions have the same denominator, so the two fractions will be equal only if their numerators are equal:

$$Cx + 5C + Dx - D = x + 17$$

By "grouping like terms," this can be rewritten (be sure you see how) as

$$(C+D)x + (5C-D) = 1x + 17,$$

and these will be equal only if $C + D = 1$ and $5C - D = 17$. Now we have two equations in two unknowns, which we know how to solve (see Section 1.6). If we add the two equations together we get $6C = 18$, so $C = 3$. Substituting this into the first equations gives $D = -2$. Thus

$$g(x) = \frac{3}{x-1} \qquad \text{and} \qquad h(x) = \frac{-2}{x+5}.$$

It is verified in the exercises that $f(x) = g(x) + h(x)$.

The method of partial fractions has many complications that can arise when the rational function f has other forms. Those of you who encounter partial fraction decomposition again will learn at that time how to alter the above technique for those forms.

Section 5.1 Exercises

To Solutions

1. Consider the functions $g(x) = 3x - 1$ and $h(x) = 2x + 5$.

   (a) Find and simplify $(g+h)(x)$.

   (b) Find and simplify $(gh)(x)$.

   (c) Find and simplify $\left(\dfrac{h}{g}\right)(x)$.

   (d) Give the domain of each of the above functions from parts (a), (b) and (c).

2. Let $f(x) = x^2 + 2x - 3$ and $g(x) = x^2 - 1$.

   (a) Find and simplify $(g - f)(x)$.

   (b) Find and simplify $(fg)(x)$.

   (c) Find and simplify $\left(\frac{f}{g}\right)(x)$.

   (d) Give the domains of each of the functions from parts (a), (b) and (c).

3. For $f(x) = \dfrac{x+5}{x-2}$ and $g(x) = \dfrac{x-4}{x-2}$, find and simplify $(f - g)(x)$. What is the domain of the function?

4. Let $f(x) = \dfrac{2x}{x-4}$ and $g(x) = \dfrac{x}{x+5}$. Find and simplify $\left(\frac{f}{g}\right)(x)$, and give its domain.

5. Let $f(x) = \dfrac{x+3}{x^2-25}$ and $g(x) = \dfrac{7}{x-5}$.

   (a) Find and simplify $(f + g)(x)$. What is the domain of the function?

   (b) Find and simplify $\left(\frac{g}{f}\right)(x)$. What is the domain of the function?

6. Letting $g(x) = \dfrac{3}{x-1}$ and $h(x) = \dfrac{-2}{x+5}$, find $(g + h)(x)$. Check your answer with the $f(x)$ from Example 5.1(e).

7. For each function f that is given, find functions g and h for which $f(x) = g(x) + h(x)$. That is, find the partial fraction decomposition of f.

   (a) $f(x) = \dfrac{4x+7}{x^2+5x+6}$

   (b) $f(x) = \dfrac{-14}{x^2-3x-10}$

   (c) $f(x) = \dfrac{11-x}{x^2-x-2}$

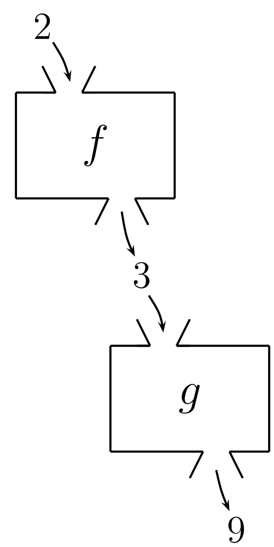
   (d) $f(x) = \dfrac{4x-10}{x^2-1}$

5.2 Compositions of Functions

Performance Criteria:

5. (d) Compute and simplify the composition of two functions.
   (e) Determine functions whose composition is a given function.

There is another way of combining two functions, called their **composition**. Compositions of functions are far more important than their sums, differences, products and quotients, for various reasons. Let's introduce the concept using the two functions $f(x) = 2x - 1$ and $g(x) = x^2$. Suppose that we are asked to find $g[f(2)]$; what this means is that we are to first find $f(2)$, then we are to find g of that result. We can see that $f(2) = 2(2) - 1 = 3$, and $g(3) = 3^2 = 9$, so $g[f(2)] = 9$. We can do all of this as one computation, as follows:



$$g[f(2)] = g[2(2) - 1] = g[3] = 3^2 = 9$$

Note that the function that is closest to 2 *in the expression* $g[f(2)]$ *is the first to act!* The picture to the right shows how this can be interpreted in terms of the "machine" idea of a function. The number two is first "fed into" machine f, which gives an output of three. That value is then "fed into" g, which outputs nine, the final result.

- **Example 5.2(a):** For the same two functions $f(x) = 2x - 1$ and $g(x) = x^2$, find $f[g(2)]$. Is $f[g(2)] = g[f(2)]$?

  $$f[g(2)] = f[2^2] = f(4) = 2(4) - 1 = 7$$

  Solution: Clearly $f[g(2)]$ is not equal to $g[f(2)]$.

This exercise illustrates an important fact:

> *Suppose we have two functions, and we apply one of them to a number and the other one to the result. Changing the order in which the two functions operates generally changes the final result.*

Consider again Example 5.2(a), where we found $f[g(2)] = 7$. We did this by first applying g to the number two, giving the result of four. We then applied f to that value to obtain the value seven. We will now create a new function that does all of that in one step, and we'll call it the **composition of** f **with** g, denoted by $f \circ g$. We get it by simply letting f act on $g(x)$:

$$(f \circ g)(x) = f[g(x)] = f(x^2) = 2(x^2) - 1 = 2x^2 - 1.$$

Note then that

$$(f \circ g)(2) = 2(2)^2 - 1 = 2(4) - 1 = 7,$$

the result that we obtained in Example 5.2(a).

◇ **Example 5.2(b):** For the same two functions $f(x) = 2x - 1$ and $g(x) = x^2$, find $(g \circ f)(x)$. Then find $(g \circ f)(2)$.

Solution: $(g \circ f)(x) = g[f(x)] = g(2x - 1) =$

$$(2x-1)^2 = (2x-1)(2x-1) = 4x^2 - 4x + 1$$

and

$$(g \circ f)(2) = 4(2)^2 - 4(2) + 1 = 4(4) - 8 + 1 = 9.$$

Before going on, let's conduct a little "thought exercise." Suppose that we made the claim that "all cows are brown." If we then went out and found one cow, and it was brown, would that prove our point? Of course not, but at least it would not disprove it either. If the first (or any) cow that we found when we went looking was black, one would have to conclude that our assertion was false.

Similarly, we can note that the value of $(g \circ f)(2)$ found in this last example is the same thing we got at the top of the previous page when we found $g[f(2)]$, except that in Example 5.2(d) we did it all with one function, rather than applying f, then g. *This does not show that the function $f \circ g$ we found is in fact correct, but at least it shows it is not incorrect either!* In other words, our function $f \circ g$ worked as it should on at least one number, so there is a reasonable chance that it is correct. This gives us a way of partially checking our answers when we create a composition.

NOTE: We sometimes speak of finding the composition of functions f and g as *composing f and g*. Some people might take this phrase to mean finding $f \circ g$, but when we use it there will be no order implied. That is, composing f and g will be taken to mean finding either $f \circ g$ or $g \circ f$, depending on the context or what is asked for.

In Chapter 3 we worked with functions like $y = 2(x-4)^2 + 1$. If we were to be given a value for x and asked to find y, we could think of it as a three step process:

(1) subtract four from x

(2) square the result of (1)

(3) take the result from (2), multiply it by two and add one

Each of the above steps is a function in its own right:

$$(1)\ \ h(x) = x - 4 \qquad\qquad (2)\ \ g(x) = x^2 \qquad\qquad (3)\ \ f(x) = 2x + 1$$

Based on these, we see that

$$(f \circ g \circ h)(x) = f\{g[h(x)]\} = f\{g[x-4]\} = f\{(x-4)^2\} = 2(x-4)^2 + 1 = y$$

We could actually break step (3) of multiplying by two and adding one into two separate steps, to get y as a composition of four functions.

◇ **Example 5.2(c):** Let $g(x) = \sqrt{x}$. Find two functions f and h such that the function $y = \dfrac{\sqrt{3x+5}}{2}$ is the composition $(f \circ g \circ h)(x)$

Solution: Here we need to think about the journey that x takes when we have a value for it and we are computing y. Before using the square root, which is function g, we need to multiply x by three and add five. Since h is the first function in $f \circ g \circ h$ to act on x, we have $h(x) = 3x+5$. Letting g act on that result gives us that $(g \circ h)(x) = \sqrt{3x+5}$. To get y we still need to divide by two, and that should be function f, so $f(x) = \frac{x}{2}$.

Section 5.2 Exercises

To Solutions

1. Consider the functions $g(x) = 3x - 1$ and $h(x) = 2x + 5$.

   (a) Find $g[h(-4)]$. Then find $(g \circ h)(x)$ and use that answer to find $(g \circ h)(-4)$. If you don't get the same thing that you did when you computed $g[h(-4)]$, go back and check your work for finding $(g \circ h)(x)$.

   (b) Find $h[g(7)]$, then find $(h \circ g)(x)$. Use the value $x = 7$ to check your answer, in the same way that you did in part (d).

2. $f(x) = x^2 - 5x$ and $g(x) = x - 1$. Find and simplify $(f \circ g)(x)$ and $(g \circ f)(x)$.

3. Let $f(x) = x-2$, $g(x) = |x|$ and $h(x) = 3x+1$. Find $(h \circ g \circ f)(x)$ and $(f \circ g \circ h)(x)$. **There is no simplifying that can be done here, so don't do any!**

4. For all parts of this exercise, let $f(x) = \sqrt{x}$.

   (a) Find a function g that can be composed with f to get the function $y = \sqrt{x+7}$. In this case, is $y = (f \circ g)(x)$, or is $y = (g \circ f)(x)$?

   (b) Find a function h that can be composed with f to get the function $y = 3\sqrt{x} - 2$. In this case, is $y = (f \circ h)(x)$, or is $y = (h \circ f)(x)$?

   (c) Find two *NEW* functions g and h such that $(g \circ f \circ h)(x) = \frac{1}{3}\sqrt{x-2} + 3$.

5. Give three functions f, g and h such that $(f \circ g \circ h)(x) = \frac{1}{2}(x-2)^3 + 1$.

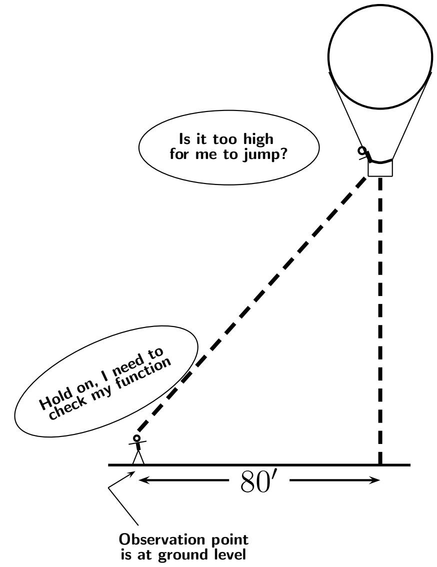
6. Let $f(x) = 3x - 5$ and $g(x) = \dfrac{x+5}{3}$.

