College Algebra

Gregg Waterman
Oregon Institute of Technology

Contents

6   Exponential and Logarithmic Functions

Exponential functions are used to model a large number of physical phenomena in engineering and the sciences. Their applications are as varied as the discharging of a capacitor to the growth of a population, the elimination of a medication from the body to the computation of interest on a bank account. In this chapter we will see a number of such applications.

In many of our applications we will have equations containing exponential functions, and we will want to solve those equations. In some cases we can do that by simply using our previous knowledge (usually with the help of a calculator). In other cases, however, we will need to invert exponential functions. This is where logarithm functions come in - they are the inverses of exponential functions.

6.1 Exponential Functions

Performance Criteria:

6. (a) Evaluate exponential functions. Recognize or create the graphs of $y = a^x$, $y = a^{-x}$ for $0 < a < 1$ and $1 < a$.

    (b) Determine inputs of exponential functions by guessing and checking, and by using a graph.

    (c) Solve problems using doubling time or half-life.

Introduction to Exponential Functions

The simplest examples of exponential functions are functions of the form $y = a^x$, where $a > 0$ and $a \neq 1$, like $y = 2^x$. To better understand such functions, let's recall some facts about exponents:

- $a^0 = 1$ for any number $a \neq 0$. 0^0 is undefined.

- $a^{-n} = \dfrac{1}{a^n}$ • $a^{\frac{1}{n}} = \sqrt[n]{a}$ • $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

⬦ **Example 6.1(a):** For $f(x) = 4^x$, find values of the function for $x = 2, 3, 1, \frac{1}{2}, 0, -1, -2, -\frac{1}{2}$ without using a calculator.

Solution: We know of course that

$$f(2) = 4^2 = 16, \qquad f(3) = 4^3 = 64, \qquad f(1) = 4^1 = 4.$$

From the above facts we also have

$$f(\tfrac{1}{2}) = 4^{\frac{1}{2}} = \sqrt{4} = 2, \qquad f(0) = 4^0 = 1, \qquad f(-1) = 4^{-1} = \frac{1}{4^1} = \tfrac{1}{4}$$

and

$$f(-2) = 4^{-2} = \frac{1}{4^2} = \tfrac{1}{16}, \qquad f(-\tfrac{1}{2}) = 4^{-\frac{1}{2}} = \frac{1}{4^{1/2}} = \frac{1}{\sqrt{4}} = \tfrac{1}{2}$$

The reason we require $a > 0$ is that a^x is undefined for $x = \frac{1}{2}$ when a is negative and a^x is also undefined for negative values of x when a is zero.

We will want to be able to evaluate such functions using our calculators. This is done by using the carat (ˆ) button or, if your calculator does not have such a button, the y^x button. You should try using your calculator to do the following computations yourself, making sure you get the results given.

◇ **Example 6.1(b):** For $f(x) = 5^x$, find $f(3.2)$ to the thousandth's place.

Solution: To evaluate this with a graphing calculator and some non-graphing calculators, simply enter $5\,\char`^\,3.2$ and hit *ENTER*. The result (rounded) should be 172.466. With some non-graphing calculators you will need to use the y^x button. Enter $5\ y^x\ 3.2$ followed by $=$ or ENTER to get the value.

◇ **Example 6.1(c):** For $g(t) = 20(7)^{-0.08t}$, find $g(12)$ to the hundredth's place.

Solution: To get the correct result here you can simply enter $20 \times 7\,\char`^\,(\,-0.08 \times 12\,)$ or $20 \times 7\ y^x\ (\,-0.08 \times 12\,)$. Note that you MUST use parentheses around the -0.08 and 12. The rounded result is 3.08.

Graphs of Exponential Functions

Suppose we wish to graph the exponential function $f(x) = 4^x$. Some values for this function were obtained in Example 6.1(a), and their decimal forms have been compiled in the table below and to the left and the ordered pairs ploteed on the graphs to the right of the table. Two graphs have been, so that we can see what is happening for values of x near zero and in the negatives, and then we see what happens as we get farther and farther from zero.

x	y
-2	0.0625
-1	0.25
-0.5	0.5
0	1
0.5	2
1	4
2	16
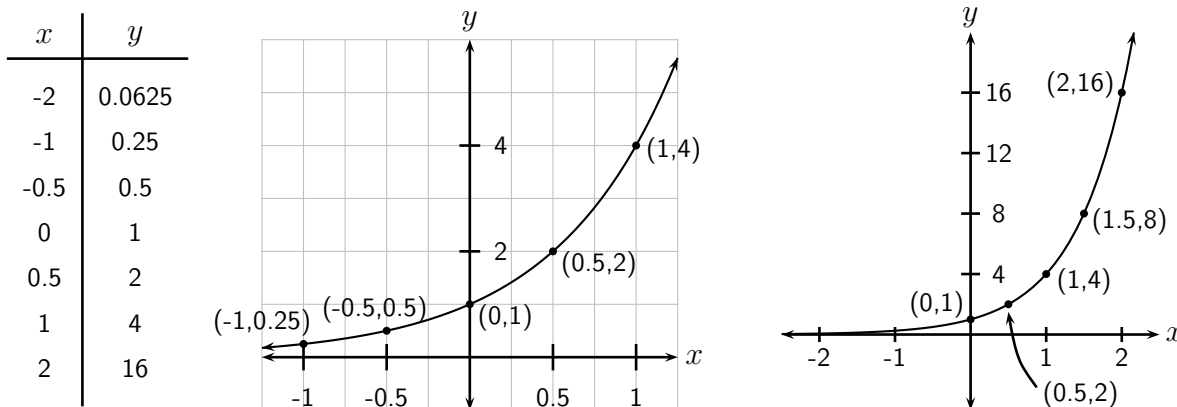
We get the point at $(1.5, 8)$ because $4^{1.5} = 4^{\frac{3}{2}} = (\sqrt{4})^3 = 2^3 = 8$. Looking at these two graphs of the function $y = 4^x$ we make the following observations:

- The domain of the function is all real numbers.

- The range of the function is $(0, \infty)$, or $y > 0$. Even when the value of x is negative, $y = 4^x$ is positive. However, we can get y to be as close to zero as we want by simply taking x to be a negative number with large absolute value.

- The function is increasing for all values of x. For negative values of x the increase is very slow, but for x positive the increase is quite rapid.

The graphs illustrate what we call **exponential growth**, where the values of y increase at a faster and faster rate as x gets larger and larger.

From the previous chapter we know that the graph of $y = f(-x)$ is the reflection of the graph of $y = f(x)$ across the y-axis. We see this in the first two graphs below, of $y = 2^x$ and $y = 2^{-x}$. The function $y = 2^{-x}$ exhibits what we call **exponential decay**, in which the y values decrease as the x values increase, but the decrease in y is less rapid as x gets larger and larger.

$$y = 2^x \qquad y = 2^{-x} \qquad y = (\tfrac{1}{2})^x \qquad y = 3^x$$
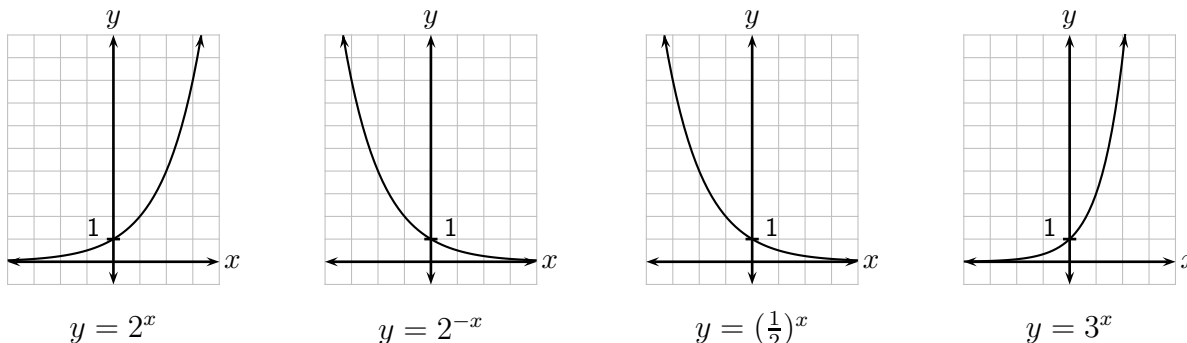
The third graph above is for $y = (\tfrac{1}{2})^x$, and we can see that the graph is the same as that of $y = 2^x$. This makes sense, because by the exponent rule $(x^a)^b = x^{ab}$ and the definition $x^{-n} = \dfrac{1}{x^n}$ we have

$$(\tfrac{1}{2})^x = (2^{-1})^x = 2^{-x}.$$

The last graph above is that of $y = 3^x$. We see that it is pretty much the same shape as the graph of $y = 2^x$, but the graph of $y = 3^x$ climbs more rapidly than that of $y = 2^x$ to the right of the y-axis, and the graph of $y = 3^x$ gets closer to zero than the graph of $y = 2^x$ as we go out in the negative direction.

Finally we note that the y-intercepts of all of these are one, by the fact that $a^0 = 1$ if $a \neq 0$. All of these things can be summarized as follows:

- For $a > 1$, the function $y = a^x$ is increasing. When such a function describes a "real life" situation, we say that the situation is one of **exponential growth**. The larger a is, the more rapidly the function increases.

- For $0 < a < 1$, the function $y = a^x$ is decreasing, and the situation is one of **exponential decay**.

- Exponential decay also occurs with functions of the form $y = a^{-x}$, where $a > 1$.

- In all cases, the range of an exponential function is $y > 0$ (or $(0, \infty)$ using interval notation).

