College Algebra

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Contents

7 Systems of Equations

Outcome/Performance Criteria:

7. Solve systems of equations. Apply systems of equations to solve problems.

    (a) Use the addition method to solve a system of three linear equations in three unknowns.

    (b) Row-reduce a matrix.

    (c) Solve a system of equations by row-reduction, without using a calculator, and using a calculator.

    (d) Know when it is possible to add, subtract or multiply two matrices. Add, subtract and multiply matrices. Multiply matrices by numbers.

    (e) Determine whether two matrices are inverses without computing the inverse of either; use the formula to find the inverse of a 2×2 matrix.

    (f) Write a system of equations as a matrix equation.

    (g) Solve a system of equations using the inverse matrix method, with or without a calculator.

    (h) Calculate the determinant of a 2×2 matrix; use Cramer's rule to solve a system of two equations in two unknowns.

7.1 Systems of Three Linear Equations

Performance Criteria:

7. (a) Use the addition method to solve a system of three linear equations in three unknowns.

Recall the following Example from Chapter 1:

◇ **Example 1.6(a):** Solve the system $\begin{aligned} 2x-4y &= 18 \\ 3x+5y &= 5 \end{aligned}$ using the addition method.

Solution: The idea is to be able to add the two equations and get either the x terms or the y terms to go away. If we multiply the first equation by 5 and the second equation by 4 the y terms will go away when we add them together. We'll then solve for x:

$$\begin{aligned} 2x-4y&=18 \quad \overset{\text{times } 5}{\Longrightarrow} \quad & 10x-20y&=90 \\ 3x+5y&=\ \ 5 \quad \overset{\text{times } -4}{\Longrightarrow} \quad & 12x+20y&=20 \\ & & 22x&=110 \\ & & x&=5 \end{aligned}$$

We then substitute this value of x into any one of our equations to get y:

$$\begin{aligned} 3(5)+5y &= \ \ 5 \\ 15+5y &= \ \ 5 \\ 5y &= -10 \\ y &= -2 \end{aligned}$$

The solution to the system is then $x=5,\ y=-2,$ or $(5,-2)$.

We should note that the system could have been solved by eliminating x, rather than y:

◇ **Example 7.1(a):** Solve the system $\begin{aligned} 2x-4y &= 18 \\ 3x+5y &= 5 \end{aligned}$ using the addition method.

Solution: If we multiply the first equation by 3 and the second equation by -2 the x terms will go away when we add them together. We'll then solve for y:

$$\begin{aligned} 2x-4y&=18 \quad \overset{\text{times } 3}{\Longrightarrow} \quad & 6x-12y&=\ \ 54 \\ 3x+5y&=\ \ 5 \quad \overset{\text{times } -2}{\Longrightarrow} \quad & -6x-10y&=-10 \\ & & -22y&=44 \\ & & y&=-2 \end{aligned}$$

We then substitute this value of y into any one of our equations to get $x=5$.

In Section 1.6 you may have solved the following system of *three equations in three unknowns*:

$$\begin{aligned} x + 3y - 2z &= -4 \\ 3x + 7y + z &= 4 \\ -2x + y + 7z &= 7 \end{aligned} \tag{1}$$

The sequence we use for such a system is the following:

- Add a multiple of the first equation to a multiple of the second equation in order to eliminate x.
- Add a multiple of the first equation to a multiple of the third equation equation in order to eliminate x.
- The first two steps results in a system of two equations in the unknowns y and z. Add a multiple of one to a multiple of the other in order to eliminate y, and solve for z.
- Substitute the value of z into either of the equations containing only y and z and solve for y.
- Substitute the values of y and z into any of the three original equations and solve for x.

Let's demonstrate:

◇ **Example 7.1(b):** Solve the above system of three equations in three unknowns.

Solution: Multiplying the first equation by -3 and adding it to the second equation results in the equation $-2y+7z = 16$. Similarly, multiplying the first equation by 2 and adding to the third equation gives $7y + 3z = -1$. We now have the system

$$\begin{aligned} -2y + 7z &= 16 \\ 7y + 3z &= -1 \end{aligned}$$

of two equations in the unknowns y and z. Multiplying the first of these by 7, the second by 2 and adding results in $55z = 110$. From this we determine that $z = 2$ and, substituting into either of the above two equations and solving for y gives us $y = -1$. Finally, we substitute our known values for y and z into any of the first three equations and solve to get $x = 3$.

Section 7.1 Exercises To Solutions

1. Consider the system of three equations in three unknowns x, y and z:

$$\begin{aligned} x + 2y - z &= -1 \\ 2x - y + 3z &= 13 \\ 3x - 2y &= 6 \end{aligned}$$

   Follow the steps below to solve the system.

   (a) Use the addition method with the first two equations to eliminate x.

   (b) Use the addition method with the first and third equations to eliminate x.

   (c) Your answers to (a) and (b) are a new system of two equations with two unknowns. Use the addition method to eliminate y and solve for z.

   (d) Substitute the value you found for z into one of the equations containing y and z to find y.

   (e) You should now know y and z. Substitute them into ANY of the three equations to find x.

   (f) You now have what we call an *ordered triple* (x, y, z). Check your solution by substituting those three numbers into each of the original equations to make sure that they make all three equations true.

2. Solve each of the following systems of equations.

   (a) $$\begin{aligned} x + 3y - z &= -3 \\ 3x - y + 2z &= 1 \\ 2x - y + z &= -1 \end{aligned}$$

   (b) $$\begin{aligned} 2x - y + z &= 6 \\ 4x + 3y - z &= 1 \\ -4x - 8y + 2z &= 1 \end{aligned}$$

   (c) $$\begin{aligned} x - 2y + 3z &= 4 \\ 2x + y - 4z &= 3 \\ -3x + 4y - z &= -2 \end{aligned}$$

7.2 Solving Systems of Equations by Row-Reduction

Performance Criteria:

7. (b) Row-reduce a matrix.
   (c) Solve a system of equations by row-reduction, without using a calculator, and using a calculator.