   (a) Find $f[g(5)]$ and $g[f(-2)]$. Give your answers clearly, using this notation.

   (b) Find $(f \circ g)(x)$ and $(g \circ f)(x)$. Again, make your answers clear.

   (c) Do you notice anything unusual here?

7. *This exercise illustrates how compositions of functions are used in applications.* A hot air balloon is released, and it rises at a rate of 5 feet per second. An observation point is situated on the ground, 80 feet from the point where the balloon is released, as shown to the right. (Picture courtesy Dr. Jim Fischer, OIT.)



   (a) Find the height h of the balloon as a function of time.

   (b) Find the distance d from the observation point to the balloon as a function of the height h.

   (c) Substitute your result from (a) into your function from (b) to determine the distance d as a function of t. What you are really doing here is finding $(d \circ h)(t)$.

5.3 Inverse Functions

Performance Criteria:

5. (f) Use composition to determine whether two functions are inverses of each other.

   (g) Compute the inverse of a function.

Introduction

In Exercise 6 of Section 5.2 you found that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$ for the functions $f(x) = 3x - 5$ and $g(x) = \dfrac{x+5}{3}$. This seems a bit unusual; what does it mean? Remember that $f \circ g$ is a single function that when applied to x is equivalent to performing g on x and then performing f on the result. The fact that $(f \circ g)(x) = x$ tells us that the "input" x is unchanged by the action of g followed by f. Clearly both f and g *DO NOT* leave their inputs unchanged, so what must be happening is that f "undoes" whatever g "does." Since we also have $(g \circ f)(x) = x$, g undoes f as well.

When this situation occurs we say that f and g are **inverse functions** of each other. In order for the computation $f[g(x)]$ to make sense we require that the range (outputs) of g coincide with the domain (inputs) of f; similarly, we also ask that the range of f coincide with the domain of g. To summarize this symbolically, we will write $f : A \to B$ to indicate that the domain of f is set A and its range is set B. We then have

Definition of Inverse Functions

Two functions $f : A \to B$ and $g : B \to A$ are inverses of each other if both

$(f \circ g)(x) = x$ for all x in B and $(g \circ f)(x) = x$ for all x in A.

The bit about sets A and B is a little technical (and has been oversimplified some here). We will sidestep that issue for now by considering only functions for which both the domains and ranges are all real numbers. In that case we can merely check to see that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

◇ **Example 5.3(a):** Are the functions $f(x) = 5x + 1$ and $g(x) = \dfrac{x-1}{5}$ inverses of each other?

Solution: We see that

$$(f \circ g)(x) = f[g(x)] = f\left[\frac{x-1}{5}\right] = 5\left(\frac{x-1}{5}\right) + 1 = (x-1) + 1 = x$$

and

$$(g \circ f)(x) = g[f(x)] = g[5x+1] = \frac{(5x+1)-1}{5} = \frac{5x}{5} = x,$$

so f and g are inverse functions of each other.

◇ **Example 5.3(b):** Are $f(x) = 3x+6$ and $g(x) = \frac{1}{3}x - 6$ inverses of each other?

Solution: Here we have

$$(f \circ g)(x) = f[g(x)] = f\left[\frac{1}{3}x - 6\right] = 3\left(\frac{1}{3}x - 6\right) + 6 = (x-18)+6 = x - 12.$$

Since $(f \circ g)(x) \neq x$, f and g are not inverses.

Note that if *EITHER* of $(f \circ g)(x)$ or $(g \circ f)(x)$ is not equal to x, the functions f and g are not inverses. That is why there was no need to check $(g \circ f)(x)$ in the previous example.

IMPORTANT: Usually if two functions are inverses we will not use f and g for their names. For a function f, we use the notation f^{-1} for its inverse. Thus, for the function $f(x) = 5x+1$ from Example 5.3(a) we have $f^{-1}(x) = \frac{x-1}{5}$.

There are a couple of questions you might have at this point:

- Do all functions have inverses?
- If I know a function f, how do I find its inverse function f^{-1}?

The answer to the first question is "NO" and we will discuss this a little more in the next section. First let's look at how we find the inverse of a function, assuming that it does have an inverse.

Finding Inverse Functions

Let's think carefully about the function $f(x) = 3x+2$, and how it "processes" an input to give an output. The first thing it does to an input is multiply it by three, *then* it adds two. The inverse must reverse this process, *including the order*. Think about dressing your feet in the morning - socks on first, then shoes. To undo this in the evening it is not socks off, then shoes off! You must not only reverse the processes of putting on both socks and shoes, but you must reverse the order as well. So to undo the function f one must first subtract two, then divide the result by three. The equation for f^{-1} is then $f^{-1}(x) = \frac{x-2}{3}$.

◇ **Example 5.3(c):** Find the inverse of $g(x) = \frac{x-3}{4}$.

Solution: If we were trying to find $g(2)$ we would start with two, subtract three and *then* divide by four. To invert this process we would first multiply by four, then add three. Therefore $g^{-1}(x) = 4x+3$.

- **Example 5.3(d):** Find the inverse of $h(x) = 4x^5 - 1$.

  Solution: To find $h(x)$ for a given value of x, one would first take that value to the fifth power, then multiply the result by four and, finally, subtract one. To invert those operations *and* the order in which they were done, we should first add one, divide that result by four, then take the fifth root of all that. The inverse function would then be $h^{-1}(x) = \sqrt[5]{\dfrac{x+1}{4}}$.

Section 5.3 Exercises To Solutions

1. Use the definition of inverse functions to determine whether each of the pairs of functions f and g are inverses of each other. (This means see if $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$. You should *NOT* find the inverse of either function.)

   (a) $f(x) = 3x + 2$, $g(x) = \dfrac{x-2}{3}$

   (b) $f(x) = 5x - 9$, $g(x) = \frac{1}{5}x + 9$

   (c) $f(x) = \sqrt[3]{\dfrac{x+1}{2}}$, $g(x) = 2x^3 - 1$

   (d) $f(x) = \frac{3}{4}x - 3$, $g(x) = \frac{4}{3}x + 4$

2. The following functions are all invertible. Find the inverse of each, and give it using correct notation.

   (a) $g(x) = \dfrac{2x-1}{5}$

   (b) $f(x) = 5x - 2$

   (c) $h(x) = 2\sqrt[3]{x-1}$

   (d) $f(x) = \dfrac{x+3}{2}$

   (e) $h(x) = x^3 + 4$

   (f) $g(x) = \left(\dfrac{x-1}{7}\right)^5$

3. Find the inverse function for each of the following variations on 2(c).

   (a) $f(x) = \sqrt[3]{2x-1}$

   (b) $g(x) = \sqrt[3]{2x} - 1$

   (c) $h(x) = 2\sqrt[3]{x} - 1$

4. The equation $F = \frac{9}{5}C + 32$ allows us to change a Celsius temperature into Fahrenheit.

   (a) Convert $25°$C to a Fahrenheit temperature.

   (b) Solve the equation $F = \frac{9}{5}C + 32$ for C.

   (c) Freezing is $32°$F, or $0°$C. Use this to check your answer to (b).

   (d) Take your answer to (a) and use your answer to (b) to convert it back into a Celsius temperature.

   (e) What is the relationship between the function $F = \frac{9}{5}C + 32$ and your answer to (b)?

5. For each given function, find functions f, g and h for which $y = (f \circ g \circ h)(x)$.

   (a) $y = \left(\dfrac{x+2}{7}\right)^5$

   (b) $y = \sqrt[3]{2x-1}$

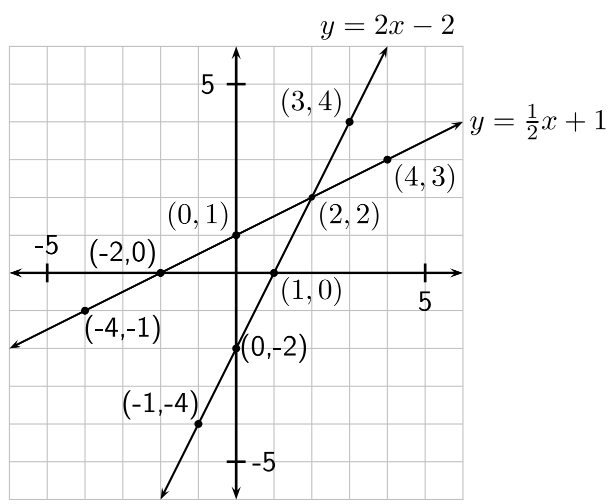
5.4 More on Inverse Functions

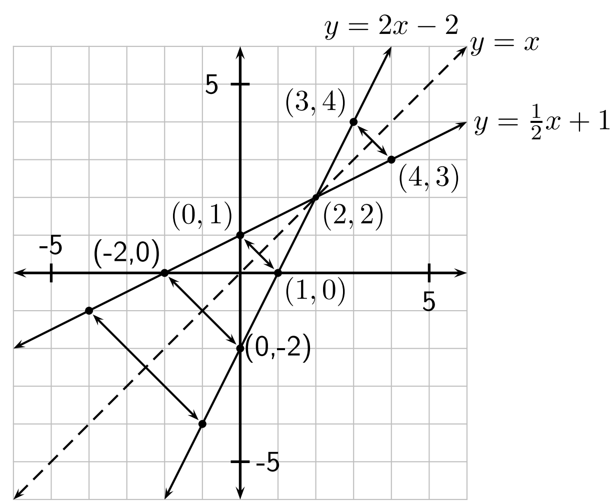
Performance Criteria:

5. (h) Given the graph of a function, draw or identify the graph of its inverse.
   (i) Determine whether a function is one-to-one based on its graph.
   (j) When necessary, restrict the domain of a function so that it has an inverse. Give the inverse and its domain.

Graphs of Inverse Functions

There is a special relationship between the graph of a function and the graph of its inverse. It can easily be shown that $y = \frac{1}{2}x + 1$ and $y = 2x - 2$ are inverses. We have graphed both functions on the same coordinate grid below and to the left. In addition we have labeled several points on each line with their coordinates. *Note that for each point* (a, b) *on the graph of* $y = \frac{1}{2}x + 1$, *the point* (b, a) *lies on the graph of* $y = 2x - 2$. On the graph below and to the right the line $y = x$ has been added, and the geometric relationship between these points with switched coordinates is shown.