Graphically we have

$$f(x) = a^x, \ (a > 1)$$

$$f(x) = a^x, \ (0 < a < 1)$$
$$f(x) = a^{-x}, \ (a > 1)$$
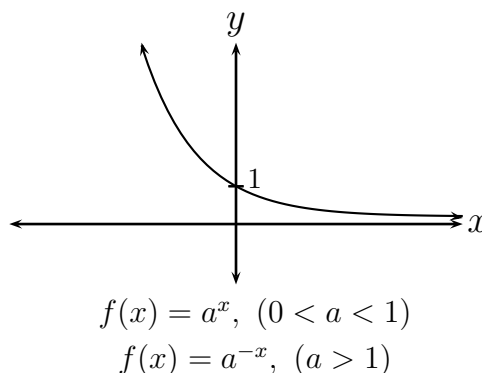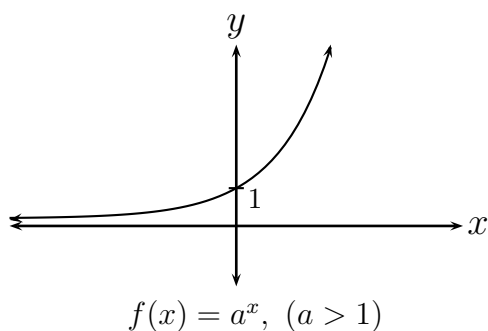
Since we have two ways to obtain exponential decay, we usually choose $a > 1$ and use the exponent $-x$ to obtain exponential decay. As a final note for this section, when graphing $f(x) = Ca^{kt}$, the y-intercept is C, rather than one.

Doubling Time and Half-Life

We conclude this section with two ideas (that are essentially the same) that arise commonly in the natural sciences and finance.

- One way to characterize the rate of exponential growth is to give the **doubling time** for the situation. This is the amount of time that it takes for the amount (usually a population or investment) to double in size. *This value remains the same at any point in the growth, not just at the beginning.*

- Exponential decay is often described in terms of its **half-life**, which is the amount of time that it takes for the amount to decrease to half the size. Like doubling time, the half-life is also the same at any point during the decay.

Let's look at a couple of examples.

◇ **Example 6.1(d):** A new radioactive element, *Watermanium*, is discovered. It has a half-life of 1.5 days. Suppose that you find a small piece of this element, weighing 48 grams. How much will there be three days later?

Solution: Note that three days is two 1.5 day periods. After a day and a half, only half of the original amount, 24 grams, will be left. After another day and a half only half of those 24 grams will be left, so there will be 12 grams after three days.

◇ **Example 6.1(e):** The population of a certain kind of fish in a lake has a doubling time of 3.5 years, and it is estimated that there are currently about 900 fish in the lake. Construct a table showing the number of fish in the lake every 3.5 years from now (time zero) up until 14 years, and a graph showing the growth of the population of fish in the lake.

Solution: We begin our table by showing that when we begin (time zero) there are 900 fish. In 3.5 years the population doubles to $2 \times 900 = 1800$ fish. 3.5 years after that, or 7 years overall, the population doubles again to $2 \times 1800 = 3600$ fish. Continuing, we get the table below and to the left and, from it, the graph below and to the right.

time in years	number of fish
0.0	900
3.5	1800
7.0	3600
10.5	7200
14.0	14,400

197
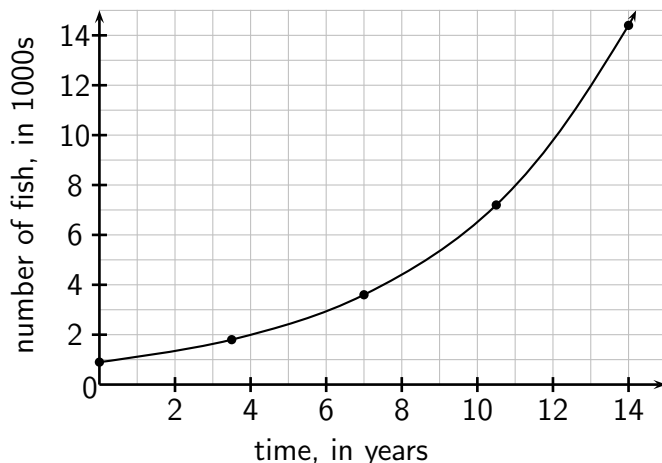
Finding "Inputs" of Exponential Functions

Suppose that we wanted to solve the equation $37 = 2^x$. We might be tempted to take the square root of both sides, but $\sqrt{2^x} \neq x$. If that were the case we would get that $x = \sqrt{37} \approx 6.08$. Checking to see if this is a solution we get $2^{6.08} \approx 67.65 \neq 37$. So how *DO* we "undo" the 2^x? That is where logarithms come in, but we are not going to get to them quite yet. For the time being, note that we could obtain an approximate solution by guessing and checking:

◇ **Example 6.1(f):** Using guessing and checking, find a value of x for which 2^x, is within one one-tenth of 37.

Solution: Since $2^3 = 8$ (I used this because most of us know it) x is probably something like 2^5; checking we find that $2^5 = 32$. This is a bit low but $2^6 = 64$, so we might guess that $x = 5.2$. $2^{5.2} = 36.75$, so we are close. But we were asked to find x to the nearest hundredth, so we need to keep going. Let's keep track of our guesses and the results in a table:

x :	5	6	5.2	5.3	5.22	5.21
2^x :	32	64	36.75	39.40	37.27	37.01

Thus we have that $x \approx 5.21$.

The above process is not very efficient, especially compared with a method you will see soon, but for certain kinds of equations this might be the only way to find a solution. There are also more efficient methods of "guessing and checking" that some of you may see someday.

We can also get approximate "input" values from a graph as well:

◇ **Example 6.1(g):** The graph below and to the left shows the growth of a $1000 investment. Determine how long you would have to wait for the investment to have a value of $5000.

Solution: Looking at the graph below and to the left, we go to $5000 on the value axis and go across horizontally to the graph of the function (see the horizontal dashed line), which is the solid curve. From the point where we meet the curve, we drop straight down to find that the value of our investment will be $5000 of winnings in about 17.5 years. This process is illustrated by the dashed lines on the graph below and to the right.
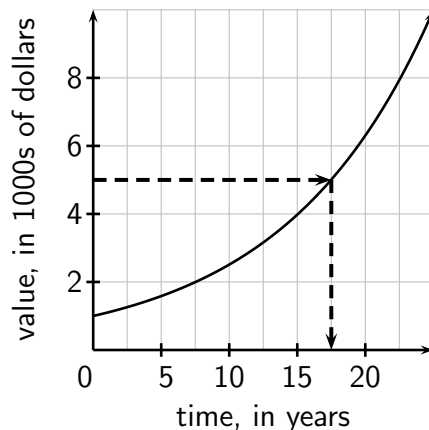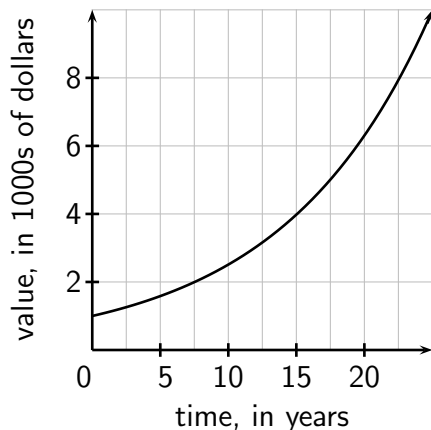
1. Let $f(t) = 150(2)^{-0.3t}$.

    (a) Find $f(8)$ to the nearest hundredth.

    (b) Use guessing and checking to determine when $f(t)$ will equal 50. Continue until you have determined t to the nearest tenth. (This means the value of t to the tenth's place that gives the value closest to 50.)

    (c) Find the average rate of change of f with respect to t from $t = 2$ to $t = 8$.

2. Graph each of the following without using a calculator of any kind. Label the y -intercept and one point on each side of the y -axis with their *exact* coordinates. You may use the facts that $a^{-n} = \frac{1}{a^n}$ and $a^{\frac{1}{n}} = \sqrt[n]{a}$.

    (a) $y = 3^x$ (b) $y = 3^{-x}$ (c) $y = (\frac{1}{5})^x$

3. Graph the function $y = 2^x$, carefully locating the points obtained when $x = 0, 1, 2$. Check your answer with *Desmos*. Then, using what you learned in Section 5.5, graph each of the following, indicating clearly the new points corresponding to the three points on the original graph. If the asymptote $y = 0$ changes, indicate the new asymptote clearly. Check each answer with *Desmos*.

    (a) $y = 2^x + 1$ (b) $y = 2^{x+1}$ (c) $y = 2^{-x}$

    (d) $y = -2^x$ (e) $y = 3(2^x)$ (f) $y = 2^{0.5x}$

4. Enter the function $y = C(2)^{kt}$ in *Desmos*, with sliders for C and k .

    (a) Which of the two parameters affects the y -intercept?

    (b) Which of the two parameters affects the rate of increase of the function more dramatically? What happens when that parameter goes from positive to negative? Give your answer using language from this section.

    (c) On the same grid, sketch the graphs obtained when $C = 1$ and $k = 1, 2, -1$, labelling each with its equation.

    (d) On a different grid from part (c), but on the same grid as each other, sketch the graphs obtained when $k = 1$ and $C = 1, 2, -1$, labelling each with its equation.

5. For $g(x) = 5^x - 3x$, use guessing and checking to approximate the value of x for which $f(x) = 100$. Find your answer to the nearest hundredth, showing all of your guesses and checks in a table.

6. The graph below and to the right shows the decay of 10 pounds of *Watermanium* (see Example 6.1(f)). Use it to answer the following.

   (a) How many pounds will be left after two days?

   (b) When will there be two pounds left?

   (c) Find the average rate of change in the amount from zero days to five days.

   (d) Find the average rate of change in the amount from two days to five days.
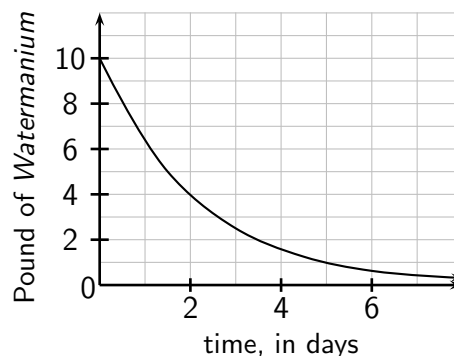
7. The half-life of ^{210}Bi is 5 days. Suppose that you begin with 80 g.

   (a) Sketch the graph of the amount of ^{210}Bi (on the vertical axis) versus time in days (on the horizontal axis), for twenty days.