In Example 7.1(b) we solved the system of equations

$$\begin{aligned} x+3y-2z &= -4 \\ 3x+7y+z &= 4 \\ -2x+y+7z &= 7 \end{aligned} \tag{1}$$

If we look carefully at our work there we find the following sequence of systems of equations:

$$\begin{aligned} x+3y-2z &= -4 \\ 3x+7y+z &= 4 \\ -2x+y+7z &= 7 \end{aligned} \quad\Rightarrow\quad \begin{aligned} x+3y-2z &= -4 \\ -2y+7z &= 16 \\ 7y+3z &= -1 \end{aligned} \quad\Rightarrow\quad \begin{aligned} x+3y-2z &= -4 \\ -2y+7z &= 16 \\ 55z &= 110 \end{aligned} \tag{2}$$

Let's put in zeros for coefficients of missing terms:

$$\begin{aligned} x+3y-2z &= -4 \\ 3x+7y+z &= 4 \\ -2x+y+7z &= 7 \end{aligned} \quad\Rightarrow\quad \begin{aligned} x+3y-2z &= -4 \\ 0x-2y+7z &= 16 \\ 0x+7y+3z &= -1 \end{aligned} \quad\Rightarrow\quad \begin{aligned} x+3y-2z &= -4 \\ 0x-2y+7z &= 16 \\ 0x+0y+55z &= 110 \end{aligned} \tag{3}$$

It turns out that the symbols x, y, z and $=$ just get in the way while we are solving the system of equations. Because of this it is convenient to arrange just the coefficients of the unknowns and the numbers from (1) on the right sides into a table of values called a **matrix**

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & 7 & 1 & 4 \\ -2 & 1 & 7 & 7 \end{bmatrix} \tag{4}$$

This is called the **augmented matrix** for the system (1). The goal is to manipulate this matrix to obtain the matrix

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 0 & 55 & 110 \end{bmatrix} \tag{5}$$

Note that this is just the matrix for the last system in (3), which is really the same as the last system in (2).

To get from (4) to (5) we use a process called **row reduction**. When working with a matrix that represents a system of equations we are allowed to do three things without affecting the solution to the system:

- Multiply each entry in a row by the same value. (We can also divide by a number, which is the same as multiplying by its reciprocal.)

- Add a row to another row (replacing the second of these two rows with the result in the process). (We can also subtract a row from another, since that is the same as first multiplying by -1 and then adding.)

- Rearrange the rows.

What we will do most often is a combination: We'll multiply a row by a number and add it to another row. To begin getting from (4) to (5) we first multiply the first row of (4) by -3 and add the result to the second row, replacing the second row with the result. We will symbolize this by

$$-3R_1 + R_2 \to R_2.$$

This says "add negative three times row one to row two, and put the result in row two." *Leave the first row in its original state.* This is how we show it and what we get:

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & 7 & 1 & 4 \\ -2 & 1 & 7 & 7 \end{bmatrix} \quad \underset{-3R_1 + R_2 \to R_2}{\Longrightarrow} \quad \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ -2 & 1 & 7 & 7 \end{bmatrix} \qquad (6)$$

Next we multiply the first row by 2 and add the result to the third row, putting the result in the third row, leaving the first two rows as they were. Often we will combine these first two steps into one:

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & 7 & 1 & 4 \\ -2 & 1 & 7 & 7 \end{bmatrix} \quad \begin{matrix} -3R_1 + R_2 \to R_2 \\ \Longrightarrow \\ 2R_1 + R_3 \to R_3 \end{matrix} \quad \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 7 & 3 & -1 \end{bmatrix} \qquad (7)$$

Now we multiply the second row by 7 and the third row by 2 and add the results, putting what we get in the third row, leaving the other two rows the same. The result should be (5) above. The symbolic representation of this is

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 7 & 3 & -1 \end{bmatrix} \quad \underset{7R_2 + 2R_3 \to R_3}{\Longrightarrow} \quad \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 0 & 55 & 110 \end{bmatrix}$$

At this point we have **row-reduced** the matrix (4) to the *row-reduced form* (5). It is possible to take this process farther to give a nicer result to work with, but it is generally not worth the effort to do so. We can get the solution to the system by first putting the final matrix (5) back into equation form:

$$\begin{aligned} x + 3y - 2z &= -4 \\ 0x - 2y + 7z &= 16 \\ 0x + 0y + 55z &= 110 \end{aligned}$$

We then solve the last equation for $z = 2$. Putting that value into the second equation and solving, we get $y = -1$, Finally we substitute our y and z values into the first equation to get $x = 3$. The full solution is therefore $(3, -1, 2)$. In Section 7.1 we saw that the solution to a system of two equations in two unknowns is a point in the plane where the two lines represented by the equations cross each other. When we deal with three linear equations in three unknowns, each equation represents a plane in three-dimensional space, and the three numbers in the solutions are the coordinates of the point in three-dimensional space where the three planes intersect.

In a moment we will see how to let our calculator do all the previous calculations for us. The calculator will take our final matrix above and continue, to obtain ones where we have 1, -2 and 55 and zeros in all the places above them. The process goes like this:

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 0 & 55 & 110 \end{bmatrix} \underset{\frac{1}{55}R_3 \to R_3}{\Longrightarrow} \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} -7R_3 + R_2 \to R_2 \\ \Longrightarrow \\ 2R_3 + R_1 \to R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \underset{-\frac{1}{2}R_2 \to R_2}{\Longrightarrow} \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \underset{-3R_2 + R_1 \to R_1}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

This last matrix is equivalent to the system

$$\begin{aligned} x + 0y + 0z &= 3 \\ 0x + y + 0z &= -1 \\ 0x + 0y + z &= 2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= 3 \\ y &= -1 \\ z &= 2 \end{aligned}$$

Row Reduction Using Your Calculator

The process of row reduction is quite tedious, and the systems of equations that I have given you so far are easier to work with than most. Let's see how to make our calculators do the "dirty work!" Your calculator is going to take the row-reduction process as far as we just did, so you will be able to just read off the values of x, y and z. As you read this you should go through the process with the system we've been working with,

$$\begin{aligned} x + 3y - 2z &= -4 \\ 3x + 7y + z &= 4 \\ -2x + y + 7z &= 7 \end{aligned} \tag{1}$$

- Select *MATRIX* (or maybe *MATRX*) somewhere.
- Select *EDIT*. At that point you will see something like 3×3 somewhere. This is the number of rows and columns your matrix is going to have. We want 3×4.
- After you have told the calculator that you want a 3×4 matrix, it will begin prompting you for the entries in the matrix, starting in the upper left corner. Here you will begin entering the values from (4), row by row. You should see the entries appear in a matrix as you enter them.