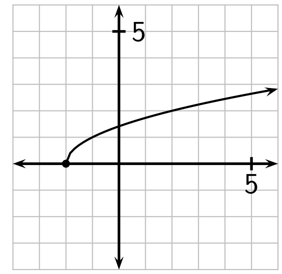


What we can see here is that for every point on the graph of a function that has an inverse (we say the function is **invertible**), there is a point on the inverse function the directly across the line $y = x$, and the same distance away from the line. We say the two graphs are **symmetric with respect to the line** $y = x$. This can be summarized as follows:

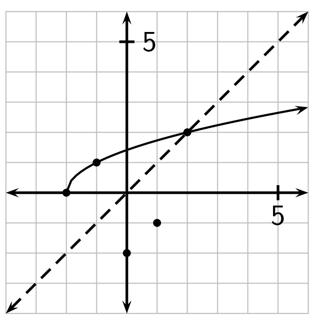
Graphs of Inverse Functions

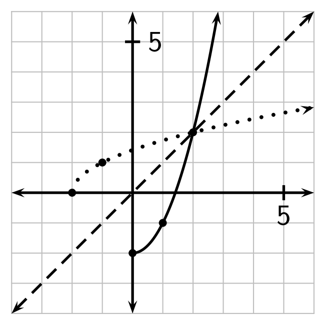
For an invertible function f, the graph of f^{-1} is the reflection of the graph of f across the line $y = x$.

◇ **Example 5.4(a):** The function $y = \sqrt{x+2}$ is invertible, and its graph is shown to the right. (Note that it is just the graph of $y = \sqrt{x}$ shifted two units to the left, since the addition of two takes place before the root acts.) Draw another graph showing $y = \sqrt{x+2}$ as a dashed curve and, on the same grid, graph the inverse function.



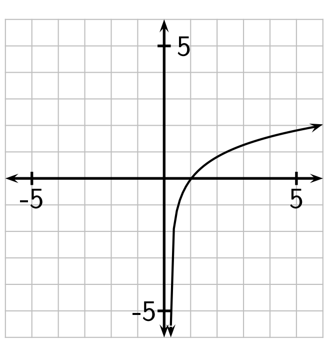
Solution: There are a couple of ways to approach this. Those with good visualization skills might just put in the line $y = x$ and simply draw the reflection of the graph over that line. Let's do that, but with a little help. First we note that the points $(-2, 0)$, $(-1, 1)$ and $(2, 2)$ are on the graph of the original function $y = \sqrt{x+2}$. This means that $(0, -2)$ and $(1, -1)$ are on the graph of the inverse, and $(2, 2)$ is on the graph of both the original function and the inverse. These points and the line $y = x$ are shown on the graph below and to the left. On the graph below and to the right, the new points have been used to draw the graph of the inverse (the solid curve) of the original function (shown as the dotted curve).

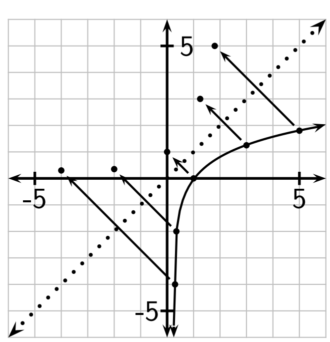


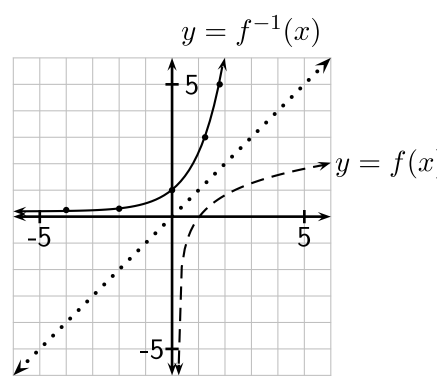


◇ **Example 5.4(b):** The graph of a function f is drawn on the grid below and to the left. Draw another graph, of the inverse function f^{-1}.

Solution: The middle grid below shows the reflection of several points on the graph of f across the line $y = x$. The right hand grid below shows how those points are connected to give the graph of the inverse function (the solid curve).







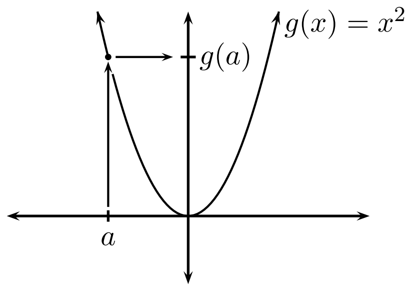
One-To-One Functions

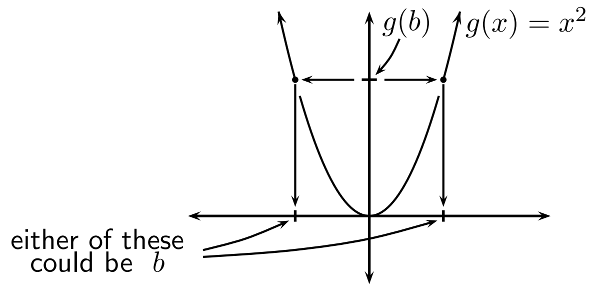
Consider the two functions $g(x) = x^2$ and $h(x) = \sqrt{x}$, and note that $h[g(5)] = h[25] = 5$. This seems to indicate that h is the inverse of g. However, $h[g(-3)] = h[9] = 3$, so maybe h is not the inverse of g! What is going on here? Well, for any function f that has an inverse f^{-1}, the inverse is suppose to take an output of the original function as its input and give out the input of the original function. That is,

$$\text{if} \quad f(a) = b, \qquad \text{then} \qquad f^{-1}(b) = a.$$

For our two functions $g(-3) = 9$ but $h(9) = 3$, so they are (sort of) not inverses.

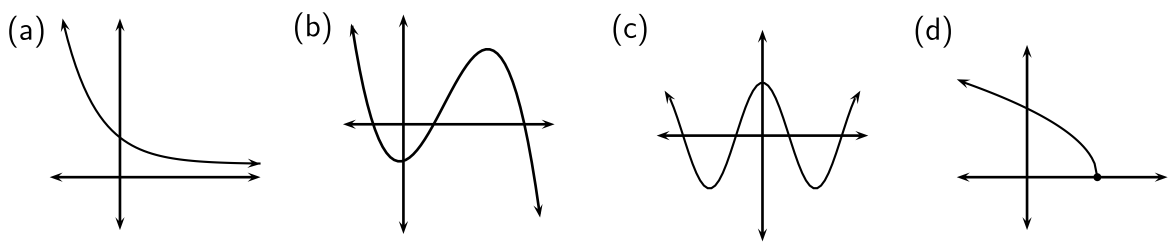
Let's look at this graphically. The graph of the function $g(x) = x^2$ is shown in both graphs below. On the graph to the left we see how to take a given input and obtain its output. The graph to the right shows how we would find the input(s) that give a particular output; note that one output comes from two different inputs.



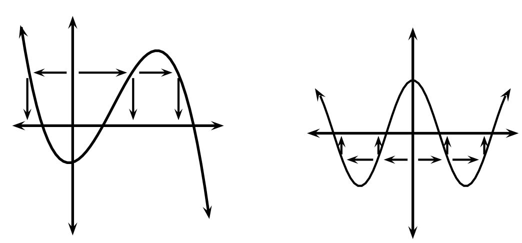


The problem here is that g is what we will call *two-to-one* for all values except zero. This means that two different inputs go to each output, so if we know an output of g, we cannot determine for certain what the input was that gave that output! For that reason $g(x) = x^2$ *is not invertible*. For a function to be invertible it is necessary for each output to have come from just one input. Such a function is called a **one-to-one** function. Graphically, no y value can have come from more than one x value in the way that it happened in the graph above and to the right.

◇ **Example 5.4(c):** The graphs of four functions are shown below. Determine which functions are one-to-one. For those that are not, indicate a y value that comes from more than one x value in the way shown with $g(x) = x^2$ above and to the right.



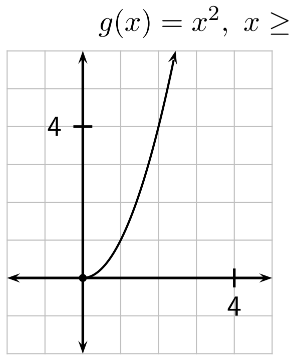
Solution: The functions shown in (b) and (c) are not one-to-one, as can be seen to the right.

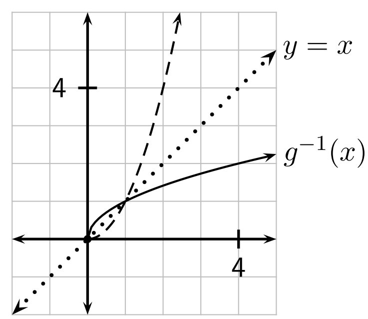


Let's get to the point now:

> **Invertibility of a Function**
>
> A function $f : A \to B$ is invertible if, and only if, f is one-to-one for all x in A.

On many occasions we want a function $f : A \to B$ to have an inverse, but the function isn't one-to-one. The key is to not let the set A be the entire domain of the function f; when we do this we say we are **restricting** the domain of the function. For example, if we consider the function $g(x) = x^2$ for only the values of x in the interval $[0, \infty)$, the graph of the function becomes the one shown to the left below. Clearly that function is one-to-one, so it has an inverse; the inverse is graphed on the grid below and to the right. As you might guess from the appearance of the graph, $g^{-1}(x) = \sqrt{x}$.

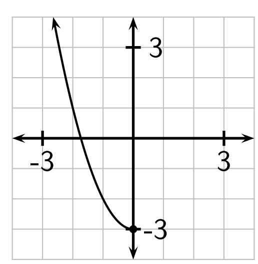




The next example combines this idea with the method we saw previously for finding the inverse of a function.

◇ **Example 5.4(d):** The graph below and to the right is for the function $f(x) = x^2 - 3$, $x \leq 0$. Find the inverse function.

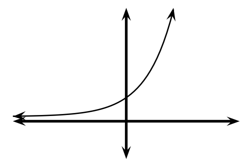
Solution: The function $f(x) = x^2 - 3$ first squares a number and then subtracts three. The inverse must first add three, then take either the positive or negative square root (we'll determine which later). So $f^{-1}(x) = \pm\sqrt{x+3}$ for now. We see that the point $(-1, -2)$ is on the graph of f, so $(-2, -1)$ must be on the graph of f^{-1}. Stated another way, $f^{-1}(-2)$ must be -1, which means that $f^{-1}(x) = -\sqrt{x+3}$.



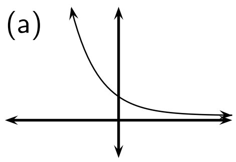
Section 5.4 Exercises

To Solutions

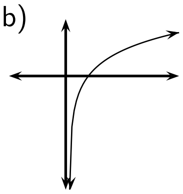
1. Look at the graph of f, shown to the right. Which of the graphs below is the graph of $f^{-1}(x)$?