   (b) Use your graph to estimate when there would be 30g of ^{210}Bi left. Give your answer as a short, but complete, sentence.

8. The half-life of ^{210}Bi is 5 days. Suppose that you begin with an unknown amount of it, and after 25 days you have 3 grams. How much must you have had to begin with?

9. Money in a certain long-term investment will double every five years. Suppose that you invest $500 in that investment.

   (a) Sketch a graph of the value of your money from time zero to 20 years from the initial investment.

   (b) Use your graph to estimate when you will have $3000.

10. Suppose that you have just won a contest, for which you will be awarded a cash prize. The rules are as follows:

   • This is a one-time cash award. After waiting whatever length of time you wish, you must take your full award all at once. You will receive nothing until that time.

   • You must choose Plan F or Plan G. If you choose Plan F, the amount of cash (in dollars) that you will receive after waiting t years is determined by the function $f(t) = 10(t+1)^2$. If you choose Plan G, your award will be determined by $g(t) = 10(2)^t$.

   Which plan should you choose?

6.2 Exponential Models, The Number e

The following example is a typical application of an exponential function.

◇ **Example 6.2(a):** A culture of bacteria is started. The number of bacteria after t hours is given by $f(t) = 60(3)^{t/2}$.

(a) How many bacteria are there to start with?

(b) How many bacteria will there be after 5 hours?

(c) Use guessing and checking to determine when there will be 500 bacteria. Get the time to enough decimal places to assure that you are within 1 bacteria of 500. **Show each time value that you guess, and the resulting number of bacteria.**

Solution: To answer part (a) we simply let $t = 0$ and find that $f(0) = 60$ bacteria. The answer to (b) is found in the same way: $f(5) = 60(3)^{5/2} = 935$ bacteria. In answering (c) we can already see that t must be less than 5 hours, by our answer to (b). We continue guessing, recording our results in a table:

time, in hours:	3.0	4.0	3.8	3.85	3.86
number of bacteria:	311.7	540.0	483.8	497.3	500.0

The next example will lead us into our next application of exponential functions.

◇ **Example 6.2(b):** Suppose that you invest $100 in a savings account at 5% interest per year. Your interest is paid at the end of each year, and you put the interest back into the account. Make a table showing how much money you will have in your account at the end of each of the first five years (right after the interest for the year is added to the account). Include the "zeroth year," at the end of which you have your initial $100.

Solution: The interest for the first year is $0.05 \times \$100 = \$5.00$, so you have $\$100 + \$5.00 = \$105.00$ at the end of the first year. The interest for the second year is $0.05 \times \$105.00 = \$5.25$, and the total at the end of that year is $\$105.00 + \$5.25 = \$110.25$. Continuing this way we get $\$115.76$, $\$121.55$ and $\$127.63$ at the ends of the third, fourth and fifth years, respectively. These results are shown in the table to the right; A stands for "amount," in dollars.

t (yrs)	A ($)
0	100.00
1	105.00
2	110.25
3	115.76
4	121.55
5	127.63

Let's call the original amount of $100 in the last example the **principal** P, which is "bankerspeak" for the amount of money on which interest is being computed (it could be debt as well as money invested), and let's let r represent the annual interest rate *in decimal form*. The amount of money after one year is then $P + Pr = P(1 + r)$. This then becomes the principal for the second year, so the amount after the end of that year is

$$[P(1+r)] + [P(1+r)]r = [P(1+r)](1+r) = P(1+r)^2.$$

Continuing like this, we would find that the amount A after t years would be $A = P(1 + r)^t$. This process of increasing the principal by the interest every time it is earned is called **compounding** the interest. In this case the interest is put in at the end of each year, which we called **compounded annually**.

In most cases the compounding is done more often than annually. Let's say it was done monthly instead. You would not earn 5% every month, but one-twelfth of that, so we would replace the r in $(1 + r)$ with $\frac{r}{12}$. If we then left the money in for five years it would get compounded twelve times a year for $12 \times 5 = 60$ times. The amount at the end of five years would then be $A = P\left(1 + \frac{r}{12}\right)^{12(5)}$. In general we have the following.

Compound Interest Formula

When a principal amount of P is invested for t years at an annual interest rate (in decimal form) of r, compounded n times per year, the total amount A is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

◇ **Example 6.2(c):** Suppose that you invest $1200 for 8 years at 4.5%, compounded semi-annually (twice a year). How much will you have at the end of the 8 years?

Another Example

Solution: The interest rate is $r = 4.5\% = 0.045$, and $n = 2$. We simply substitute these values, along with the time of 8 years, into the formula to get

$$A = 1200\left(1 + \frac{0.045}{2}\right)^{2(8)}.$$

You can compute this on your calculator by entering $1200 \times (1 + .045 \div 2) \char94 (2 \times 8)$, giving a result of $1713.15.

Here are a couple of comments at this point:

- Try the above calculation yourself to make sure you can get the correct value using your calculator. *Be sure to put parentheses around the* 2×8. You should learn to enter it all at once, rather than making intermediate computations and rounding, which will cause errors in your answer.

- *The power of compounding is significant!* If your interest was just calculated at the end of the eight years (but giving you 4.5% per year), your amount would be $A = 1200 + 1200(.045)(8) = \$1632.00$. The difference is about $\$80$, which is significant for the size of the investment. Interest computed without compounding is called **simple interest**.

The following example shows that although we are better off having our money compounded as often as possible, there is a limit to how much interest we can earn as the compounding is done more and more often.

◇ **Example 6.2(d):** Suppose that you invest $\$1000$ at 5% annual interest rate for two years. Find the amounts you would have at the end of the year if the interest was compounded quarterly, monthly, weekly, daily and hourly. (Assume there are 52 weeks in a year, and 365.25 days in a year.)

Solution: Using the formula $A = 1000 \left(1 + \dfrac{0.05}{n}\right)^{2n}$ for $n = 4, 12, 52, 365.25$ and 8766 we get $\$1104.49, \$1104.94, \$1105.12, \$1105.16$ and $\$1105.17$.

What we would find as we compounded even more often than hourly is that the amount would continue to increase, but by less and less. Rounded, the amounts for compounding more often would all round to $\$1105.17$. Using language and notation from the last chapter, we could say

$$\text{As } n \to \infty, \ A \to \$1105.17$$

Let's see (sort of) why that happens. Using the property of exponents that $x^{ab} = (x^a)^b$, and remembering that the value 0.05 is our interest rate r, the equation for the amount A can be rewritten as

$$A = 1000 \left(1 + \frac{0.05}{n}\right)^{2n} = 1000 \left[\left(1 + \frac{0.05}{n}\right)^n\right]^2 = 1000 \left[\left(1 + \frac{r}{n}\right)^n\right]^2$$

It turns out that as $n \to \infty$, the quantity $\left(1 + \frac{r}{n}\right)^n$ goes to the value $(2.718...)^r$, where $2.718...$ is a decimal number that never repeats or ends. That number is one of the four most important numbers in math, along with $0, 1$ and π, and it gets its own symbol, e. The final upshot of all this is that

$$\text{as } n \to \infty, \ A = P \left(1 + \frac{r}{n}\right)^{nt} = P \left[\left(1 + \frac{r}{n}\right)^n\right]^t \to P(e^r)^t = Pe^{rt}.$$

The last expression is the amount one would have if the interest was **compounded continuously**, which is a little hard to grasp, but sort of means that the interest is compounded "every instant" and added in to the principal as "instantaneously." This is summarized as follows.

> **Continuous Compound Interest Formula**
>
> When P dollars are invested for t years at an annual interest rate of r, compounded continuously, the amount A of money that results is given by
>
> $$A = Pe^{rt}$$

⬦ **Example 6.2(e):** If you invest $500 at 3.5% compounded continuously for ten years, how much will you have at the end of that time?

Solution: The trick here is not the setup, but understanding how to use your calculator. The amount is given by $A = 500e^{0.035(10)}$. All calculators allow us to take e to a power (which is all we ever do with e); this operation is found above the LN button. To do this computation with a graphing calculator you will type in $500 \times$ 2nd LN ($.035 \times 10$) and hit ENTER. With other calculators, you might do $.035 \times 10 =$ 2nd LN $\times 500$. The result is $709.53.

In an earlier example we used the exponential model $f(t) = 60(3)^{t/2}$, which can be rewritten as $f(t) = 60(3)^{0.5t} = 60(3^{0.5})^t$. It turns out that $3^{0.5} = e^{0.549...}$ (you can verify this with your calculator), so we could then write $f(t) = 60(e^{0.549...})^t = 60e^{0.549...t}$. This sort of thing can always be done, so we often use e as the base for our exponential models.

Exponential Growth and Decay Model

When a quantity starts with an amount A_0 or a number N_0 and grows or decays exponentially, the amount A or number N after time t is given by

$$A = A_0 e^{kt} \qquad \text{or} \qquad N = N_0 e^{kt}$$

for some constant k, called the **growth rate** or **decay rate**. When k is a growth rate it is positive, and when k is a decay rate it is negative.

Section 6.2 Exercises   To Solutions

1. (a) You invest $500 at an annual interest rate of 4 %. If it is compounded monthly, how much will you have after 3 years?

   (b) If you invest $2500 at an annual interest rate of 1.25%, compounded weekly, how much will you have after 7 years?

   (c) You invest $10,000 at an annual interest rate of 6.25%, compounded semi-annually (twice a year). How much will you have after 30 years?

   (d) For the scenario of part (c), what is your average rate of increase over the 30 years?

2. You are going to invest some money at an annual interest rate of 2.5%, compounded quarterly. In six years you want to have $1000. How much do you need to invest? **Note that here you are given** A, **not** P.

3. How much will be in an account after $2500 is invested at 8% for 4 years, compounded continuously?

4. (a) How much money will result from continuous compounding if $3000 is invested for 9 years at $5\frac{1}{4}\%$ annual interest?

   (b) What is the average rate of increase over those nine years?