- After you enter the matrix, you need to select *MATH* under the *MATRIX* menu. Select *rref* (this stands for **row-reduced echelon form**) and you should see *rref* (on your calculator screen.
- Select *NAMES* under the *MATRIX* menu. Highlight A and hit enter, then enter again. At that point you should see the matrix(8), from which the solution can be read off.

Section 7.2 Exercises

To Solutions

1. Fill in the blanks in the second matrix with the appropriate values after the first step of row-reduction. Fill in the long blanks with the row operations used.

(a)

$$\begin{bmatrix} 1 & 5 & -7 & 3 \\ -5 & 3 & -1 & 0 \\ 4 & 0 & 8 & -1 \end{bmatrix} \quad \begin{matrix} \underline{\hspace{3cm}} \\ \Longrightarrow \\ \underline{\hspace{3cm}} \end{matrix} \quad \begin{bmatrix} \_\_ & \_\_ & \_\_ & \_\_ \\ 0 & \_\_ & \_\_ & \_\_ \\ 0 & \_\_ & \_\_ & \_\_ \end{bmatrix}$$

(b)

$$\begin{bmatrix} 2 & -8 & -1 & 5 \\ 0 & -2 & 3 & 1 \\ 0 & 6 & -5 & 2 \end{bmatrix} \quad \begin{matrix} \Longrightarrow \\ \\ \underline{\hspace{3cm}} \end{matrix} \quad \begin{bmatrix} \_\_ & \_\_ & \_\_ & \_\_ \\ 0 & \_\_ & \_\_ & \_\_ \\ 0 & 0 & \_\_ & \_\_ \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 3 & 5 & -2 \\ 0 & 2 & -8 & 1 \end{bmatrix} \quad \begin{matrix} \Longrightarrow \\ \\ \underline{\hspace{3cm}} \end{matrix} \quad \begin{bmatrix} \_\_ & \_\_ & \_\_ & \_\_ \\ 0 & \_\_ & \_\_ & \_\_ \\ 0 & 0 & \_\_ & \_\_ \end{bmatrix}$$

2. Find x, y and z for the system of equations that reduces to the each of the matrices shown.

(a) $\begin{bmatrix} 1 & 6 & -2 & 7 \\ 0 & 8 & 1 & 0 \\ 0 & 0 & -2 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 6 & -2 & 7 \\ 0 & 2 & -5 & -13 \\ 0 & 0 & 3 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -4 & 8 \end{bmatrix}$

3. Use row operations on an augmented matrix to solve each system of equations.

(a) $\begin{aligned} x - 2y - 3z &= -1 \\ 2x + y + z &= 6 \\ x + 3y - 2z &= 13 \end{aligned}$ (b) $\begin{aligned} -x - y + 2z &= 5 \\ 2x + 3y - z &= -3 \\ 5x - 2y + z &= -10 \end{aligned}$

(c) $\begin{aligned} x + 2y + 4z &= 7 \\ 2x + 3y + 3z &= 7 \\ -x + y + 2z &= 5 \end{aligned}$

4. Use your calculator to solve each of the systems of equations from Exercise 3.

7.3 The Algebra of Matrices

Performance Criteria:

7. (d) Know when it is possible to add, subtract or multiply two matrices. Add, subtract and multiply matrices. Multiply matrices by numbers.

In this section we see how to add, subtract and multiply matrices. *It is not possible to divide one matrix by another.* Before beginning we need to know a tiny bit of language of matrices. The individual numbers in a matrix are called the **entries** of the matrix. The **rows** and **columns** of a matrix should be obvious - the rows are the *horizontal* rows of entries, and the columns are the *vertical* columns of entries. The numbers of rows and columns of a matrix, *in that order*, are the **dimensions** of the matrix. For example, a matrix with 3 rows and four columns has dimensions 3×4, which we read as "three by four."

I will use the matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

to demonstrate the algebra of matrices. (*Note that we use CAPITAL letters to name matrices.*) Matrices like these, with the same number of rows as columns, are called **square matrices**. *Note that we use CAPITAL letters to name matrices.* I am using letters rather than numbers for the entries of the matrices because it will make it easier to follow what is happening. First we show how to add and subtract two matrices:

$$A + B = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \qquad A - B = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

In other words, we add and subtract matrices "entry by entry". We should make note of a couple things:

- To add or subtract two matrices, they must have the same dimensions.
- Whenever we are using an operation to put two mathematical objects together to form one, we should always ask whether the order of the two objects matters. If it doesn't, we say the operation is **commutative**. As with numbers, addition of matrices is commutative and subtraction is not.

We can also multiply a matrix by a number - we simply multiply every entry of the matrix by that number. For example,

$$5A = 5\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5a & 5b \\ 5c & 5d \end{bmatrix}$$

◇ **Example 7.3(a):** For $A = \begin{bmatrix} -5 & 2 \\ 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 7 \\ -4 & 0 \end{bmatrix}$, find $-A + 2B$.

$$-A + 2B = -\begin{bmatrix} -5 & 2 \\ 1 & 8 \end{bmatrix} + 2\begin{bmatrix} 3 & 7 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -1 & -8 \end{bmatrix} + \begin{bmatrix} 6 & 14 \\ -8 & 0 \end{bmatrix} = \begin{bmatrix} 11 & 12 \\ -9 & -8 \end{bmatrix}$$

Adding and subtracting matrices is neither interesting nor very useful in applications. Much more important (and more complicated to do) is multiplying two matrices. I will try to describe how to do it here, but it is best learned "in person". Consider again the matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

We will find that *matrices do not have to be the same dimensions to multiply them. There ARE certain other restrictions on their dimensions in order to be able to multiply them, however.* The only time we can multiply two matrices with the same dimensions is when they are both square as well, and the resulting matrix is also square with those dimensions. When we multiply A and B we will get a 2 by 2 matrix as a result.

Since our result is going to be a 2 by 2 matrix, it will have four entries. The question is then "How do we get each entry of the resulting matrix?" To get the first entry we "multiply" the first *row* of the first matrix by the first *column* of the second matrix. Actually we multiply the entries (two multiplications) and add the results:

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & * \\ * & * \end{bmatrix}.$$

Here $*$ simply means an entry that is not being shown, to avoid cluttering things up!