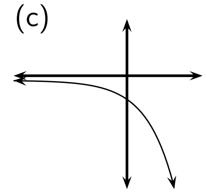
(a)



(b)



(c)



2. Do the following for each of the function/inverse function pairs below.

   - Graph the function and the line $y = x$ together with *Desmos*.
   - Sketch the graph on your paper and, on the same grid, graph what you think the graph of the inverse function would look like.
   - Plot the inverse function with *Desmos* (on the same grid with the other two graphs) and use the result to check your graph. For part (b), recall that $\sqrt[3]{x} = x^{\frac{1}{3}}$. *We have not yet worked with the functions in parts* (c) *and* (d), *but you should still be able to draw graphs of the inverse functions. Use the underline* (_) *to get a subscript.*

   (a) $f(x) = 2x - 3, \quad f^{-1}(x) = \dfrac{x+3}{2}$

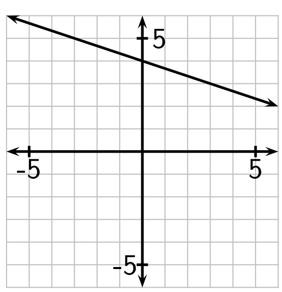
   (b) $g(x) = \sqrt[3]{x} + 1, \quad g^{-1}(x) = (x-1)^3$

   (c) $h(x) = 2^x, \quad h^{-1}(x) = \log_2 x$

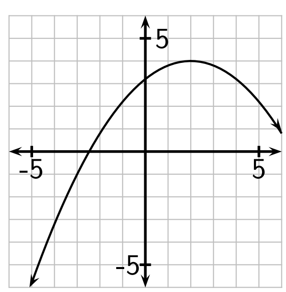
   (d) $f(x) = e^{-x}, \quad f^{-1}(x) = -\ln x$

3. Determine which of the functions below are one-to-one. For those that are, draw the graph of the function as a dashed curve on separate paper and, on the same grid, draw the graph of the inverse function as a solid curve. For those that are not, give a mathematical statement of the form $f(a) = f(b) = c$ for specific values of a, b and c.

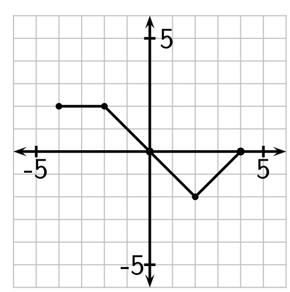
(a)



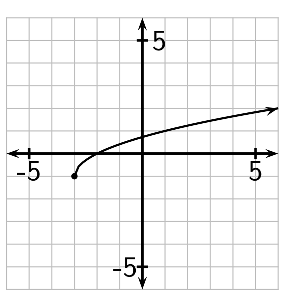
(b)



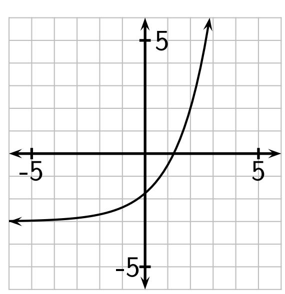
(c)



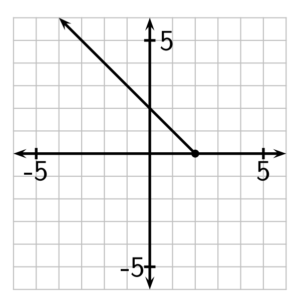
(d)



(e)



(f)



4. (a) Recall that the graph of $f(x) = (x-3)^2$ is a shift of the graph of $y = x^2$. Do you remember which way and how far? Refer to Section 2.6 if you need to. Sketch the graph, then check your answer by graphing the function with *Desmos*.

   (b) The graph indicates that f is not a one-to-one function. How would we restrict the domain in order to get just the right side of the parabola? Add $\{\ restriction\ \}$ to the *Desmos* command line for graphing f to see the graph with the restriction.

   (c) On a new grid, sketch the restricted function. On the same grid, sketch the graph of the inverse function f^{-1}.

   (d) Compute the inverse in the way that you learned in Section 5.3. Graph the inverse function with *Desmos*, and see if the graph matches your answer to (c). If it doesn't, figure out what is wrong and fix it!

5. (a) Sketch the left half of graph of $f(x) = (x-3)^2$. What restriction is this? Graph with *Desmos* to check your answer.

   (b) Sketch the graph of the inverse function on the same grid.

   (c) Determine algebraically the inverse function. Graph it with *Desmos* to check your answer - if it does not match your answer to (b), determine which is incorrect and fix it.

5.5 Transformations of Functions

Performance Criteria:

5. (k) Identify the graphs of $y = x^2$, $y = x^3$, $y = \sqrt{x}$, $y = |x|$ and $y = \frac{1}{x}$.

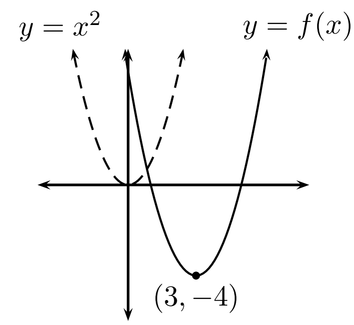
   (l) Given the graph of a function, sketch or identify various transformations of the function.

Some of the ides in this section were seen previously in Section 2.6; we will repeat those ideas here, reinforcing them with things you saw in Sections 3.2 and 5.2.

Introduction, Shifts of Graphs of Functions

We know that the graph of the function

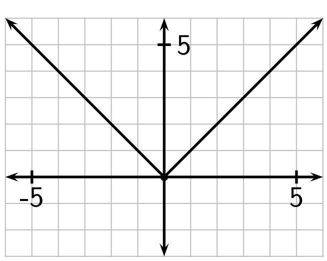
$$f(x) = (x-3)^2 - 4$$



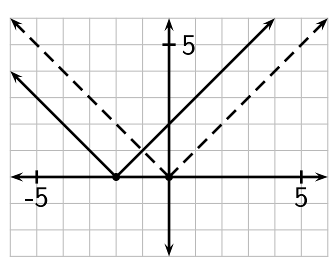
is a parabola opening upward, with vertex at $(3, -4)$. We also know that it has the exact same shape as $y = x^2$. The graph to the right shows f graphed on the same axes as $y = x^2$. One can see that the graph of f can be obtained by "sliding" the graph of $y = x^2$ three units to the right and four units down.

We should note that the function f is really a composition of three functions; first we subtract 3, then we square, then we subtract four. Here we will want to consider the squaring to be the "main function," with the actions of subtracting three and subtracting four occurring before and after the main function, respectively.

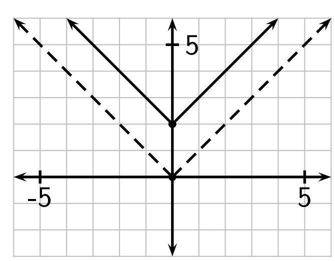
Let's now consider the absolute value function $y = |x|$, whose graph is shown below and to the left, and the two functions $g(x) = |x+2|$ and $h(x) = |x| + 2$. Note the difference between the two functions. With g, we add two *before* the main function of absolute value, and with h we add two *after* the absolute value. The graphs of g and h are shown on the middle and right hand grids below. For both, the graph of $y = |x|$ is shown with dashed lines.



$y = |x|$



$g(x) = |x+2|$



$h(x) = |x| + 2$

What we see here is that when we add two *before* applying the main function of absolute value, the graph of the function is that of $y = |x|$ shifted two units *to the left*, or in the negative direction. This may seem a bit counterintuitive. The right way to think about it is that the value

$x = -2$ results in $|0| = 0$, which is where the "vertex" of the absolute value function occurs. (The term vertex is generally reserved for parabolas, but it certainly would make sense to think of the point of the absolute value function as sort of a vertex as well.) When we add two *after* the absolute value, the result is that the graph is shifted up by two. What is really happening is that the y values are obtained by first computing $|x|$, then two units are added to each y, moving the whole graph up by two.

The following summarizes what you have been seeing. When reading it, think of the function f as being something like squaring, or absolute value.

Shifts of Graphs:

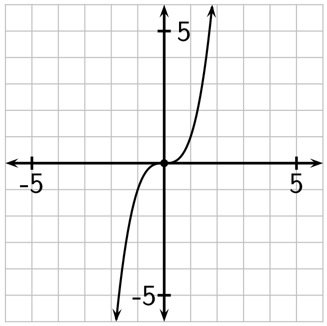
Let a and b be *positive* constants and let $f(x)$ be a function.

- The graph of $y = f(x + a)$ is the graph of $y = f(x)$ shifted a units to the left. That is, what happened at zero for $y = f(x)$ happens at $x = -a$ for $y = f(x + a)$.
- The graph of $y = f(x - a)$ is the graph of $y = f(x)$ shifted a units to the right. That is, what happened at zero for $y = f(x)$ happens at $x = a$ for $y = f(x - a)$.
- The graph of $y = f(x) + b$ is the graph of $y = f(x)$ shifted up by b units.
- The graph of $y = f(x) - b$ is the graph of $y = f(x)$ shifted down by b units.

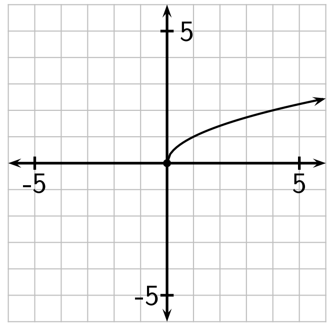
Note that when we add or subtract the value a *before* the function acts, the shift is in the x direction, and the direction of the shift is the opposite of what our intuition tells us should happen. When we add or subtract b *after* the function acts, the shift is in the y direction and the direction of the shift is up for positive and down for negative, as our intuition tells us it should be.

Some Basic Functions

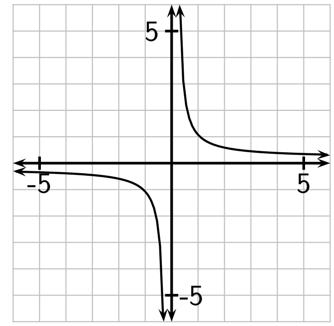
Before continuing, we need to pause to consider the graphs of some common functions. You should by now be quite familiar with the graph of $y = x^2$, and we just saw the graph of $y = |x|$. Three other graphs you should have committed to memory are those of $y = x^3$, $y = \sqrt{x}$ and $y = \frac{1}{x}$, all of which are shown at the top of the next page.