5. How much would have to be invested at 12%, compounded continuously, in order to have $50,000 in 18 years?

6. You are going to invest $500 for 10 years, and it will be compounded continuously. What interest rate would you have to be getting in order to have $1000 at the end of those 10 years? **You will have to use guessing and checking to answer this. Find your answer to the nearest tenth of a percent.**

7. In 1985 the growth rate of the population of India was 2.2% per year, and the population was 762 million. Assuming continuous exponential growth, the number of people t years after 1985 is given by the equation $N(t) = 762e^{0.022t}$, where N **is in millions** of people.

   (a) What will the population of India be in 2010? Round to the nearest million.

   (b) Based on this model, what is the average rate of change in the population, with respect to time, from 1985 to 2010?

8. The height h (in feet) of a certain kind of tree that is t years old is given by
$$h = \frac{120}{1 + 200e^{-0.2t}}.$$

   (a) Use this to find the height of a tree that is 30 years old.

   (b) This type of equation is called a *logistic equation*. It models the growth of things that start growing slowly, then experience a period of rapid growth, then level off. Sketch the graph of the function for the first 60 years of growth of a tree. Use your graphing calculator, or find values of h for $t = 0, 10, 20, ..., 60$.

   (c) Around what age does the tree seem to be growing most rapidly?

9. For the variety of tree from the previous exercise,

   (a) What is the average rate of growth for the first 30 years?

   (b) What is the average rate of growth from 20 years to 30 years?

10. The expression e^u is greater than zero, no matter what u is. (This is actually saying the range of $y = a^x$ is $(0, \infty)$ for all $a > 0$, $a \neq 1$, as noted in Section 6.1.) This is a very important fact - use it to do the following.

   (a) Find all values of x for which $f(x) = 0$, where $f(x) = x^2e^x - 9e^x$. How do you know that you have given ALL such values?

   (b) Find all values of x for which $f(x) = 0$, where $f(x) = x^2e^{-x} - 9xe^{-x}$.

6.3 Logarithms

The Definition of a Logarithm Function

Recall the graphs of $y = a^x$ for $a > 1$ and $0 < a < 1$:
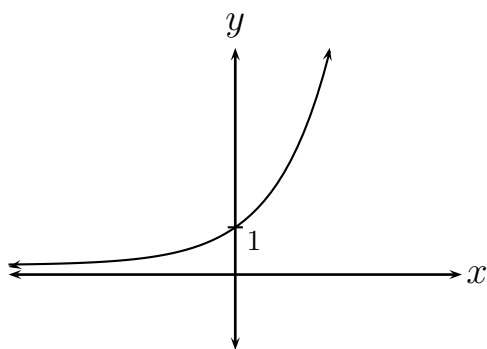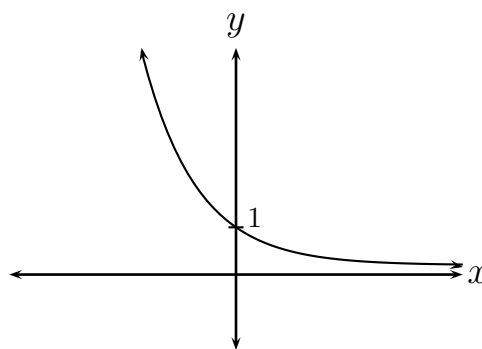
$$f(x) = a^x, \ (a > 1) \qquad\qquad f(x) = a^x, \ (0 < a < 1)$$

In both cases we can see that the function is one-to-one, with range $y > 0$. Therefore an inverse function exists, with domain $x > 0$. We call the inverse of $y = a^x$ the **logarithm base** a, denoted as $y = \log_a x$. Remembering that the graph of the inverse of a function is its reflection across the line $y = x$, we can see that the graph of $y = \log_a x$ can have the two appearances below, depending on whether a is greater than or less than one.
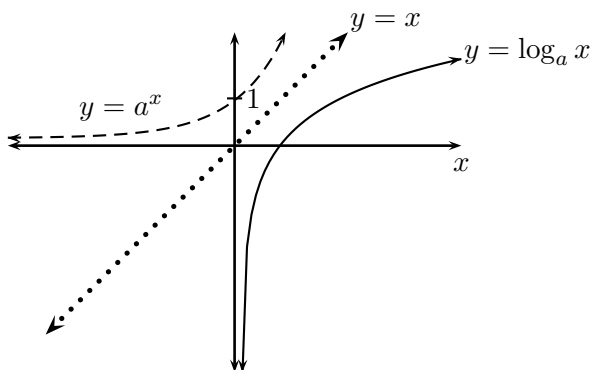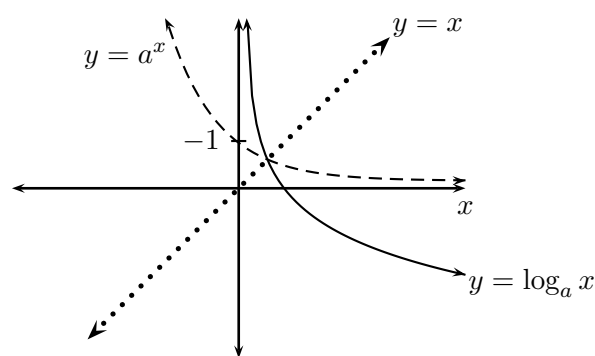
$$y = \log_a x, \ (a > 1) \qquad\qquad y = \log_a x, \ (0 < a < 1)$$

Remember that square root is the inverse function for "squared," at least as long as the values being considered are greater than or equal to zero. Based on this, the square root of a number x is the (non-negative) value that can be squared to get x. Similarly, the logarithm base a of $x > 0$ (because the domain of the logarithm is $(0, \infty)$) is the exponent to which a can be taken to get x. Clear as mud? Let's look at two examples:

⬦ **Example 6.3(a):** Find $\log_2 8$ and $\log_3 \frac{1}{9}$.

Solution: $\log_2 8$ the the power that 2 can be taken to get 8. Since $2^3 = 8$, $\log_2 8 = 3$. For the second one, we are looking for a power of 3 that equals $\frac{1}{9}$. Because $3^2 = 9$, $3^{-2} = \frac{1}{9}$ and we have $\log_3 \frac{1}{9} = -2$.

Note that what we were essentially doing in this example was solving the two equations $2^x = 8$ and $3^x = \frac{1}{9}$, which shows that every question about a logarithm can always be restated as a question about an exponent. All of this is summarized in the following definition.

> **Definition of the Logarithm Base a, Version 1**
>
> Suppose that $a > 0$ and $a \neq 1$. For any $x > 0$, $\log_a x$ is the power that a must be raised to in order to get x. Another way of saying this is that
>
> $$y = \log_a x \quad \text{is equivalent to} \quad a^y = x$$

Here x and y are variables, but a is a constant, the **base** of the logarithm. There are infinitely many logarithm functions, since there are infinitely many values of a that we can use. (Remember, though, that a must be positive and not equal to one.) Note that the domain of any logarithm function is $(0, \infty)$ because that set is the range of any exponential function.

The easiest way to apply the above when working with an equation with a logarithm in it is to get it into the form $\log_a B = C$ and then use a "circular motion" to transform it into an exponential equation, as I've attempted to show below:

$$\log_a B = C \quad \Longrightarrow \quad a^C = B \tag{1}$$

"a to the C equals B"

⬦ **Example 6.3(b):** Change $3 = \log_5 125 = 3$ and $\log_2 R = x^4$ to exponential form.

Solution: To work with $3 = \log_5 125 = 3$ we rewrite it as $\log_5 125 = 3$ and apply (1) above to get $5^3 = 125$. Doing the same with the second equation results in $2^{x^4} = R$.

⬦ **Example 6.3(c):** Solve the equation $\log_3 4x + 5 = 7$.

Solution: In order to apply (1) we must first get the equation in the form $\log_3 A = B$, which we do by subtracting five from both sides.

$$\begin{aligned}
\log_3 4x + 5 &= 7 \\
\log_3 4x &= 2 \\
3^2 &= 4x \\
9 &= 4x \\
x &= \tfrac{9}{4}
\end{aligned}$$

Note that if the equation from the previous example woud have been instead $\log_3(4x+5) = 7$ we could have applied (1) right away.

◇ **Example 6.3(d):** Solve the equation $\log_3(4x+5) = 7$.

Solution:

$$\begin{aligned} \log_3(4x+5) &= 7 \\ 3^7 &= 4x+5 \\ 2187 &= 4x+5 \\ 2182 &= 4x \\ x &= \tfrac{2182}{4} \end{aligned}$$

◇ **Example 6.3(e):** Solve the equation $\log_4 \frac{1}{64} = 2x$.

Solution: Changing the equation into exponential form results in $4^{2x} = \frac{1}{64}$. Noting that $\frac{1}{64} = 4^{-3}$, we must have $2x = -3$ and $x = -\frac{3}{2}$.

Usually we will wish to change from logarithm form to exponential form, as in the last example, but we will occasionally want to go the other way.

◇ **Example 6.3(f):** Change $4^{x-1} = 256$ to logarithm form.   Another Example

Solution: In this case, from the definition of a logarithm we have $a = 4$, $y = x - 1$ and $x = 256$. Putting these into $y = \log_a x$ results in $x - 1 = \log_4 256$.

Logarithms With a Calculator

In general, we can't find logarithms "by hand" - we need to enlist the help of our calculator. Before discussing how to do this, recall that there is a logarithm function *for every positive number except* 1, and your calculator is not going to be able to have a key for every one of them! There are two logarithms that are used most often:

- $\log_{10}$, "log base 10". This is called the **common logarithm**, and we don't generally write the base with it. It is the key on your calculator just labeled LOG.

- $\log_e$, "log base e". This is called the **natural logarithm**, and we write "ln" for it, with no base indicated. It is the key on your calculator labeled LN.

◇ **Example 6.3(g):** Find $\log 23.4$ and $\ln 126$ to the nearest thousandth.

Solution: $\log 23.4 = 1.369$ and $\ln 126 = 4.836$.

You will find that we can do just about everything we ever want to with just those two logarithms, but should you ever wish to find the logarithm of a number with some base other than 10 or e there are a couple options. Some calculators will compute logarithms with other bases, or you can find sites online that will do this. The other option is to use the following, the derivation of which we'll see later.