To get the second entry (second entry of the first row) we multiply the first row of the first matrix (again) by the *second* column of the second matrix:

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ * & * \end{bmatrix}.$$

To get the second row of the resulting matrix we multiply the second row of the first matrix by the first and second columns of the second matrix, respectively. The final result is

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}.$$

Let's look at a specific example.

◇ **Example 7.3(b):** Consider $A = \begin{bmatrix} -5 & 2 \\ 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 7 \\ -4 & 0 \end{bmatrix}$. Find AB and BA.

$$AB = \begin{bmatrix} -5 & 2 \\ 1 & 8 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} (-5)(3) + (2)(-4) & (-5)(7) + (2)(0) \\ (1)(3) + (8)(-4) & (1)(7) + (8)(0) \end{bmatrix} = \begin{bmatrix} -23 & -35 \\ -29 & 7 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 3 & 7 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} (3)(-5)+(7)(1) & (3)(2)+(7)(8) \\ (-4)(-5)+(0)(1) & (-4)(2)+(0)(8) \end{bmatrix} = \begin{bmatrix} -8 & 62 \\ 20 & -8 \end{bmatrix}$$

There are two very important comments to be made at this point:

- As can bee seen in this example, **matrix multiplication is not generally commutative!** That is, in most cases $AB \neq BA$. In fact it is sometimes possible to compute only one of AB or BA, but not the other.
- Although it is not true in general that $AB = BA$, it *can* be true sometimes. You will see some examples soon.

A matrix with the same number of rows as columns is called a **square matrix**. All of our examples so far have been square matrices. If we have a square matrix A, it can be multiplied by itself, so for a square matrix A it makes sense to talk about A^2. *This is only meaningful for square matrices!*

◇ **Example 7.3(c):** If $A = \begin{bmatrix} -5 & 2 \\ 1 & 8 \end{bmatrix}$, find A^2.

$$A^2 = \begin{bmatrix} -5 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} (-5)(-5)+(2)(1) & (-5)(2)+(2)(8) \\ (1)(-5)+(8)(1) & (1)(2)+(8)(8) \end{bmatrix} = \begin{bmatrix} 27 & 6 \\ 3 & 66 \end{bmatrix}$$

The next example looks at when matrices can be multiplied and when they cannot.

◇ **Example 7.3(d):** For the following matrices, give an example of each of the following:

- Two non-square matrices that can be multiplied.
- A square matrix and a non-square matrix that can be multiplied.
- Two matrices that cannot be multiplied.
- A matrix that can be squared.
- A matrix that cannot be squared.

$$A = \begin{bmatrix} 0 & 5 \\ -3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -7 & 3 \\ -2 & 0 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} -5 \\ 4 \\ -7 \end{bmatrix}$$

Solution: B and C are both non-square, and BC exists. Note also that CB does not exist. This takes care of the first and third examples. A is square, B is non-square, and AB exists. Finally, A can be squared, and B and C cannot.

◇ **Example 7.3(e):** For the matrices A and B from the previous example, find AB.

$$\begin{aligned} AB &= \begin{bmatrix} 0 & 5 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 4 & -7 & 3 \\ -2 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} (0)(4)+(5)(-2) & (0)(-7)+(5)(0) & (0)(3)+(5)(5) \\ (-3)(4)+(1)(-2) & (-3)(-7)+(1)(0) & (-3)(3)+(1)(5) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{aligned}$$

$$IB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (1)(a)+(0)(c) & (1)(b)+(0)(d) \\ (0)(a)+(1)(c) & (0)(b)+(1)(d) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If you understand how to multiply two matrices, you should be able to determine whether two matrices can be multiplied without relying on any rules. You should also be able to determine the dimensions of the resulting matrix. However, to summarize the facts:

> If matrix A has dimensions $m \times n$ and matrix B has dimensions $p \times q$, then AB exists only if $n = p$. That is, *the number of columns in the first matrix must equal the number of rows in the second.* In that case, the matrix AB has dimensions $m \times q$.

◇ **Example 7.3(f):** Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Find BI and IB.

$$BI = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (a)(1)+(b)(0) & (a)(0)+(b)(1) \\ (c)(1)+(d)(0) & (c)(0)+(d)(1) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$IB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (1)(a)+(0)(c) & (1)(b)+(0)(d) \\ (0)(a)+(1)(c) & (0)(b)+(1)(d) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The matrix I from this last example is called the 2×2 **identity matrix**. In math, an identity is something that can be combined, via some operation, with all other things without changing them. Technically speaking, we must give the operation as well as the identity:

- For numbers, zero is the *additive* identity because $a+0 = 0+a = a$ for every number a.
- For numbers, one is the *multiplicative* identity because $a \cdot 1 = 1 \cdot a = a$ for every number a.
- For matrices, the matrix O of all zeros (called the zero matrix) is the additive identity matrix because it can be added to any other matrix (of the same dimensions) without changing it.

It turns out that the additive identity for matrices has little or no use, so when we are working with matrices and talk about the identity matrix, we mean multiplicative identity. There are many identity matrices of various sizes, but each satisfies the following:

- They are square matrices. (There is then *one* identity matrix of each size of square matrix.)
- Their entries consist of ones on the diagonal (from upper left to lower right) and zeroes elsewhere.
- For any square matrix A of the same size as I, $AI = IA = A$.

Even though the last statement is about square matrices A, the products AI and IA always equal A *whenever they can be carried out*. However, if A is not square each of these products will use a different sized identity matrix. It will not be immediately apparent to you why we care about the identity matrices, but we do!