$y = x^3$



$y = \sqrt{x}$

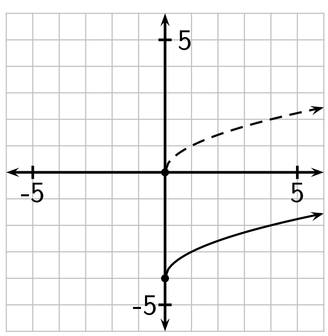


$y = \dfrac{1}{x}$

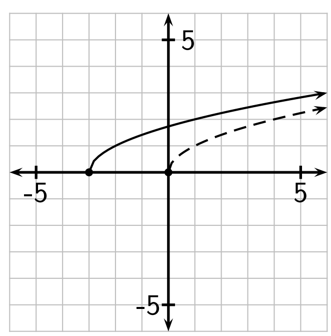
We will use the graphs of $y = x^2$, $y = |x|$, $y = x^3$ and $y = \sqrt{x}$ to demonstrate the principles of this section. Using these basic functions, we can now look at a few more examples of shifts of functions.

◇ **Example 5.5(a):** Sketch the graphs of $y = \sqrt{x} - 4$ and $y = \sqrt{x+3}$, putting the graph of $y = \sqrt{x}$ on the same grid with each.

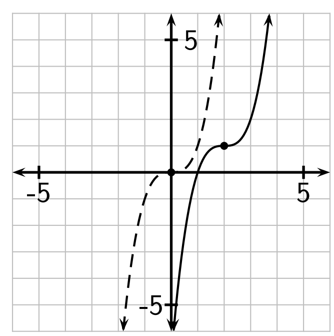
Solution: For the function $y = \sqrt{x} - 4$, the subtraction of four is taking place *after* the square root, so it will shift the graph of $y = \sqrt{x}$ vertically. Vertical shifts are as they appear, so the shift will be down. The result is shown in the graph below and to the left. $y = \sqrt{x} - 4$ is the solid curve, $y = \sqrt{x}$ is the dashed. For $y = \sqrt{x+3}$, the addition of three is *before* the square root, so the shift will be in the x-direction and opposite of what the sign indicates, so to the left. Again, this is because the square root will be acting on zero at $x = -3$. The graph of this function is on the middle graph below, as a solid curve. Again, the dashed curve is $y = \sqrt{x}$.



$y = \sqrt{x} - 4$



$y = \sqrt{x+3}$



$y = (x-2)^3 + 1$

◇ **Example 5.5(b):** Sketch the graph of $y = (x-2)^3 + 1$ on the same grid with the graph of $y = x^3$.

Solution: Here the function $y = x^3$ is composed with two functions, a subtraction of two before cubing, and an addition of one after cubing. The subtraction of two before the cubing causes a shift to the right of two, and the addition of one after causes a shift of one upward. The final result is seen in the right hand graph above.

Stretches and Shrinks of Graphs of Functions

Next we look at the effect of multiplying, which causes the graph of a function to "stretch" or "shrink" in either the x- or y-direction, again depending on whether it occurs before or after the function acts.

Stretches and Shrinks of Graphs:

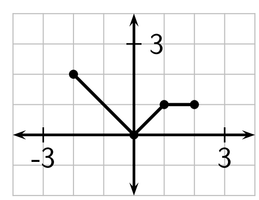
Let a and b be *positive* constants and let $f(x)$ be a function.

- If $a > 1$, the graph of $y = f(ax)$ is the graph of $y = f(x)$ shrunk by a factor of $\frac{1}{a}$ in the x direction, toward the y-axis. If $a < 1$, the graph of $y = f(ax)$ is the graph of $y = f(x)$ stretched by a factor of $\frac{1}{a}$ in the x direction, away from the y-axis.
- If $b > 1$, the graph of $y = bf(x)$ is the graph of $y = f(x)$ stretched by a factor of b in the y direction, away from the x-axis. If $b < 1$, the graph of $y = bf(x)$ is the graph of $y = f(x)$ shrunk by a factor of b in the y direction, toward the x-axis.

Here we see again the phenomenon that an algebraic change before f acts causes a change in the graph in the x direction, but the change is again the opposite of what we think intuitively should happen. An algebraic change *after* f acts results in a change in the graph in the vertical direction, and in the way that our intuition tells us it should happen.

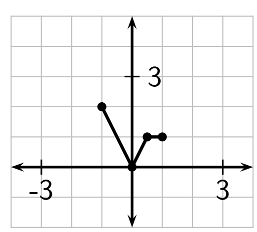
For all of the basic functions that we are currently familiar with, the form $y = f(ax)$ can always be converted to $y = bf(x)$. The form $y = f(ax)$ is quite important in trigonometry, however, so let's take a look at it here. There is no need to have an equation in order to illustrate the principles we are talking about, though, so we'll just go with a graph.

◇ **Example 5.5(c):** The graph of a function $y = f(x)$ is shown to the right. Draw the graphs of $y = f(2x)$ and $y = 2f(x)$ on separate graphs.

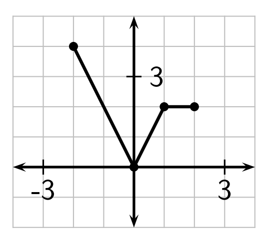


$y = f(x)$

Solution: For the function $y = f(2x)$, the multiplication by two is taking place before the function f acts, so the effect is in the horizontal, or x, direction. Since all horizontal effects are the opposite of what our intuition tells us, this means the graph shrinks toward the y-axis by a factor of one half. This is shown in the first graph to the right.



$y = f(2x)$



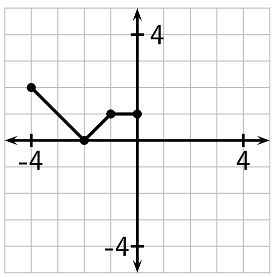
$y = 2f(x)$

For the function $y = 2f(x)$, the multiplication takes place after f acts. Therefore the effect is vertical and exactly as it seems - the graph is stretched vertically away from the x-axis by a factor of two, as shown in the second graph above and to the right.

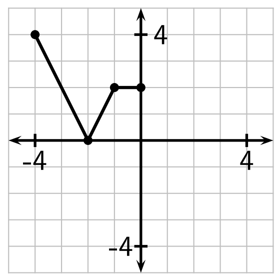
We can combine stretches or shrinks with shifts:

◇ **Example 5.5(d):** For the function f from the previous exercise, sketch the graph of $y = 2f(x + 2) - 3$.

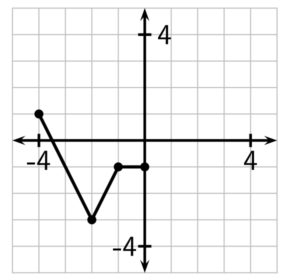
Solution: For function like this, we want to think about the order of operations that are applied to x. In this case, we add two before f acts, so that shifts the function two units to the left, as shown in the left hand graph below. Next f acts, then we multiply by two *after* f has acted, so this causes a stretch by a factor of two away from the x-axis. This is shown on the middle graph below. Finally we subtract three, resulting in a downward shift by three units. The final graph is shown below and to the right.



$y = f(x + 2)$



$y = 2f(x + 2)$



$y = 2f(x + 2) - 3$

Reflections (Flips) of Graphs

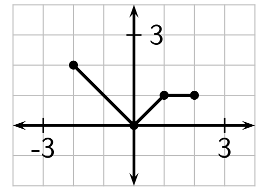
When we multiply by a negative, the graph of the function gets flipped over the x- or y-axis, depending on whether the multiplication takes place before or after the function acts.

Reflections of Graphs:

For a function $y = f(x)$ and *negative* numbers a and b, we have the following:

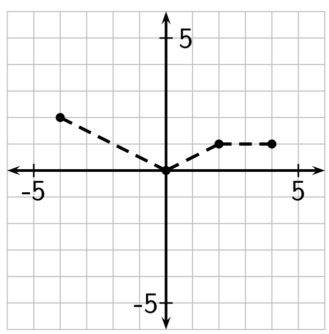
- The effect of a in $y = f(ax)$ is the same as when a is positive, except that the graph not only stretches or shrinks away from or toward the y-axis, but it "flips" over the axis as well. We say that the graph **reflects** across the y-axis.
- The effect of b in $y = bf(x)$ is the same as when b is positive, but in this case there is a reflection across the x-axis.

◇ **Example 5.5(e):** The graph of a function $y = f(x)$ is shown to the right. Draw the graphs of $y = f(-\frac{1}{2}x)$ and $y = -\frac{1}{2}f(x)$ on separate graphs.

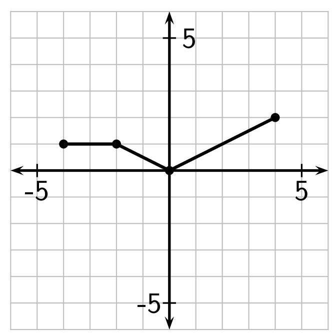


$y = f(x)$

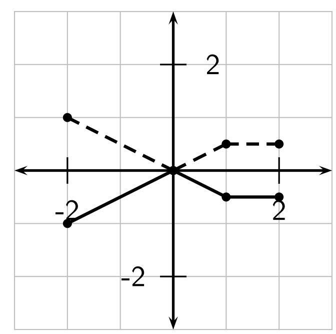
Solution: For the function $y = f(-\frac{1}{2}x)$, the multiplication by one-half is taking place before the function f acts, so the effect is in the horizontal, or x, direction. Since all horizontal effects are the opposite of what our intuition tells us, this means the graph stretches away from the y-axis by a factor of two. This is shown by the dashed graph below and to the left. But the fact that we are really multiplying by *negative* one-half means that the graph also reflects in the x-direction, across the y-axis, so the final graph is the solid one shown in the middle below.



$y = f(\frac{1}{2}x)$



$y = f(-\frac{1}{2}x)$

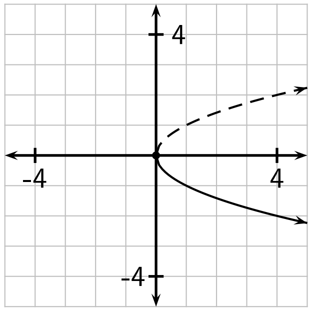


$y = -\frac{1}{2}f(x)$

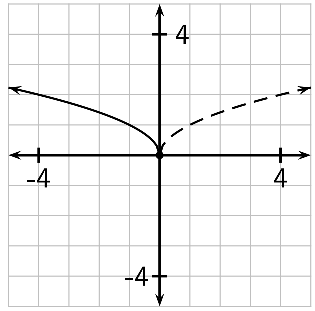
For the function $y = -\frac{1}{2}f(x)$, the multiplication takes place after f acts, so the graph shrinks to half its original distance from the x-axis, as shown by the dashed graph on the right grid above. (Note the change in scale.) But the multiplication by a negative causes the graph to reflect vertically, across the x-axis, so the final result is as shown by the solid graph on the right grid above.