Change of Base Formula

$$\log_a B = \frac{\log B}{\log a}$$

◇ **Example 6.3(h):** Find $\log_3 17.2$ to the nearest hundredth.

Solution: Applying the formula we get $\log_3 17.2 = \dfrac{\log 17.2}{\log 3} = 2.59.$

Logarithms as Inverses of Exponential Functions

Use your calculator to find $\log 10^{3.49}$. You should get 3.49, because the common logarithm "undoes" 10^x and vice versa. That is, the inverse of $f(x) = 10^x$ is $f^{-1}(x) = \log x$. Similarly, the inverse of $g(x) = e^x$ is $g^{-1}(x) = \ln x$. This is the true power of logarithms, which can be summarized for all logarithms as follows:

Inverse Property of Logarithms, Part I

$$\log_a(a^u) = u \qquad \text{and} \qquad a^{\log_a u} = u$$

Notice that 10^x and LOG share the same key on your calculator, as do e^x and LN. This is, of course, no coincidence!

◇ **Example 6.3(i):** Find $\log_5 5^\pi$ and $\log_7 4^{1.2}$.

Solution: $\log_5 5^\pi = \pi$. We cannot do the same for $\log_7 4^{1.2}$ because the base of the logarithm does not match the base of the exponent. We would need to use the change of base formula to find its value; we'll skip that, since that was not the point of this example.

There is another formulation of the definition of logarithms that we can use to find inverses of functions that contain exponential or logarithm functions:

> **Definition of a Logarithm Base a, Version 2**
>
> The inverse of $y = \log_a x$ is $y = a^x$, and vice-versa.

◇ **Example 6.3(j):** Find the inverse of $f(x) = \log_2 3x + 5$.

Solution: As we did in Chapter 5, we determine the operations, and the order in which they are performed, for f. x is first multiplied by 3, then $\log_2$ is performed on that result and, finally, five is added. For the inverse we must first subtract five, and we take three to the resulting power. Finally, that result is divided by two or, equivalently, multiplied by $\frac{1}{2}$:

$$f^{-1}(x) = \frac{3^{x-5}}{2} = \tfrac{1}{2}(3)^{x-5}.$$

◇ **Example 6.3(j):** Find the inverse of $g(x) = 5^{x+1}$.

Solution: The function adds one, then raises five to the resulting power. For the inverse we must first take the logarithm base five, then subtract one, so

$$g^{-1}(x) = \log_5 x - 1.$$

Section 6.3 Exercises   To Solutions

1. Find any of the following that are possible.

   (a) $\log_4 1$ (b) $\log_3 \frac{1}{81}$ (c) $\log_4(-16)$ (d) $\log_9 3$

2. Change each of the following equations containing logarithms to their equivalent exponential equations.

   (a) $\log_2 \frac{1}{8} = -3$ (b) $\log_{10} 10,000 = 4$ (c) $\log_9 3 = \frac{1}{2}$

3. Change each of the following exponential equations into equivalent logarithm equations.

   (a) $4^3 = 64$ (b) $10^{-2} = \frac{1}{100}$ (c) $8^{2/3} = 4$

4. Use the definition of a logarithm to find **two** ordered pairs that satisfy $y = \log_5 x$.

5. Use the definition of a logarithm to solve   (a) $\log_4 x = \frac{3}{2}$ (b) $\log_9 x = -2$

6. Change $7^x = 100p$ to logarithmic form.   7. Find $\log_4 \frac{1}{16}$.

8. Use your calculator to find each of the following. Round to the nearest hundredth.

    (a) $\log 12$ (b) $\ln 17.4$ (c) $\ln 0.035$ (d) $\log 4.6$

9. Find each of the following.

    (a) $\log(10^{6.2})$ (b) $10^{\log 0.89}$ (c) $e^{\ln 4}$ (d) $\ln(e^{7.1})$

10. Find each of the following.

    (a) $\log_3 3^{7x}$ (b) $\ln e^{x^2}$ (c) $10^{\log 12.3}$ (d) $7^{\log_7(x-5)}$

11. Find each, rounded to three significant digits: (a) $\log_3 4.9$ (b) $\log_8 0.35$

12. Read Example 6.3(h), then compute $\dfrac{\ln 17.2}{\ln 3}$. How does the result compare to what was obtained in the example?

13. Solve each logarithm equation for the given unknown.

    (a) $\log_2 3x = 1, \ \ x$ (b) $2 = \log_3(w-4), \ \ w$

    (c) $\log_{10}\left(\dfrac{7}{t}\right) = 0, \ \ t$ (d) $\ln(2x-1) = 3, \ \ x$

    (e) $d = \log_5(ax) + b, \ \ x$ (f) $\log_5(ax + b) = d, \ \ x$

    (g) $\log_5(ax + b) = d, \ \ b$

14. Give the inverse of each function.

    (a) $f(x) = \ln(3x + 1)$ (b) $y = 2^x + 5$

    (c) $g(x) = 2(3)^x$ (d) $h(x) = \dfrac{\log_2(x + 1)}{7}$

    (e) $y = \log(3x - 4)$ (f) $f(x) = 3^{2x+1}$

6.4 Some Applications of Logarithms

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Performance Criteria:

6. (h) Use the inverse property of the natural logarithm and common logarithm to solve problems.

</div>

In this section we will see some applications of logarithms; a large part of our interest in logarithms is that they will allow us to solve exponential equations for an unknown that is in the exponent of the equation without guessing and checking, as we did earlier. The key is the inverse property of logarithms, adapted specifically to the natural logarithm:

$$\log_a(a^u) = u \text{ so, in particular, } \ln e^u = u$$

◇ **Example 6.4(a):** Solve $1.7 = 5.8e^{-0.2t}$ for t, to the nearest hundredth.

$$
\begin{array}{rcll}
1.7 & = & 5.8e^{-0.2t} & \text{the original equation} \\
\frac{1.7}{5.8} & = & e^{-0.2t} & \text{divide both sides by } 5.8 \\
\ln\left(\frac{1.7}{5.8}\right) & = & \ln e^{-0.2t} & \text{take the natural logarithm of both sides} \\
\ln\left(\frac{1.7}{5.8}\right) & = & -0.2t & \text{use the fact that } \ln e^u = u \\
\frac{\ln\left(\frac{1.7}{5.8}\right)}{-0.2} & = & t & \text{divide both sides by } -0.2 \\
6.14 & = & t & \text{compute the answer and round as directed}
\end{array}
$$

You should try this last computation on your calculator, to make sure that you know how to do it.

◇ **Example 6.4(b):** The growth of a culture of bacteria can be modeled by the equation $N = N_0 e^{kt}$, where N is the number of bacteria at time t days, N_0 is the number of bacteria at time zero, and k is the growth rate. If the doubling time of the bacteria is 4 days, what is the growth rate, to two significant digits?

Solution: Although we don't know the actual number of bacteria at any time, we DO know that when $t = 4$ days the number of bacteria will be twice the initial amount, or $2N_0$. We can substitute this into the equation along with $t = 4$ and solve for k:

$$
\begin{aligned}
2N_0 &= N_0 e^{k(4)} && \text{the original equation} \\
2 &= e^{4k} && \text{divide both sides by } N_0 \\
\ln 2 &= \ln e^{4k} && \text{take the natural logarithm of both sides} \\
\ln 2 &= 4k && \text{use the fact that } \ln e^u = u \\
\frac{\ln 2}{4} &= k && \text{divide both sides by } 4 \\
0.17 &= k && \text{compute the answer and round as directed}
\end{aligned}
$$

There are many, many applications of exponential functions in science and engineering; although perhaps not as common, logarithms functions have a number of applications as well. One that some of you may have seen already is the idea of pH, used to measure the acidity or basicity of a a chemical substance, in liquid form. The pH of a liquid is defined to be

$$\mathrm{pH} = -\log[\mathrm{H^+}],$$

where $\mathrm{H^+}$ is the hydrogen ion concentration in moles per liter. Another application is in the study of sound intensity, which is measured in something called decibels. Sound intensity is really just a pressure, as are waves that travel through the earth after an earthquake, so the Richter scale for describing the intensity of an earthquake is also based on logarithms.

Let's consider a specific example. *Ultrasound imaging* is a method of looking inside a person's body by sending sound waves into the body, where they bounce off of tissue boundaries and return to the surface to be recorded. The word *ultrasound* means that the sound is at frequencies higher than we can hear. The sound is sent and received by a device called a *transducer*. As the sound waves travel through tissues they lose energy, a process called *attenuation*. This loss of energy is expressed in units of **decibels**, which is abbreviated as dB. We have the following equation:

$$\text{attenuation} = 10\log\frac{I_2}{I_1} \tag{1}$$

where I_1 and I_2 are the intensities at two points 1 and 2, with point 2 being beyond point 1 on the path of travel of the ultrasound signal. The units of intensity are in units of power per unit of area (commonly watts per square centimeter).

⬦ **Example 6.4(c):**   The intensity of a signal from a 12 megahertz (MHz) transducer at a depth of 2 cm under the skin is one-fourth of what it is at the surface. What is the attenuation at that depth, to the hundredth's place?

Solution: Here I_1 is the intensity at the surface of the skin and I_2 is the intensity at 2 centimeters under the surface. We are given that $I_2 = \frac{1}{4}I_1$, so

$$\text{attenuation} = 10\log\frac{I_2}{I_1} = 10\log\frac{\frac{1}{4}I_1}{I_1} = 10\log\frac{1}{4} = -6.02 \text{ dB}$$

The attenuation is -6.02 decibels.

Attenuation comes up in many other settings that involve transmission of signals through some medium like air, wire, fiber optics, etc.

We can use the inverse property of logarithms to solve equations that begin as logarithm equations as well, as illustrated in the next example.

◇ **Example 6.4(d):** A 12MHz (megahertz) ultrasound transducer is applied to the skin, and at a depth of 5.5 cm the intensity is $8.92 \times 10^{-3} \mu\text{W}/\text{cm}^2$ (micro-watts per square centimeter). If this is an attenuation of -34 decibels, what is the intensity of the signal at the transducer, on the skin? Round to three significant figures.