Section 7.3 Exercises To Solutions

1. Consider the matrices shown below. There are *THIRTEEN* multiplications possible, including squaring. Find and compute each possible multiplication *by hand*. (**Hint:** Be systematic - try A times each matrix (including itself, since you are looking for matrices that can be squared also), *in that order*. Then try B times every other matrix (including A, since AB is not necessarily equal to BA), in order, and so on.)

$$A = \begin{bmatrix} 0 & 5 \\ -3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -7 & 3 \\ -2 & 0 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} -5 \\ 4 \\ -7 \end{bmatrix}$$

$$D = \begin{bmatrix} 6 & 0 & 3 \\ -5 & 4 & 2 \\ 1 & 1 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 5 & -1 & 2 \end{bmatrix} \qquad F = \begin{bmatrix} 2 & -1 \\ 6 & 9 \end{bmatrix}$$

2. Fill in the blanks:

$$\begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & -7 \\ -5 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 8 & -2 \\ -1 & -3 & 5 \\ 6 & 0 & 4 \end{bmatrix} = \begin{bmatrix} \_\_\_ & * & * \\ * & * & \_\_\_ \\ * & * & * \end{bmatrix}$$

3. Let $A = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Find AI and IA.

4. Let $A = \begin{bmatrix} 3 & 0 & 2 \\ -1 & 4 & 5 \\ 1 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Find AB and AX.

5. Let $A = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$. Find$AC$ and CA.

7.4 Solving Systems of Equations With Inverse Matrices

Performance Criteria:

7. (e) Determine whether two matrices are inverses without computing the inverse of either; use the formula to find the inverse of a 2×2 matrix.
   (f) Write a system of equations as a matrix equation.
   (g) Solve a system of equations using the inverse matrix method, with or without a calculator.

Inverse Matrices

Let's begin with an example.

◇ **Example 7.4(a):** Find AC and CA for $A = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$.

$$AC = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} (5)(3)+(7)(-2) & (5)(-7)+(7)(5) \\ (2)(3)+(3)(-2) & (2)(-7)+(3)(5) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$CA = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} (3)(5)+(-7)(2) & (3)(7)+(-7)(3) \\ (-2)(5)+(5)(2) & (-2)(7)+(5)(3) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

What just occurred is special; it is very much like the fact that $\frac{1}{3}(3) = 1$, the multiplicative identity. The fact that $\frac{1}{3}(3) = 1$ is because $\frac{1}{3}$ and 3 are **multiplicative inverses**. In the above example the matrices A and C are multiplicative inverses as well, since their product is the multiplicative identity.

Definition of Inverse Matrices

Suppose that for matrices A and B we have $AB = BA = I$. Then we say that A and B are **inverse matrices**.

Notationally we write $B = A^{-1}$ or $A = B^{-1}$. Note that in order for us to be able to do both multiplications AB and BA, both matrices must be square and of the same dimensions. It also turns out that that to test two square matrices to see if they are inverses we only need to multiply them in one order:

Test for Inverse Matrices

To test two *square* matrices A and B to see if they are inverses, compute AB. If it is the identity, then the matrices are inverses.

Here are a few notes about inverse matrices:

- Not every square matrix has an inverse, but many do. If a matrix does have an inverse, it is said to be **invertible**.
- The inverse of a matrix is unique, meaning there is only one.
- Matrix multiplication *IS* commutative for inverse matrices.

Two questions that should be occurring to you now are

1) How do we know whether a particular matrix has an inverse?
2) If a matrix does have an inverse, how do we find it?

There are a number of ways to answer the first question; here is one:

> **Test for Invertibility of a Matrix**
>
> A square matrix A is invertible if *rref* $(A) = I$.

Here is the answer to the second question in the case of a 2×2 matrix:

> **Inverse of a 2×2 Matrix**
>
> The inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

For square matrices larger than 2×2 there is a fairly simple method for finding the inverse that we will not go into here.

◇ **Example 7.4(b):** Find the inverse of $A = \begin{bmatrix} -2 & 7 \\ 1 & -5 \end{bmatrix}$.

$$A^{-1} = \frac{1}{(-2)(-5) - (1)(7)} \begin{bmatrix} -5 & -7 \\ -1 & -2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -5 & -7 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} & -\frac{7}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

Matrices and Systems of Equations

Once again we begin with an example.

$\diamond$ **Example 7.4(c):** Let $A = \begin{bmatrix} -1 & 3 & 1 \\ 2 & 5 & 0 \\ 3 & 1 & -2 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Find AX

$$AX = \begin{bmatrix} -1 & 3 & 1 \\ 2 & 5 & 0 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x + 3y + z \\ 2x + 5y \\ 3x + y - 2z \end{bmatrix}$$

Because of what you have just seen, a system of equations like

$$\begin{aligned} -1x + 3y \ + \ z &= 2 \\ 2x + 5y \qquad &= 1 \\ 3x + \ y - 2z &= -4 \end{aligned}$$

can be written as

$$\begin{bmatrix} -1 & 3 & 1 \\ 2 & 5 & 0 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$$

or $AX = B$, where

$$A = \begin{bmatrix} -1 & 3 & 1 \\ 2 & 5 & 0 \\ 3 & 1 & -2 \end{bmatrix}, \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \text{and} \qquad B = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$$

The following summarizes this for a system of three linear equations in three unknowns, but of course the same thing can be done for larger systems as well.

Matrix Equation Form of a System

A system of three linear equations in three unknowns can be written as $AX = B$ where A is the 3×3 **coefficient matrix** of the system, X is the 3×1 matrix consisting of the three unknowns and B is the 3×1 matrix consisting of the right-hand sides of the equations, as shown below.

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned} \iff \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

A system of two equations in two unknowns can be written the same way, except that A will then be 2×2 and X and B will be 2×1.

This form of a system of equations can be used, as you will soon see, in another method (besides row-reduction) for solving a system of equations.

◇ **Example 7.4(d):** Give the matrix form of the system

$$\begin{aligned} x + 3y - 2z &= -4 \\ 3x + 7y + z &= 4 \\ -2x + y + 7z &= 7 \end{aligned}$$

Solution: The matrix form of the system is $\begin{bmatrix} 1 & 3 & -2 \\ 3 & 7 & 1 \\ -2 & 1 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix}$

Suppose now that we have a system of equations and put it in the matrix form $AX = B$. We then solve it in much the same way that we would solve $3x = 5$, but with one small difference. Where we would ordinarily divide both sides of $3x = 5$ by 3, we can accomplish the same thing by multiplying by $\frac{1}{3}$, *the multiplicative inverse of* 3. In the matrix version we multiply by A^{-1}, the multiplicative inverse of A. Let's do the two side-by-side:

$$\begin{aligned} 3x &= 5 \\ \tfrac{1}{3} \cdot 3x &= \tfrac{1}{3} \cdot 5 \\ 1x &= \tfrac{5}{3} \\ x &= \tfrac{5}{3} \end{aligned} \qquad\qquad \begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

◇ **Example 7.4(e):** Use the above method to solve the system $\begin{aligned} 5x + 4y &= 25 \\ -2x - 2y &= -12 \end{aligned}$.