• **Example 5.5(f):** Sketch the graph of $y = \sqrt{x}$ as a dashed curve. Then, on the same grid, graph $y = -\sqrt{x}$. Repeat this for $y = \sqrt{-x}$ and $y = \sqrt{-x} + 2$.

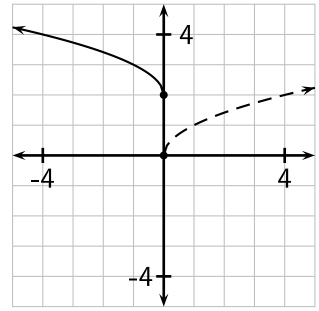
Solution: For $y = -\sqrt{x}$, the negative is applied after the function, so the reflection is vertical, over the x axis. This is shown on the left graph at the top of the next page. The reflection is horizontal, or over the y-axis, for $y = \sqrt{-x}$, as shown in the middle graph at the top of the next page.



$y = -\sqrt{x}$



$y = \sqrt{-x}$



$y = \sqrt{-x} + 2$

For the function $y = \sqrt{-x} + 2$, the graph first reflects across the y-axis, then moves up two units, as shown in the third graph above.

Conclusion

In this section we have seen how certain changes to a common function change the appearance of its graph. we can summarize the basic principles as follows:

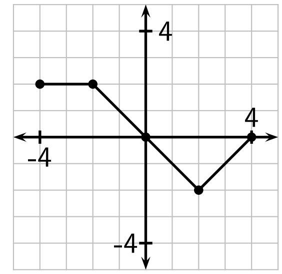
Transformations of Graphs:

Variations of a function $f(x)$ result in transformations of the graph of $f(x)$ according to the following principles:

- Addition or subtraction cause shifts of the graph.
- Multiplication causes stretches or shrinks of the graph.
- Multiplication by a negative causes a reflection of the graph.
- All of the above affect the graph in the x-direction (horizontally) if they occur before the main function acts. They affect the graph in the y-direction (vertically) if they occur after the main function acts.
- Horizontal effects are the opposite of what they seem they should be, and vertical effects are as they seem they should be.

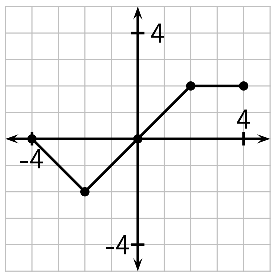
You've seen examples of all these things. You've also seen, in a few of the examples, how combinations of more than one of the actions affects the graph of the main function. There will be more of those in the exercises, and we'll look at one last example of such in a moment. One thing we won't do at this point is combine a multiplication and an addition (or subtraction) before the function acts. The effect of that is a bit more complicated to analyze than what we have looked at so far.

◇ **Example 5.5(g):** The graph of a function $y = g(x)$ is shown to the right. Sketch the graph of $y = \frac{1}{2}g(-x) - 3$.

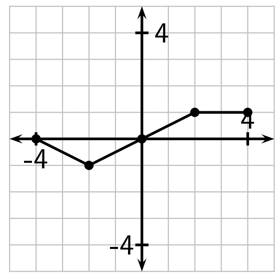


$y = g(x)$

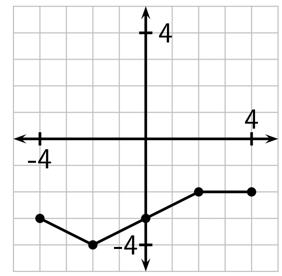
Solution: Here there are three actions taking place: multiplication by a negative one before g acts, and Multiplying by $\frac{1}{2}$ and subtraction three after g acts. We can get the desired graph by accumulating those three actions one at a time, as shown in the graphs at the top of the next page.



$y = g(-x)$



$y = \frac{1}{2}g(-x)$



$y = \frac{1}{2}g(-x) - 3$

Section 5.5 Exercises

To Solutions

1. For each of the quadratic functions below,

   - sketch the graph of $y = x^2$ **accurately** on a coordinate grid,
   - **on the same grid**, sketch what you think the graph of the given function would look like,
   - graph $y = x^2$ and the given function together on your calculator to check your answer

   (a) $h(x) = (x+1)^2$ (b) $y = x^2 - 4$

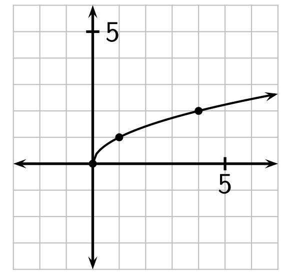
   (c) $g(x) = \frac{1}{4}(x+2)^2 - 1$ (d) $y = -2(x-4)^2 + 2$

2. For each of the following, graph $h(x) = x^3$ and the given function on the same graph. Check your answers using a graphing calculator or an online grapher.

   (a) $y = -x^3$ (b) $y = \frac{1}{2}x^3 + 4$ (c) $y = (x-1)^3$

   (d) $y = (x+3)^3 + 1$ (e) $y = -(x+2)^3$

3. The graph of $y = \sqrt{x}$ is shown to the right, with the three points $(0,0)$, $(1,1)$ and $(4,2)$ indicated with dots. For each of the following, sketch what you think the graph of each would look like, along with a dashed copy of the graph of $y = \sqrt{x}$. On the graph of the new function, indicate three points showing the new positions of the three previously mentioned points. Check your answers in the back, making sure to see that you have the correct points plotted.



$y = \sqrt{x}$

(a) $y = \sqrt{x-1}$ (b) $y = \sqrt{x} - 3$ (c) $y = 2\sqrt{x}$

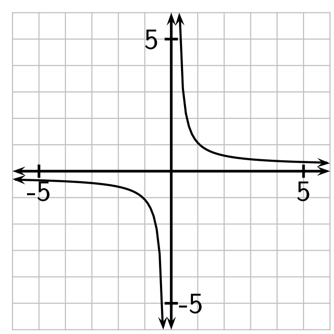
(d) $y = 2\sqrt{x} - 3$ (e) $y = \sqrt{-x} + 1$ (f) $y = -\sqrt{x-2} + 4$

4. For each of the following, graph $f(x) = \frac{1}{x}$ (its graph is shown below and to the right) and the given function on the same graph, with the graph of $f(x) = \frac{1}{x}$ dashed.

(a) $y = \dfrac{1}{x+3}$ (b) $y = \dfrac{1}{x} + 3$

(c) $y = -\dfrac{1}{x}$ (d) $y = \dfrac{1}{-x}$

(e) $y = -\dfrac{1}{x-2}$ (f) $y = \dfrac{1}{x+2} + 1$



$y = \dfrac{1}{x}$

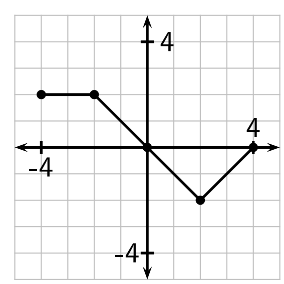
5. Consider again the function $y = g(x)$ from Example 5.2(g), shown below and to the right. Use it to graph each of the following.

(a) $y = 2g(x)$ (b) $y = g(-2x)$

(c) $y = g(x) - 2$ (d) $y = g(x-2)$

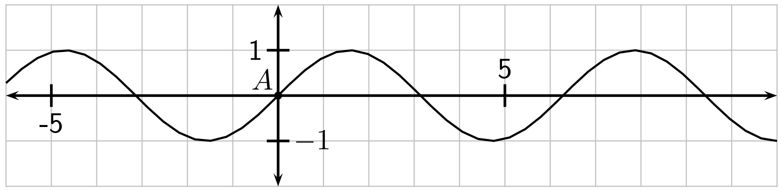
(e) $y = \frac{1}{2}g(x+3)$ (f) $y = g(\frac{1}{2}x) - 3$

(g) $y = -g(x-1) + 2$ (h) $y = -\frac{1}{2}g(x-3)$

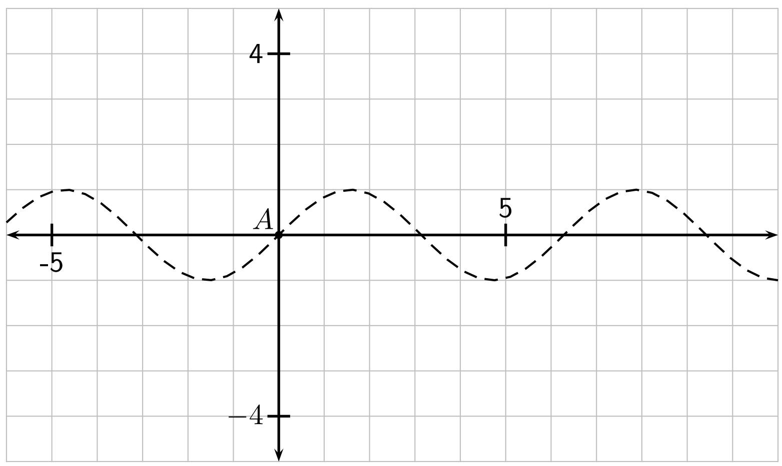


$y = g(x)$

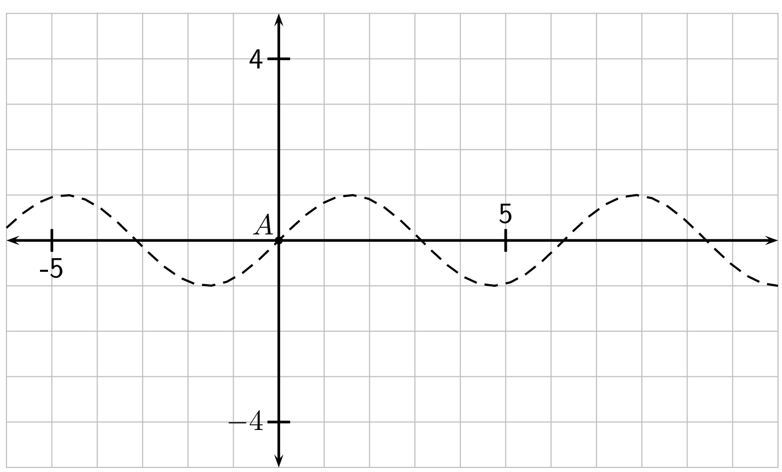
6. Some of you will go on to take trigonometry, in which you will study functions that are called *periodic*. This means several things; one characteristic of periodic functions is that their graphs consist of one portion repeated over and over. One of the most basic periodic functions is called the *sine function*, which is denoted $f(x) = \sin x$. Its graph is shown below. The point labeled A is just for reference when doing the exercises below.