Solution: Here $I_2 = 8.92 \times 10^{-3}$, attenuation is -34, and we are trying to determine I_1. The answer is obtained by substituting the known values into equation (1) and solving:

$$-34 = 10 \log \frac{8.92 \times 10^{-3}}{I_1} \qquad \text{equation (1) with values substituted}$$

$$-3.4 = \log \frac{8.92 \times 10^{-3}}{I_1} \qquad \text{divide both sides by } 10$$

$$10^{-3.4} = \frac{8.92 \times 10^{-3}}{I_1} \qquad \text{apply the definition of a logarithm}$$

$$10^{-3.4} I_1 = 8.92 \times 10^{-3} \qquad \text{multiply both sides by } I_1$$

$$I_1 = \frac{8.92 \times 10^{-3}}{10^{-3.4}} \qquad \text{divide both sides by } 10^{-3.4}$$

$$I_1 = 22.4 \qquad \text{compute the answer and round as directed}$$

The intensity at the transducer is $22.4 \mu\text{W}/\text{cm}^2$.

Section 6.4 Exercises

1. Solve $280 = 4e^{2.3k}$, rounding your answer to the nearest hundredth.

2. Solve $15 = 3000e^{-.04t}$, rounding your answer to two significant figures.

3. You are going to invest $3000 at 3.5% interest, compounded continuously. How long do you need to invest it in order to have $5000?

4. The growth of a population of rabbits in a ten square mile plot is modeled by $N = 1300e^{0.23t}$, with t in years.

   (a) When will the population of rabbits reach 10,000? Round to the nearest tenth of a year.

   (b) What is the doubling time for the population, to the nearest tenth of a year?

5. Recall the equation $N(t) = 762e^{0.022t}$ for the number of people in India t years after 1985, where N **is in millions** of people.

   (a) According to this model, when should the population of India will reach 2 billion?

   (b) Compute what the current population of India should be, based on this model. Then look up the population of India. Has the rate of population growth increased, or decreased from what it was when this model was developed?

6. Previously we saw that the height h (in feet) of a certain kind of tree that is t years old is given by
$$h = \frac{120}{1 + 200e^{-0.2t}} \ .$$
When, to the nearest whole year, will such a tree reach a height of 100 feet?

7. The intensity of a particular 8MHz ultrasound transducer at the point where it is applied to the skin is $20\mu\mathrm{W/cm^2}$. The attenuation for such a transducer is -4dB per centimeter of depth into soft tissue. Based on equation (1) (the equation for attenuation), what intensity will a signal from the transducer have at a depth of 3 centimeters? Give your answer to two significant figures; as usual, include units!

8. Consider again the transducer from Exercise 6. If we let d represent depth below the skin, in centimeters, then the attenuation at depth d is $-4d$. From the attenuation formula we then get
$$-4d = 10 \log \frac{I_2}{20}$$
as an equation relating the intensity and depth. Solve this equation for I_2, to get I_2 as a function of d.

9. The depth at which the intensity of an ultrasound signal has decreased to half the intensity at the skin is called the **half-intensity depth**. What is the attenuation at that depth? Round to the nearest tenth of a decibel. (**Hint:** I_2 is what fraction of I_1 in this case?)

10. The growth model obtained for the bacteria in Example 6.4(b) is $N = N_0 e^{0.17t}$, Using this model, when will the population of bacteria be triple what it was to begin?

11. $3000 is invested at an annual rate of $5\frac{1}{4}\%$, compounded continuously. How long will it take to have $5000?

12. The half-life of a medication is three hours. Find the rate of decay, k.

13. Assume that the number of mice in a barn is modelled by an equation of the form $N = N_0 e^{kt}$.

   (a) At the time that is being considered to be time zero there are estimated to be 500 mice in the barn. This tells us that when _____ $= 0$, _____ $= 500$.

(b) Use your answers to (a), along with the equation $N = N_0 e^{kt}$ to find N_0. It is a constant, so once its value is determined it remains the same, so you can put that value into the equation. Write down what you know the equation looks like at this point.

(c) Two months later it is estimated that there are 590 mice in the barn. Use this information and your equation from (b) to determine the value of k, to the thousandth's place. k is also a constant, so once you know its value you can put into the equation as well - write down the result.

(d) Now that you know N_0 and k, determine how many mice we'd expect to be in the barn a year after time zero.

14. A person takes 150mg of a medication, and 4 hours later there are 120mg in their system. Assuming that the amount A at any time t is given by $A = A_0 e^{kt}$, use a method like that of the previous exercise to determine how many milligrams will be in the person's body after 10 hours, to the nearest milligram. (Again find k to the thousandth's place.)

6.5 Properties of Logarithms

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Performance Criteria:

6. (i) Use properties of logarithms to write a single logarithm as a linear combination of logarithms, or to write a linear combination of logarithms as a single logarithm.

   (j) Apply properties of logarithms to solve logarithm equations.

</div>

In the last section we saw how to use the inverse property of logarithms to solve equations like $1.7 = 5.8e^{-0.2t}$. In certain applications like compound interest we will instead wish to solve an equation like $5000 = 1000(1.0125)^{4t}$, and we will not be able to use the inverse property of the logarithm directly because our calculators do not have a $\log_{1.0125}$. In this section we will see some algebraic properties of logarithms, one of which will allow us to solve such problems. We begin with the following.

◇ **Example 6.5(a):** Determine whether, in general, $\log_a(u+v) = \log_a u + \log_a v$.

Solution: Let's test this equation for $\log_{10}$, with $u = 31$ and $v = 7$:

$$\log(u+v) = \log 38 = 1.58 \quad \text{and} \quad \log u + \log v = \log 31 + \log 7 = 1.49 + 0.85 = 2.34$$

This shows that $\log_a(u+v) \neq \log_a u + \log_a v$.

This should not be surprising. About the only thing we can do to a sum that is equivalent to doing it to each item in the sum and *THEN* adding is multiplication:

$$z(x+y) = zx + zy, \quad \text{but} \quad (x+y)^2 \neq x^2 + y^2, \quad \sqrt{x+y} \neq \sqrt{x} + \sqrt{y}, \quad a^{x+y} \neq a^x + a^y.$$

When you go into trigonometry and use a function called "cos" (that is, the *cosine* function) $\cos(x+y) \neq \cos x + \cos y$. All is not lost however! Even though $a^{x+y} \neq a^x + a^y$, we *DO* have the property that $a^{x+y} = a^x a^y$, and a similar thing happens with logarithms.

◇ **Example 6.5(b):** Determine which of $\log_a(u+v)$, $\log_a(u-v)$, $\log_a(uv)$ and $\log_a\left(\dfrac{u}{v}\right)$ seem to relate to $\log_a u$ and $\log_a v$, and how.

Solution: Let's again let $a = 10$, $u = 31$ and $v = 7$, and compute the values of all the above expressions, rounding to the nearest hundredth:

$$\log u = 1.49, \qquad \log v = 0.85, \qquad \log(u+v) = 1.58,$$

$$\log(u-v) = 1.38, \qquad \log(uv) = 2.34, \qquad \log\left(\frac{u}{v}\right) = 0.65$$

Note that $\log u + \log v = 2.34$ and $\log u - \log v = 0.64$, which are (allowing for rounding) equal to the values of $\log(uv)$ and $\log(\frac{u}{v})$, respectively. From this, it appears that

$$\log_a uv = \log_a u + \log_a v \qquad \text{and} \qquad \log_a\left(\frac{u}{v}\right) = \log_a u - \log_a v$$

The results of this example were not a fluke due to the numbers we chose. There are three algebraic properties of logarithms we'll use, two of which you just saw. Including the third, they can be summarized as follows.

Properties of Logarithms

For $a > 0$ and $a \neq 1$, and for $u > 0$ and $v > 0$,

- $\log_a uv = \log_a u + \log_a v$

- $\log_a \left(\dfrac{u}{v} \right) = \log_a u - \log_a v$

- $\log_a u^r = r \log_a u$

In this section we'll see some ways these properties can be used to manipulate logarithm expressions, and we'll use the first two along with the definition of a logarithm to solve equations containing logarithms. In the next section we'll see that the third property is an especially powerful tool for solving exponential equations.

◇ **Example 6.5(c):** Apply the properties of logarithms to write $\log_4 \dfrac{\sqrt{x}}{y^3}$ in terms of logarithms of x and y.

Solution: First we note that $\sqrt{x} = x^{\frac{1}{2}}$. It should be clear that we will apply the second and third properties, but the order may not be so clear. We apply the second property first, and there are two ways to see why this is:

1) Since the numerator and denominator have different exponents, we don't have something of the form $\log_a u^r$. Instead we have $\log_a \left(\dfrac{u}{v} \right)$, where $u = x^{\frac{1}{2}}$ and $v = y^3$.

2) If we knew values for x and y and wanted to compute the logarithm, we would apply the exponents first, then divide, then perform the logarithm. In the same manner that we invert functions, we have to undo the division first here, then undo the exponents.

So, undoing the division with the second property, then undoing the exponents with the third, we get

$$\log_4 \frac{\sqrt{x}}{y^3} = \log_4 \frac{x^{\frac{1}{2}}}{y^3} = \log_4 x^{\frac{1}{2}} - \log_4 y^3 = \frac{1}{2} \log_4 x - 3 \log_4 y$$

◇ **Example 6.5(d):** Apply the properties of logarithms to write $5 \ln x + 2 \ln(x + 3)$ as a single logarithm. Another Example

Solution: You might recognize that what we are doing here is reversing the procedure from the previous exercise:

$$5 \ln x + 2 \ln(x + 3) = \ln x^5 + \ln(x + 3)^2 = \ln x^5 (x + 3)^2$$

This second example gives us a hint as to how we solve an equation like

$$\log_2 x + \log_2(x + 2) = 3 \ .$$

The procedure goes like this:

- Use the above properties to combine logarithms and get just one logarithm.

- Apply the definition of a logarithm to get an exponential equation.

- Evaluate the exponent and solve the resulting equation.

- Check to see that the logarithms in the original equation are defined for the value(s) of x that you found.

Note that you can't simply discard a value of x because it is negative; you need to see if it causes a logarithm to be evaluated for a negative number.