Solution: The matrix form of the system is $\begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 25 \\ -12 \end{bmatrix}$, so $A = \begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix}$.

We can use the formula for the inverse of a 2×2 matrix to find

$$A^{-1} = \frac{1}{(5)(-2) - (-2)(4)}\begin{bmatrix} -2 & -4 \\ 2 & 5 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -2 & -4 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -\frac{5}{2} \end{bmatrix}$$

We can now solve the matrix equation by multiplying both sides by A^{-1} *on the left*.

$$\begin{aligned} \begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 25 \\ -12 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 \\ -1 & -\frac{5}{2} \end{bmatrix}\begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ -1 & -\frac{5}{2} \end{bmatrix}\begin{bmatrix} 25 \\ -12 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 \\ 5 \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 \\ 5 \end{bmatrix} \end{aligned}$$

The solution to the system is $x = 1$, $y = 5$ or $(1, 5)$.

Section 7.4 Exercises

To Solutions

1. (a) Determine whether $A = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & -4 \\ -3 & 2 \end{bmatrix}$ are inverses without finding the inverse of either.

   (b) Determine whether $A = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \\ -4 & 0 & 2 \end{bmatrix}$ and $B = \frac{1}{14}\begin{bmatrix} 2 & 0 & -2 \\ 0 & 14 & 0 \\ 4 & 0 & 3 \end{bmatrix}$ are inverses without finding the inverse of either.

2. As you just saw, a system of equations can be written in the form $AX = B$ using matrices. For each of the following systems of equations, determine the matrices A, X and B.

   (a) $\begin{aligned} 3x + 2y &= -1 \\ 4x + 5y &= 1 \end{aligned}$ (b) $\begin{aligned} -2x + 2y + 3z &= -1 \\ x - y &= 0 \\ y + 4z &= 4 \end{aligned}$ (c) $\begin{aligned} x + 2y + 3z &= -3 \\ -2x + y &= -2 \\ 3x - y + z &= 1 \end{aligned}$

3. Solve each of the following systems using the method of Example 7.6(e), showing all steps in the manner that was done there. Use the formula for the inverse of a 2×2 matrix to find A^{-1} in each case, but from step two to step three be sure to multiply $A^{-1}A$ to make sure that you calculated the inverse correctly.

   (a) $\begin{aligned} 5x + 7y &= -4 \\ 2x + 3y &= 1 \end{aligned}$ (b) $\begin{aligned} 3x + 2y &= -1 \\ 4x + 5y &= 1 \end{aligned}$ (c) $\begin{aligned} 2x + 5y &= 3 \\ 3x + 8y &= -2 \end{aligned}$

4. Use the information below to solve each of the following systems of equations using inverse matrices. Do these by hand; you can go straight to the solution $X = A^{-1}B$, but do the computation first. (**Hint:** Multiply the matrices first, then multiply by the number.)

   (a) $\begin{aligned} -1x + 3y + z &= 2 \\ 2x + 5y &= 1 \\ 3x + y - 2z &= -4 \end{aligned}$ (b) $\begin{aligned} -2x + 2y + 3z &= -1 \\ x - y &= 0 \\ y + 4z &= 4 \end{aligned}$ (c) $\begin{aligned} x + 2y + 3z &= -3 \\ -2x + y &= -2 \\ 3x - y + z &= 1 \end{aligned}$

$$\text{If } A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}, \text{ then } A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -5 & -3 \\ 2 & -8 & -6 \\ -1 & 7 & 5 \end{bmatrix}$$

$$\text{If } A = \begin{bmatrix} -1 & 3 & 1 \\ 2 & 5 & 0 \\ 3 & 1 & -2 \end{bmatrix}, \text{ then } A^{-1} = \frac{1}{9}\begin{bmatrix} -10 & 7 & -5 \\ 4 & -1 & 2 \\ -13 & 10 & -11 \end{bmatrix}$$

$$\text{If } A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}, \text{ then } A^{-1} = \frac{1}{3}\begin{bmatrix} -4 & -5 & 3 \\ -4 & -8 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

7.5 Determinants and Cramer's Rule

Performance Criteria:

7. (h) Calculate the determinant of a 2×2 matrix; use Cramer's rule to solve a system of two equations in two unknowns.

Let's consider the system of equations $\begin{array}{rcl} 5x+3y & = & 1 \\ 4x+2y & = & 2 \end{array}$. To solve for x we can multiply the first equation by 2 and the second equation by -3 to obtain

$$\begin{array}{rcl} 10x+6y & = & 2 \\ -12x-6y & = & -6 \end{array}.$$

We then add the two equations to obtain the equation $-2x=-4$, so $x=2$. Note that we could instead multiply the first equation by 2, the second by 3, and then *subtract* the two equations to get

$$\begin{array}{rcl} 5x+3y=1 & \Longrightarrow & 10x+6y=2 \\ 4x+2y=2 & \Longrightarrow & \underline{12x+6y=6} \\ & & -2x=-4, \end{array}$$

solving the resulting equation to get $x=2$. Using this process on a general system of two equations we get

$$\begin{array}{rcl} ax+by=e & \Rightarrow & adx+bdy=ed \\ cx+dy=f & \Rightarrow & \underline{bcx+bdy=bf} \\ & & adx-bcx=ed-bf \end{array} \quad \Rightarrow\ (ad-bc)x=ed-bf \ \Rightarrow\ x=\frac{ed-bf}{ad-bc} \quad (1)$$

We can also multiply the top equation by c and the bottom equation by a and then subtract the top equation from the bottom one to get

$$\begin{array}{rcl} ax+by=e & \Rightarrow & acx+bcy=ce \\ cx+dy=f & \Rightarrow & \underline{acx+ady=af} \\ & & ady-bcy=af-ce \end{array} \quad \Rightarrow\ (ad-bc)y=af-ce \ \Rightarrow\ y=\frac{af-ce}{ad-bc} \quad (2)$$

Note the matrix form of the system of equations:

$$\begin{array}{c} ax+by=E \\ cx+dy=f \end{array} \qquad \text{and} \qquad \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right]=\left[\begin{array}{c} e \\ f \end{array}\right].$$

If we look carefully at the expressions for x and y that were obtained in (1) and (2) above, we see that they both have the same denominator $ad-bc$, and that the numbers a, b, c and d are the entries in the coefficient matrix $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$. We will return to (1) and (2) after introducing a new concept called the determinant of a matrix.