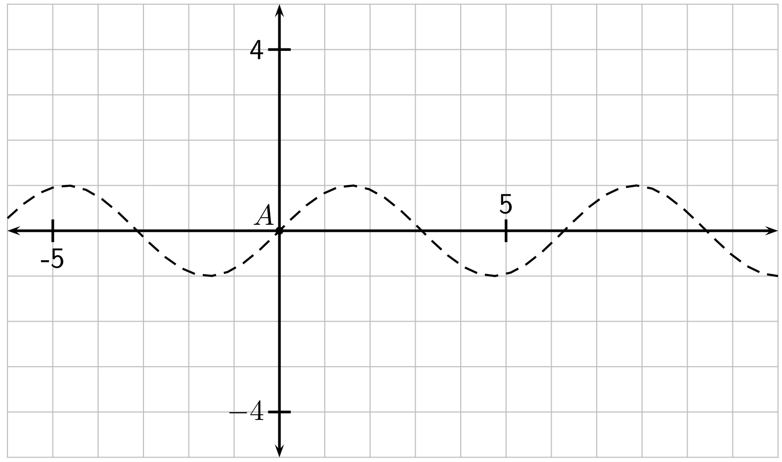
(a) Sketch the graph of $g(x) = \sin(x - 2)$ on the grid below. The dashed curve is the graph of $y = g(x)$. Label the new position of the point A from the graph above as A'.



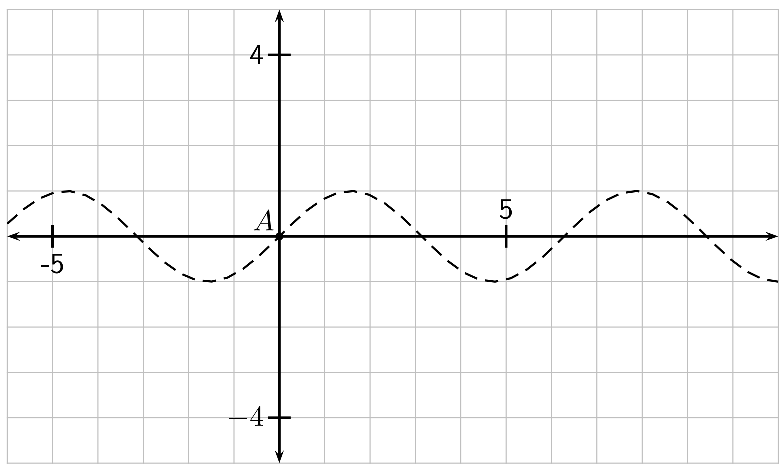
(b) Sketch the graph of $g(x) = \sin(x) - 2$ on the grid below.



(c) Sketch the graph of $h(x) = 3\sin x$ on the grid below.



(d) Sketch the graph of $y = 2\sin(x+1)$, again labeling the new position of A as A'. Again, check with your calculator.



Understanding how the graph of the sine function and other similar functions are shifted and stretched is a major part of trigonometry.

A Solutions to Exercises

A.5 Chapter 5 Solutions

Section 5.1 Solutions Back to 5.1 Exercises

1. (a) $(g+h)(x) = 5x+4$ (b) $(gh)(x) = 6x^2+13x-5$ (c) $\left(\frac{h}{g}\right)(x) = \frac{2x+5}{3x-1}$

   (d) $\mathrm{Dom}(g+h) = \mathrm{Dom}(gh) = \mathbb{R}$, $\mathrm{Dom}\left(\frac{h}{g}\right) = \{x \mid x \neq \frac{1}{3}\}$

2. (a) $(g-f)(x) = -2x+2$ (b) $(fg)(x) = x^4+2x^3-4x^2-2x+3$ (c) $\left(\frac{f}{g}\right)(x) = \frac{x+3}{x+1}$

   (d) $\mathrm{Dom}(g-f) = \mathrm{Dom}(fg) = \mathbb{R}$, $\mathrm{Dom}\left(\frac{f}{g}\right) = \{x \mid x \neq -1, 1\}$

3. $(f-g)(x) = \frac{9}{x-2}$, $\mathrm{Dom}(f-g) = \{x \mid x \neq 2\}$

4. $\left(\frac{f}{g}\right)(x) = \frac{2x+10}{x-4}$, $\mathrm{Dom}\left(\frac{f}{g}\right) = \{x \mid x \neq -5, 0, 4\}$

5. (a) $(f+g)(x) = \frac{8x+38}{x^2-25}$, $\mathrm{Dom}(f+g) = \{x \mid x \neq -5, 5\}$

   (b) $\left(\frac{g}{f}\right)(x) = \frac{7x+35}{x+3}$, $\mathrm{Dom}\left(\frac{g}{f}\right) = \{x \mid x \neq -5, -3, 5\}$

7. Note that g and h can be switched in any of the following.

   (a) $g(x) = \frac{5}{x+3}$, $h(x) = \frac{-1}{x+2}$ (b) $g(x) = \frac{-2}{x-5}$, $h(x) = \frac{2}{x+2}$

   (c) $g(x) = \frac{-4}{x+1}$, $h(x) = \frac{3}{x-2}$ (d) $g(x) = \frac{7}{x+1}$, $h(x) = \frac{-3}{x-1}$

Section 5.2 Solutions Back to 5.2 Exercises

1. (a) $g[h(-4)] = -10$, $(g \circ h)(x) = 6x+14$ (b) $h[g(7)] = 45$, $(h \circ g)(x) = 6x+3$

2. $(f \circ g)(x) = x^2-7x+6$, $(g \circ f)(x) = x^2-5x-1$

3. $(h \circ g \circ f)(x) = 3|x-2|+1$, $(f \circ g \circ h)(x) = |3x+1|-2$

4. (a) $g(x) = x+7$, $y = (f \circ g)(x)$ (b) $h(x) = 3x-2$, $y = (h \circ f)(x)$

(c) $h(x) = x - 2,\ \ g(x) = \frac{1}{3}x + 3$

5. $h(x) = x - 2,\ \ g(x) = x^3,\ \ f(x) = \frac{1}{2}x + 1$

6. (a) $f[g(5)] = 5,\ g[f(-2)] = -2$ (b) $(f \circ g)(x) = x,\ (g \circ f)(x) = x$

   (c) When we "do" g to a number, then "do" f to the result, we end up back where we started. The same thing happens if we do f first, then g.

7. (a) $h = 5t$ (b) $d = \sqrt{h^2 + 6400}$ (c) $d = \sqrt{(5t)^2 + 6400} = \sqrt{25t^2 + 6400}$

Section 5.3 Solutions

Back to 5.3 Exercises

1. (a) inverses (b) not inverses (c) inverses (d) inverses

2. (a) $g^{-1}(x) = \dfrac{5x+1}{2}$ (b) $f^{-1}(x) = \dfrac{x+2}{5}$ (c) $h^{-1}(x) = (\frac{x}{2})^3 + 1$ or $h^{-1}(x) = \frac{1}{8}x^3 + 1$

   (d) $f^{-1}(x) = \dfrac{2x}{x-4}$ (e) $h^{-1}(x) = \sqrt[3]{x-4}$ (f) $g^{-1}(x) = \dfrac{4x+3}{2x-1}$

3. (a) $f^{-1}(x) = \dfrac{x^3+1}{2}$ (b) $g^{-1}(x) = \dfrac{(x+1)^3}{2}$ (c) $h^{-1}(x) = \left(\dfrac{x+1}{2}\right)^3$

4. (a) 77°F (b) $C = \frac{5}{9}F - \frac{160}{9}$ (d) 25°C

   (e) $F = \frac{9}{5}C + 32$ and $C = \frac{5}{9}F - \frac{160}{9}$ are inverse functions.

5. (a) $f(x) = x^5,\ \ g(x) = \dfrac{x}{7},\ \ h(x) = x + 2$ (b) $f(x) = \sqrt[3]{x},\ \ g(x) = x - 1,\ \ h(x) = 2x$

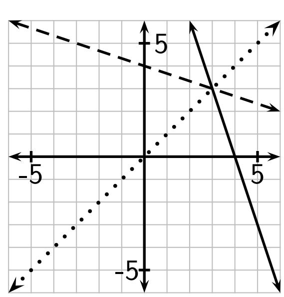
Section 5.4 Solutions

Back to 5.4 Exercises

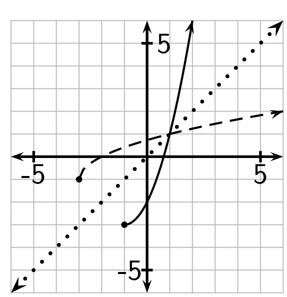
1. (b)

3. The functions graphed in (a), (d), (e) and (f) are invertible. The original graphs are shown below as dashed curves, and their inverses are given as solid curves. (Any of the "curves" might be straight lines.)

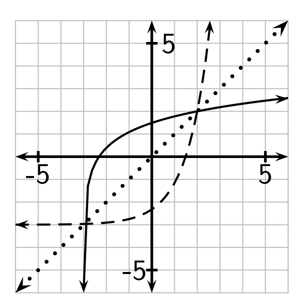
(a)



(d)



(e)



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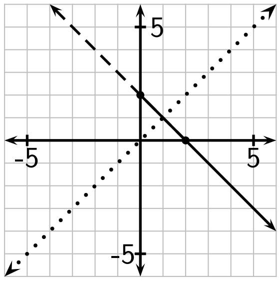
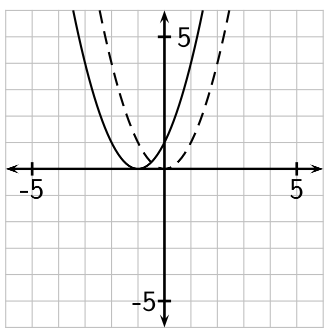


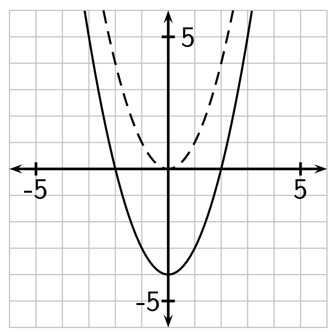
The functions graphed in (b) and (c) are not invertible. For (b) we can see that, for example, $f(-1) = f(5) \approx 2.1$, or $f(0) = f(4) \approx 4.1$. For (c), $f(0) = f(4) = 0$, $f(1) = f(3) = -1$.