◇ **Example 6.5(e):** Solve the equation $\log_2 x + \log_2(x + 2) = 3$. <mark>Another Example</mark>

Solution: Applying the above procedure we get

$$
\begin{aligned}
\log_2 x + \log_2(x + 2) &= 3 && \text{the original equation} \\
\log_2 x(x + 2) &= 3 && \text{apply the first property to} \\
&&& \text{combine the two logarithms} \\
x(x + 2) &= 2^3 && \text{apply the definition of a logarithm} \\
x^2 + 2x &= 8 && \text{distribute and square} \\
x^2 + 2x - 8 &= 0 && \text{continue solving...} \\
(x + 4)(x - 2) &= 0
\end{aligned}
$$

Here we have the solutions $x = -4, \ 2$, but if $x = -4$ both $\log_2 x$ and $\log_2(x+2)$ are undefined, so the only solution is $x = 2$.

◇ **Example 6.5(f):** Solve $\ln(8 - x) = \ln(3x - 2) + \ln(x + 2)$. <mark>Another Example</mark>

<mark>Another Example</mark>

Solution: There are two ways to solve this problem. First we'll do it like the one above, which requires that we get all terms on one side and apply the properties of logarithms to turn them all into one logarithm:

$$
\begin{aligned}
\ln(8 - x) &= \ln(3x - 2) + \ln(x + 2) \\
0 &= \ln(3x - 2) + \ln(x + 2) - \ln(8 - x) \\
0 &= \ln \frac{(3x - 2)(x + 2)}{(8 - x)} \\
e^0 &= \frac{(3x - 2)(x + 2)}{(8 - x)}
\end{aligned}
$$

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$$1 = \frac{(3x-2)(x+2)}{(8-x)}$$

$$8 - x = 3x^2 + 4x - 4$$

$$0 = 3x^2 + 5x - 12$$

$$0 = (3x-4)(x+3)$$

Here we have the solutions $x = -3$, $\frac{4}{3}$, but if $x = -3$ both $\ln(3x-2)$ and $\ln(x+2)$ are undefined, so the only solution is $x = \frac{4}{3}$.

The other way to solve this problem is to combine the two terms on the right side and note that if $\ln A = \ln B$, then $A = B$ because logarithms are one-to-one:

$$\ln(8-x) = \ln(3x-2) + \ln(x+2)$$

$$\ln(8-x) = \ln(3x-2)(x+2)$$

$$(8-x) = (3x-2)(x+2)$$

$$8 - x = 3x^2 + 4x - 4$$

The remaining steps are the same as for the other method.

---

1. Apply the above properties to write each of the following logarithms in terms of logarithms of x and y.

   (a) $\log_4 \dfrac{x}{y^2}$

   (b) $\log_5 x^3 \sqrt{y}$

   (c) $\ln \sqrt[3]{x^2 y}$

   (d) $\ln \dfrac{\sqrt{x}}{y^3}$

2. Apply the above properties to write each of the following expressions as one logarithm, **without negative or fractional exponents**.

   (a) $2 \log x + \log y$

   (b) $\frac{1}{2} \log_2 x - 3 \log_2 y$

   (c) $\ln x - \ln y - \ln z$

   (d) $\log_8 y - \frac{3}{2} \log_8 z$

   (e) $\ln(x+1) - \ln(x^2 - 1)$

   (f) $2 \log x + 4 \log x$

3. Solve each of the following logarithm equations.

   (a) $\log_3 x + \log_3 9 = 1$

   (b) $\log_6 x + \log_6(x-1) = 1$

   (c) $\log_4(x-2) - \log_4(x+1) = 1$

   (d) $\log_8 x = \frac{2}{3} - \log_8(x-3)$

   (e) $\log_2 x + \log_2(x-2) = 3$

   (f) $\log 100 - \log x = 3$

4. Solve $\log_2(x - 3) + \log_2(x - 1) = 3$ for x.

5. Write $\log_3\left(\dfrac{x}{y^5}\right)$ in terms of $\log_3 x$ and $\log_3 y$.

6. Write $\frac{1}{2}\log x + 3\log y$ as one logarithm, without negative or fractional or negative exponents.

7. Solve $\log_2(x + 7) + \log_2 x = 3$.

8. Write $\log\left(\dfrac{\sqrt{x}}{yz^3}\right)$ in terms of $\log x$, $\log y$ and $\log z$.

6.6 More Applications of Exponential and Logarithm Functions

> **Performance Criteria:**
>
> 6. (k) Solve an applied problem using a given exponential or logarithmic model.
>
>   (l) Create an exponential model for a given situation.

Solving General Exponential Equations

Suppose that you were to invest $1000 at an annual interest rate of 5%, compounded quarterly. If you wanted to find out how many years it would take for the value of your investment to reach $5000, you would need to solve the equation $5000 = 1000(1.0125)^{4t}$. The method used is similar to what we did in Example 6.4(a) to solve $1.7 = 5.8e^{-0.2t}$, but different in one important way. Where we took the natural logarithm of both sides in that example, we would need to take $\log_{1.0125}$ of both sides here! Instead, we take the common logarithm $(\log_{10})$ of both sides and apply the rule

$$\log_a u^r = r \log_a u$$

that we saw in the last section. Here's how it's done:

◇ **Example 6.6(a):** Solve the equation $5000 = 1000(1.0125)^{4t}$.

$$
\begin{aligned}
5000 &= 1000(1.0125)^{4t} && \text{the original equation} \\
5 &= 1.0125^{4t} && \text{divide both sides by } 1000 \\
\log 5 &= \log 1.0125^{4t} && \text{take the common logarithm of both sides} \\
\log 5 &= 4t \log 1.0125 && \text{apply the rule } \log_a u^r = r \cdot \log_a u \\
\frac{\log 5}{4 \log 1.0125} &= t && \text{divide both sides by } 4 \log 1.0125 \\
32.39 &= t && \text{compute the final solution}
\end{aligned}
$$

It will take about 32.4 years for an investment of $1000 to reach a value of $5000 when invested at 5%, compounded quarterly.

Note that we could have used the natural logarithm everywhere that the common logarithm was used above, but we'll generally save the natural logarithm for situations dealing with the number e.

Remember the change of base formula $\log_a B = \dfrac{\log B}{\log a}$? Well, now we are ready to see where this comes from. Suppose that we want to find $\log_a B$; we begin by setting $x = \log_a B$. If we then solve this equation for x in the same way as just done, we'll have the value of $\log_a B$:

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$$
\begin{aligned}
x &= \log_a B & \text{the original equation} \\
a^x &= B & \text{apply the definition of a logarithm} \\
& & \text{to change to exponential form} \\
\log a^x &= \log B & \text{take the common logarithm of both sides} \\
x \log a &= \log B & \text{apply the rule } \log_a u^r = r \cdot \log_a u \\
x &= \frac{\log B}{\log a} & \text{divide both sides by } \log a
\end{aligned}
$$

Creating Exponential Models

Recall, from Section 6.2, the general forms of exponential growth and exponential decay models:

> **Exponential Growth and Decay Model**
>
> When a quantity starts with an amount A_0 or a number N_0 and grows or decays exponentially, the amount A or number N after time t is given by
>
> $$A = A_0 e^{kt} \qquad \text{or} \qquad N = N_0 e^{kt}$$
>
> for some constant k, called the **growth rate** or **decay rate**. When k is a growth rate it is positive, and when it is a decay rate it is negative.

The variables in the above models are N and t, or A and t, depending on which model is being used. (You should be able to see by now that they are really the same model!) The values N_0, A_0 and k are what we call parameters. They vary from situation to situation, but for a given situation they remain constant. When working with any exponential growth or decay problem the first thing we need to do is determine the values of these parameters, if they are not given to us. The next example shows how this is done.

◇ **Example 6.6(b):** A new radioactive element, *Watermanium*, is discovered. You have a sample of it that you know weighed 54 grams when it was formed, and there are 23 grams of it 7 days later. Determine the equation that models the decay of *Watermanium*.

Solution: We use the model $A = A_0 e^{kt}$, knowing that $A_0 = 54$. Because there are 23 grams left at 7 days, we can put $A = 23$ and $t = 7$ into the equation and it must be true: $23 = 54e^{7k}$. Solving this equation for k gives us $k = -0.12$. Our model is then $A = 54e^{-0.12t}$.

Note that the constant k depends on the material, in this case *Watermanium*, and A_0 is the initial amount, which can of course change with the scenario.

1. Solve $1500 = 500(2)^{5t}$ for t. Round to the hundredth's place.

2. In Example 6.6(b) we found that the amount of *Watermanium* left in a 54 gram sample after t days is given by $A = 54e^{-0.12t}$.

   (a) To the nearest gram, how much *Watermanium* will be left in 10 days?

   (b) How many days would it take a 347 gram sample of *Watermanium* to decay to 300 grams? Round your answer to the nearest tenth of a day. (**Hint:** The decay constant $k = -0.12$ does not depend on the original amount or its units.)

   (c) What is the half-life of Watermanium? Round to the nearest tenth of a day.

   (d) How many days, to the nearest tenth, will it take any amount of *Watermanium* to decay to 10% of the original amount?

3. Recall the culture of bacteria from Example 6.2(a), whose growth model was the function $f(t) = 60(3)^{t/2}$. Find the length of time that it would take for the number of bacteria to grow to 4000. Give your answer to the nearest hundredth of an hour.

4. Suppose that you invest some money at an annual percentage rate of 8%, compounded semi-annually. How long would it take for you to double your money?

5. The population of a certain kind of fish in a lake has a doubling time of 3.5 years. A scientist doing a study estimates that there are currently about 900 fish in the lake.

   (a) Using only the the doubling time, estimate when there will be 5000 fish in the lake. Do this by saying that "It will take between _____ and _____ years for there to be 5000 fish in the lake."

   (b) Create an exponential model for the fish population. Round k to the thousandth's place.

   (c) Use your model from (b) to determine when, to the nearest tenth of a year, the fish population in the lake will reach 5000. Check with your answer to (a).

6. Use the half-life of 5.8 days from Exercise 2(c) to find a function of the form $W = C(2)^{kt}$ that gives the amount of *Watermanium* left after t days in a sample that was originally 37.5 ounces.