Determinants of Matrices

There are situations in math where we attach a number to an object, with that number somehow giving us some information about the object. An example you are familiar with is that every non-vertical line has a number attached to it, its slope. That number describes the steepness of the line (and whether it slopes upward or downward as we go left to right). There is a method for finding the slope of any line, which essentially boils down to "rise over run."

Every *square* matrix has a number associated with it called its **determinant**. The method for computing the determinant of a 2×2 matrix is fairly straightforward:

Determinant of a 2×2 Matrix

The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A) = ad - bc$.

Suppose that we have the square matrix $A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$. It should be clear from the definition that the determinant of this matrix is $(3)(2) - (2)(-1) = 17$. There are two ways we generally indicate the determinant of a matrix: the first is to simply write $\det$ before the matrix, and the second is to replace the brackets $[\ ]$ around the matrix with vertical bars $|\ |$. So for our matrix we could write

$$\det \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} = 17 \qquad \text{or} \qquad \begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix} = 17.$$

Thus $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$ is the matrix itself, whereas $\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix}$ is the determinant of the matrix, a single number.

Finding determinants of 3×3 matrices by hand is a little complicated, and finding determinants of larger matrices is yet more complicated. They are easily found by graphing calculators or online tools, however.

Now what does the determinant tell us a bout matrix? Well, here is the most basic fact about determinants:

Determinants and Invertibility

A square matrix has an inverse if, and only if, its determinant is *NOT* zero.

One way of interpreting the above is that *if the determinant of a matrix is zero, then the matrix does not have an inverse*. Now recall the following computations from the previous section:

$$\begin{aligned} 3x &= 5 \\ \tfrac{1}{3} \cdot 3x &= \tfrac{1}{3} \cdot 5 \\ 1x &= \tfrac{5}{3} \\ x &= \tfrac{5}{3} \end{aligned} \qquad\qquad \begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

Note that is our original equation on the left had been $0x = 5$ we would not have been able to use the method shown because $\frac{1}{0}$ is undefined. Similarly, if A does not have an inverse, we can't solve the equation $AX = B$ in the way shown above and to the right. Therefore, the matrices with determinant zero (and there are infinitely many of them) are to all other matrices as the number zero is to all other numbers.

Cramer's Rule

We will now see what determinants have to do with results (1) and (2) from earlier in this section. Recall that we had the system of equations with the standard and matrix forms

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned} \qquad \text{and} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}.$$

The solution to the system was given by the two expressions

$$x = \frac{ed - bf}{ad - bc} \qquad \text{and} \qquad y = \frac{af - ce}{ad - bc}.$$

We can now see that the denominator of each of the expressions is the determinant of the coefficient matrix of the system, but what are the numerators? They also appear to be determinants of 2×2 matrices. but what matrices?

The numerator of the expression for x contains d and b just like the numerator, so the second column of the matrix we are seeking could be the same as that of the coefficient matrix. If we were to replace the first column of the coefficient matrix with the right hand side of the matrix form of the system we get

$$\begin{bmatrix} e & b \\ f & d \end{bmatrix},$$

and the determinant of this matrix is the numerator of the expression for x. Note that *the first column of the coefficient matrix is the coefficients of x.* Similarly, if we take the coefficient matrix and replace the *second* (y) column with the right hand side numbers we get

$$\begin{bmatrix} a & e \\ c & f \end{bmatrix},$$

and the determinant of this is $af - ce$, the numerator of the expression for y. Using the "bar notation" for the determinant, we now have

Cramer's Rule

The system of two equations in two unknowns with standard and matrix forms

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned} \qquad \text{and} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}.$$

has solution given by

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad \text{and} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

We reiterate: the denominators of both of these fractions are the determinant of the coefficient matrix. The numerator for finding x is the determinant of the matrix obtained when the coefficient matrix has its first column (the coefficients of x) replaced with the numbers to the right of the equal signs. Let's see an example:

◇ **Example 7.5(a):** Use Cramer's rule to solve the system of equations $\begin{array}{rcl} 5x+3y & = & 1 \\ 4x+2y & = & 2 \end{array}$.

Solution: Cramer's Rule gives us

$$x = \frac{\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 4 & 2 \end{vmatrix}} = \frac{1\cdot 2 - 2\cdot 3}{5\cdot 2 - 4\cdot 3} = \frac{-4}{-2} = 2 \qquad y = \frac{\begin{vmatrix} 5 & 1 \\ 4 & 1 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 4 & 2 \end{vmatrix}} = \frac{5\cdot 2 - 4\cdot 1}{5\cdot 2 - 4\cdot 3} = \frac{6}{-2} = -3$$

Section 7.5 Exercises

To Solutions

1. Find the determinant of each of the following 2×2 matrices.

   (a) $\begin{bmatrix} -2 & 5 \\ 4 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 8 & -3 \\ 4 & -5 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 7 \\ 0 & 4 \end{bmatrix}$

2. Use Cramer's Rule to solve each of the following systems of equations.

   (a) $\begin{array}{rcl} -2x+5y & = & 13 \\ 4x+7y & = & 25 \end{array}$ (b) $\begin{array}{rcl} 1x-3y & = & -17 \\ -2x+5y & = & 29 \end{array}$

   (c) $\begin{array}{rcl} 8x-3y & = & 32 \\ 4x-5y & = & 16 \end{array}$ (d) $\begin{array}{rcl} 1x+7y & = & 48 \\ 4y & = & 28 \end{array}$

3. Solve the following systems using Cramer's Rule.

   (a) $\begin{array}{rcl} x-3y & = & 6 \\ -2x+5y & = & -5 \end{array}$ (b) $\begin{array}{rcl} 2x-3y & = & -6 \\ 3x-y & = & 5 \end{array}$ (c) $\begin{array}{rcl} x+y & = & 3 \\ 2x+3y & = & -4 \end{array}$