Section 5.5 Solutions Back to 5.5 Exercises

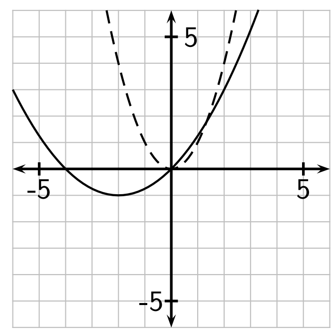
1. (a)



(b)



(c)



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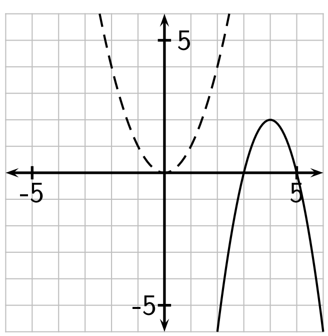
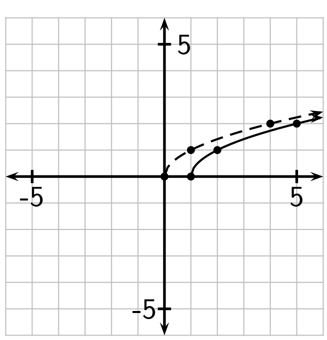
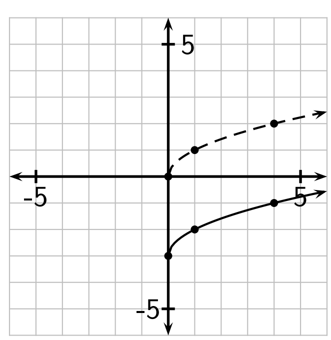


2. (a)



(b)



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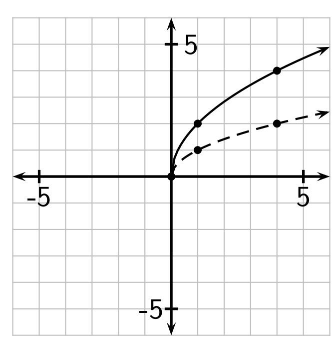

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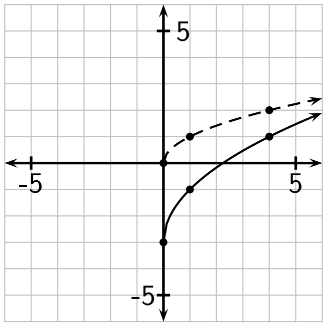

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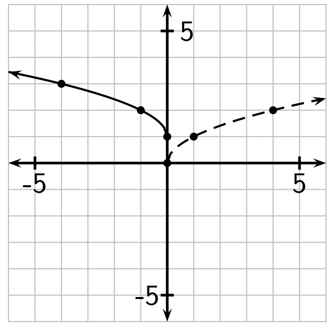

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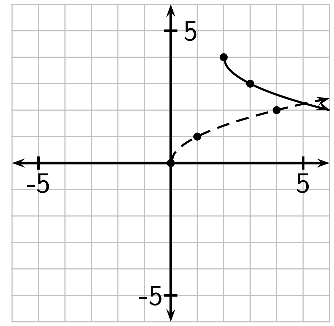

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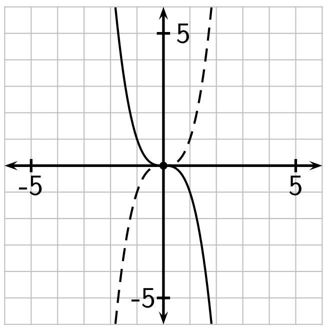

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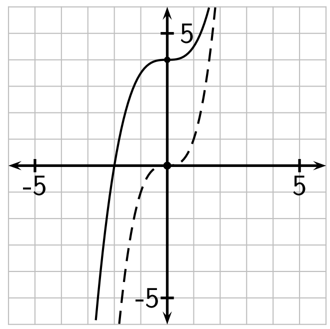

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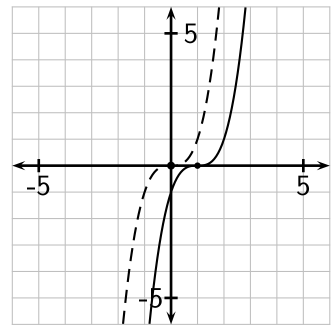

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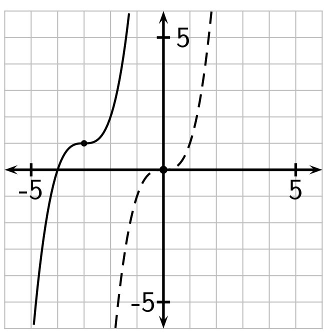

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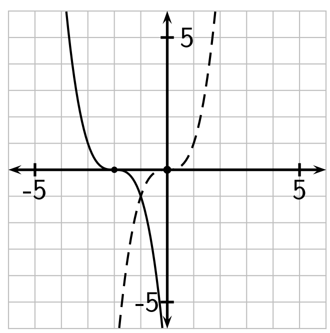

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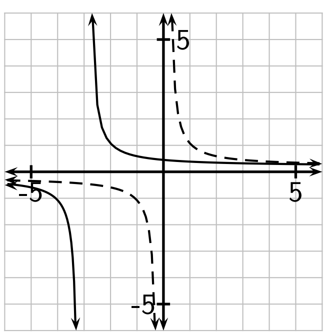

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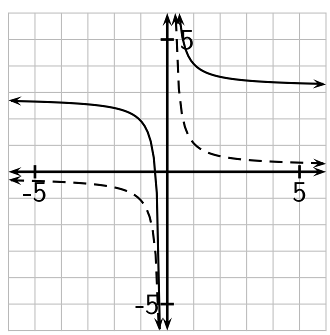

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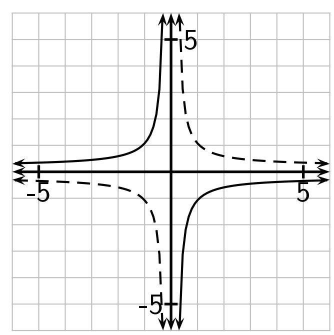

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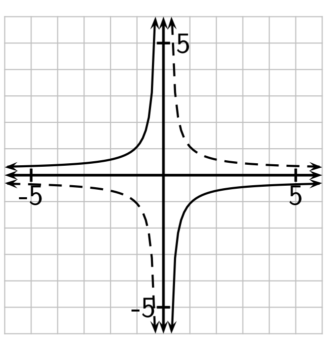

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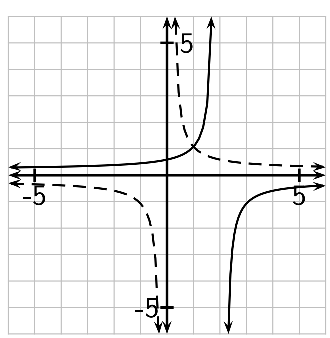

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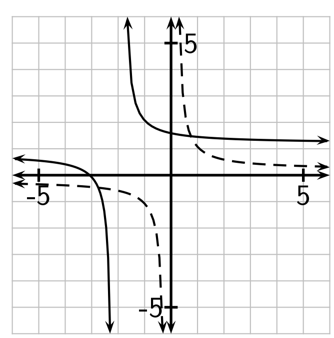
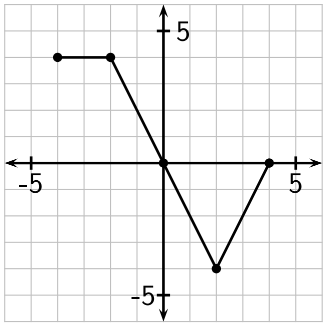


5. (a)



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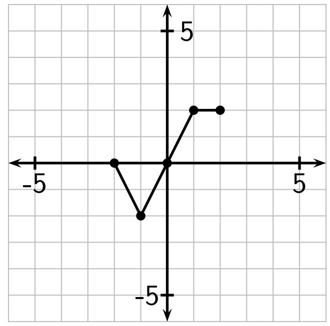

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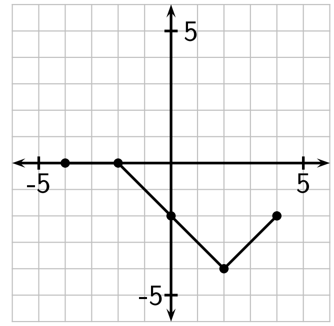

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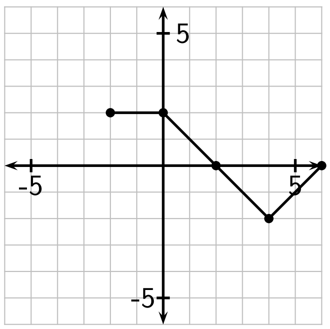

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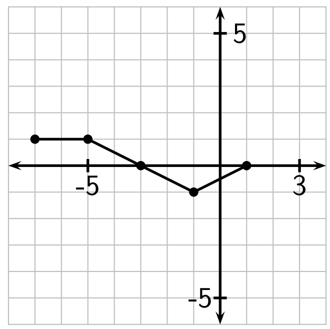

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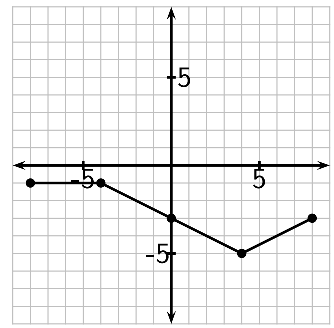
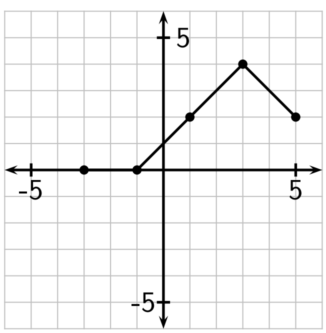
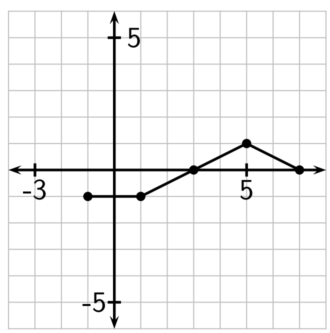


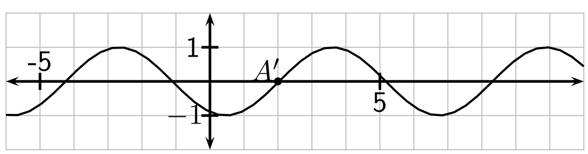
(g)



(h)



6. (a)



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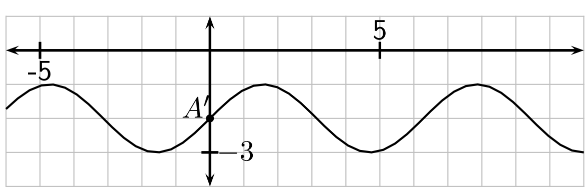

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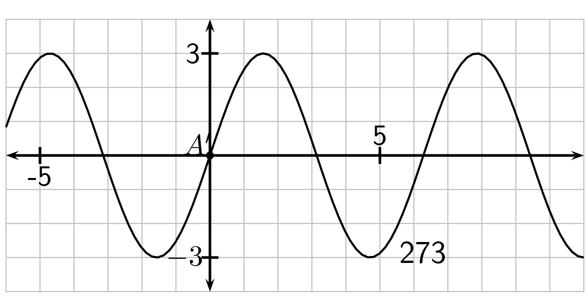

(d)