7. Solve $1500 = 500(2)^{5t}$ for t, **to the nearest hundredth**.

8. Suppose that you invest $1500 at an annual percentage rate of 8%, compounded semi-annually. How long would it take for you to double your money?

9. If we begin with 80 milligrams (mg) of radioactive ^{210}Bi (bismuth), the amount A after t days is given by $A = 80(2)^{-t/5}$. Determine when there will be 12 milligrams left.

10. You invest some money at an annual interest rate of 5%, compounded quarterly. How long will it take to triple your money?

11. A culture of bacteria began with 1000 bacteria. An hour later there were 1150 bacteria. Assuming exponential growth, how many will there be 12 hours after starting? (**Hint:** You will need to create an exponential model.) Round to the nearest whole bacterium.

12. A 25 pound sample of radioactive material decays to 24.1 pounds in 30 days. How much of the original will be left after two years?

13. A drug is eliminated from the body through urine. Suppose that for an initial dose of 20 milligrams, the amount $A(t)$ in the body t hours later is given by $A(t) = 20(0.85)^t$. Determine when there will be 1 milligram left.
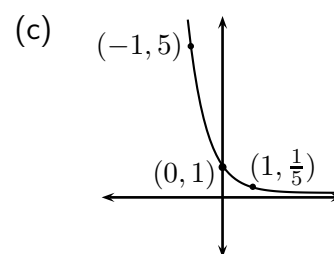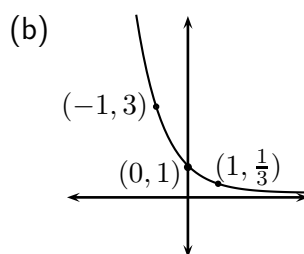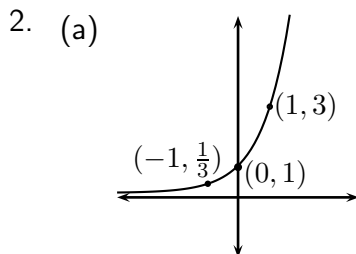
A Solutions to Exercises

A.6 Chapter 6 Solutions

1. (a) $f(8) = 28.42$ (b) $t = 5.3$ (c) -11.76

2. (a)

(b)

(c)

5. $x = 2.91,\quad g(x) = 99.41$

6. (a) After two days there will be four pounds left.

   (b) There will be 2 pounds left after about 3.5 days.

   (c) The average rate of change from 0 days to 5 days is -1.8 pounds per day.

   (d) The average rate of change from 2 days to 5 days is -1 pounds per day.

7. (a) See graph to the right.

   (b) There will be 30 grams of ^{210}Bi left after about seven days.
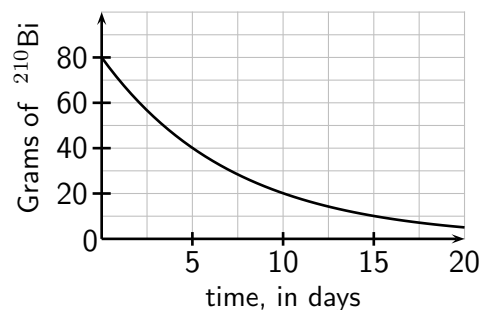
8. You began with 96 grams. (After 20 days there were 6 grams, after 15 days there were 12 grams, ...)

9. (a) See graph to the right.
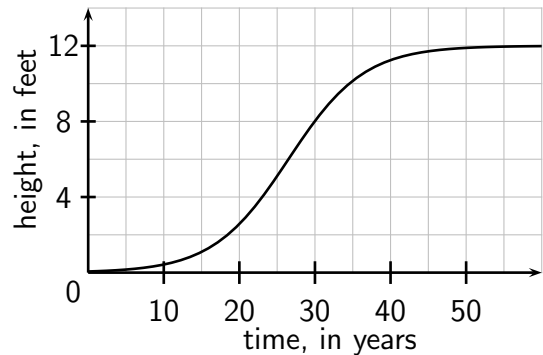
   (b) You will have $3000 after about thirteen years.
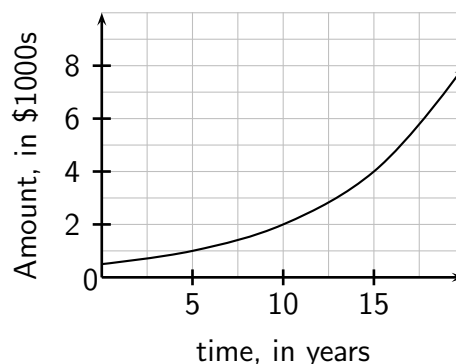
Section 6.2 Solutions

1. (a) $563.64   (b) $2728.58   (c) $63,363.29   (d) $1778.78 per year

2. $861.11   3. $3442.82

4. (a) $4812.00   (b) $201.33 per year

5. $5766.26   6. 6.9%

7. (a) 1321 million people   (b) 22.36 million people per year

8. (a) 80.2 feet

   (b) See graph to the right.

   (c) The tree seems to be growing most rapidly when it is about 25 to 30 years old.

9. (a) 2.65 feet per year   (b) 25.7 feet per year

10. $x = 3, \ \ x = -3$ 11. $x = 0, \ \ x = 9$

Section 6.3 Solutions

1. (a) 0 (b) -4 (c) does not exist   (d) $\frac{1}{2}$

2. (a) $2^{-3} = \frac{1}{8}$ (b) $10^4 = 10,000$ (c) $9^{1/2} = 3$

3. (a) $\log_4 64 = 3$ (b) $\log_{10} \frac{1}{100} = -2$ (c) $\log_8 4 = \frac{2}{3}$

4. any two of $(1,0)$, $(5,1)$, $(25,2)$, $(\frac{1}{5}, -1)$, $(\frac{1}{25}, -2)$, ...

5. (a) $4^{3/2} = x \implies x = (\sqrt{4})^3 = 8$ (b) $9^{-2} = x \implies x = \frac{1}{9^2} = \frac{1}{81}$

6. $\log_7 100p = x$ 7. $4^x = \frac{1}{16} \implies x = -2$

8. (a) 1.08   (b) 2.86   (c) -3.35 (d) 0.66

9. (a) 6.2   (b) 0.89   (c) 4   (d) 7.1

10. (a) $7x$ (b) x^2 (c) 12.3   (d) $x - 5$

11. (a) 1.447 (b) -0.505

12. The result is the same whether you use the common logarithm (base ten) or the natural logarithm.

13. (a) $x = \frac{2}{3}$ (b) $w = 13$ (c) $t = 7$ (d) $x = \dfrac{e^3 + 1}{2}$

     (e) $x = \dfrac{5^{d-b}}{a}$ (f) $x = \dfrac{5^d - b}{a}$ (g) $b = 5^d - ax$

14.   (a) $f^{-1}(x) = \dfrac{e^x - 1}{3}$ (b) $y = \log_2(x - 5)$

     (c) $g^{-1}(x) = \log_3 \frac{x}{2}$ (d) $h^{-1}(x) = 2^{7x} - 1$

     (e) $y = \dfrac{10^x + 4}{3}$ (f) $f^{-1}(x) = \dfrac{\log_3 x - 1}{2}$

Section 6.4 Solutions   Back to 6.4 Exercises

1. $k = 1.85$ 2. $t = 130$ 3. 14.6 years

4.   (a) 8.9 years   (b) 3.0 years

5.   (a) 43.9 years after 1985, or in October of 2028   6.   35 years

     (b) The rate of population growth has decreased since when the model was developed.

7. $I_2 = 1.3\mu\text{W/cm}^2$ 8. $I_2 = 20(10)^{-0.4d}$ 9. -3.0 dB

10. 6.46 days   11. 9.73 years   12. $k = -0.23$

13.   (a) $t = 0$, $N = 500$ (b) $N_0 = 500$, $N = 500e^{kt}$ (c) $k = 0.083$, $N = 500e^{0.083t}$

     (d) $N = 1354$ mice after one year

14. $A_0 = 150$, so $A = 150e^{kt}$, $k = -0.056$, so $A = 150e^{-0.056t}$, $A = 86$mg after
     10 hours

Section 6.5 Solutions   Back to 6.5 Exercises

1.   (a) $\log_4 x - 2\log_4 y$ (b) $3\log_5 x + \frac{1}{2}\log_5 y$

     (c) $\frac{2}{3}\ln x + \frac{1}{3}\ln y$ (d) $\frac{1}{2}\ln x - 3\ln y$

2.   (a) $\log x^2 y$ (b) $\log_2 \dfrac{\sqrt{x}}{y^3}$ (c) $\ln \dfrac{x}{yz}$

     (d) $\log_8 \dfrac{y}{\sqrt{z^3}}$ (e) $\ln \dfrac{1}{x - 1}$ (f) $6\log x$

3.   (a) $x = \frac{1}{3}$ (b) $x = 3$; you should also get $x = -2$, but it "doesn't check"

     (c) no solution; you should get $x = -2$ but it doesn't check

     (d) $x = 4$, $x = -1$ doesn't check   (e) $x = 4$, $x = -2$ doesn't check   (f)
     $x = \frac{1}{10}$

4. $x = \frac{1}{3}$ 5. $\log_3 - 5\log_3 y$ 6. $\log \sqrt{x}\,y^3$

7. $x = 1$ (-8 doesn't check)   8. $\frac{1}{2}\log x - \log y - 3\log z$

1. $t = 0.32$

2. (a) 16 g   (b) 1.2 days   (c) 5.8 days   (d) 19.2 days

3. 7.65 hours   4. 8.8 years

5. (a) between 7 and 10.5 years   (b) $N = 900e^{0.198t}$ (c) $t = 8.7$ years

6. $W = 37.5(2)^{-0.17t}$ 7. $t = 0.32$ 8. $t = 8.8$ years

9. $t = 13.7$ days   10. $t = 22.1$ years

11. Model: $N = 1000e^{0.14t}$, $N = 5,366$ bacteria

12. Model: $A = 25e^{-0.0012t}$, $A = 10.4$ pounds   13. $t = 18.4$ hours