   (d) $\begin{array}{rcl} 7x-6y & = & 13 \\ 6x-5y & = & 11 \end{array}$ (e) $\begin{array}{rcl} 5x+3y & = & 7 \\ 3x-5y & = & -23 \end{array}$ (f) $\begin{array}{rcl} 5x-3y & = & -11 \\ 7x+6y & = & -12 \end{array}$

4. Attempt to solve the following two systems of equations, using Cramer's Rule. What goes wrong?

   (a) $\begin{array}{rcl} 2x-5y & = & 3 \\ -4x+10y & = & 1 \end{array}$ (b) $\begin{array}{rcl} 2x-5y & = & 3 \\ -4x+10y & = & -6 \end{array}$

A Solutions to Exercises

A.7 Chapter 7 Solutions

Section 7.1 Solutions Back to 7.1 Exercises

1. (a) $-2y+7z=16$ (b) $7y+3z=-1$ (c) $z=2$ (d) $y=-1$ (e) $x=3$

2. (a) $(-2,1,4)$ (b) $(1,\frac{1}{2},\frac{9}{2})$ (c) $(4,3,2)$

Section 7.2 Solutions Back to 7.2 Exercises

1. (a) $\left[\begin{array}{rrrr} 1 & 5 & -7 & 3 \\ -5 & 3 & -1 & 0 \\ 4 & 0 & 8 & -1 \end{array}\right]$ $\begin{array}{c} 5R_1+R_2 \longrightarrow R_2 \\ \Longrightarrow \\ -4R_1+R_3 \longrightarrow R_3 \end{array}$ $\left[\begin{array}{rrrr} 1 & 5 & -7 & 3 \\ 0 & 28 & -36 & 15 \\ 0 & -20 & 36 & -13 \end{array}\right]$

   (b) $\left[\begin{array}{rrrr} 2 & -8 & -1 & 5 \\ 0 & -2 & 0 & 0 \\ 0 & 6 & -5 & 2 \end{array}\right]$ $\begin{array}{c} \Longrightarrow \\ \\ 3R_2+R_3 \longrightarrow R_3 \end{array}$ $\left[\begin{array}{rrrr} 2 & -8 & -1 & 5 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 4 & 5 \end{array}\right]$

   (c) $\left[\begin{array}{rrrr} 1 & -2 & 4 & 1 \\ 0 & 3 & 5 & -2 \\ 0 & 2 & -8 & 1 \end{array}\right]$ $\begin{array}{c} 2R_2+(-3)R_3 \longrightarrow R_3 \\ \text{OR} \\ (-2)R_2+3R_3 \longrightarrow R_3 \end{array}$ $\left[\begin{array}{cccc} 1 & -2 & 4 & 1 \\ 0 & 3 & 5 & -2 \\ 0 & 0 & 34 \text{ or } -34 & -7 \text{ or } 7 \end{array}\right]$

2. (a) $(-4,\frac{1}{2},-4)$ (b) $(33,-4,1)$ (c) $(7,0,-2)$

3. (a) $(2,3,-1)$ (b) $(-2,1,2)$ (c) $(-1,2,1)$

Section 7.3 Solutions Back to 7.3 Exercises

1. $A^2=\left[\begin{array}{rr} -15 & 5 \\ -3 & -14 \end{array}\right]$ $AB=\left[\begin{array}{rrr} -10 & 0 & 25 \\ -14 & 21 & -4 \end{array}\right]$ $AF=\left[\begin{array}{rr} 30 & 45 \\ 0 & 12 \end{array}\right]$

   $BC=\left[\begin{array}{r} -69 \\ -25 \end{array}\right]$ $BD=\left[\begin{array}{rrr} 62 & -25 & -2 \\ -7 & 5 & -6 \end{array}\right]$ $CE=\left[\begin{array}{rrr} -25 & 5 & -10 \\ 20 & -4 & 8 \\ -35 & 7 & -14 \end{array}\right]$

   $DC=\left[\begin{array}{r} -51 \\ 27 \\ -1 \end{array}\right]$ $D^2=\left[\begin{array}{rrr} 39 & 3 & 18 \\ -48 & 18 & -7 \\ 1 & 4 & 5 \end{array}\right]$ $EC=\left[\begin{array}{r} -43 \end{array}\right]$

   $ED=\left[\begin{array}{rrr} 37 & -2 & 13 \end{array}\right]$ $FA=\left[\begin{array}{rr} 3 & 9 \\ -27 & 39 \end{array}\right]$ $FB=\left[\begin{array}{rrr} 10 & -14 & 1 \\ 6 & -42 & 63 \end{array}\right]$

   $F^2=\left[\begin{array}{rr} -2 & 11 \\ 66 & 75 \end{array}\right]$

2. $23,\ -32$

3. $AI=IA=A$

4. $AB=\left[\begin{array}{r} -6 \\ 23 \\ -3 \end{array}\right]$ $AX=\left[\begin{array}{r} 3x+0y+2z \\ -1x+4y+5z \\ 1x+1y+0z \end{array}\right]$

5. $AC=CA=I$

Section 7.4 Solutions

Back to 7.4 Exercises

1. (b) Since $AB = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, A and B are not inverses.

   (c) Since $AB = \frac{1}{14}\begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, A and B are inverses.

2. (a) $A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$

3. (a) $x = -19,\ y = 13$ (b) $x = -1, y = 1$ (c) $x = 34,\ y = -13$

4. (a) $x = \frac{7}{9},\ y = -\frac{1}{9},\ z = \frac{28}{9}$ (b) $x = \frac{16}{3},\ y = \frac{16}{3},\ z = -\frac{1}{3}$ (c) $x = 2,\ y = 2,\ z = -3$

Section 7.5 Solutions

Back to 7.5 Exercises

1. (a) -34 (b) 0 (c) -28 (d) 4

2. (a) $(1, 3)$ (b) $(-2, 5)$ (c) $(4, 0)$ (d) $(-1, 7)$

For Exercise 3, check your answers by substituting them into *BOTH* of the original equations. In Exercise 4 the problem is that the determinant in the bottoms of the fractions are zero.