Performance Criteria:

1. Compute and interpret average rates of change and their limits.
   (a) Determine an average rate of change numerically.
   (b) Determine an average rate of change of a function from its graph. Draw a secant line whose slope represents the average rate of change between two points.
   (c) Determine the average rate of change of a function algebraically, from its equation.
   (d) Find and simplify a difference quotient.

Average Rate of Change

◇ **Example 1:** Suppose that you left Klamath Falls at 10:30 AM, driving north on on Highway 97. At 1:00 PM you got to Bend, 137 miles from Klamath Falls. How fast were you going on this trip?

Since the trip took two and a half hours, we need to divide the distance 137 miles by 2.5 hours, with the units included in the operation:

$$\text{speed} = \frac{137 \text{ miles}}{2.5 \text{ hours}} = 54.8 \text{ miles per hour}$$

Of course we all know that you weren't going 54.8 miles per hour the entire way from Klamath Falls to Bend; this speed is the *average* speed during your trip. When you were passing through all the small towns like Chemult and La Pine you likely slowed down to 30 or 35 mph, and between towns you may have exceeded the legal speed limit. 54.8 miles per hour is the speed you would have to go to make the trip in two and a half hours driving at a constant speed for the entire distance. The speed that you see any moment that you look at the speedometer of your car is called the *instantaneous* speed.

Speed is a quantity that we call a **rate of change**. It tells us the change in distance (in the above case, measured in miles) for a given change in time (measured above in hours). That is, for every change in time of one hour, an additional 54.8 miles of distance will be gained. In general, we consider rates of change when one quantity depends on another; we call the first quantity the **dependent variable** and the second quantity the **independent variable**. In the above example, the distance traveled depends on the time, so the distance is the dependent variable and the time is the independent variable. We find the average rate of change as follows:

$$\text{average rate of change} = \frac{\text{change in dependent variable}}{\text{change in independent variable}}$$

The changes in the variables are found by subtracting. The order of subtraction does not matter for one of the variables, but *the values of the other variable must be subtracted in the corresponding order*. This should remind you of the process for finding a slope. In fact, we will see soon that every average rate of change can be interpreted as the slope of a line.

The following example will illustrate what we have just talked about.

◇ **Example 2:** Again you were driving, this time from Klamath Falls to Medford. At the top of the pass on Highway 140, which is at about 5000 feet elevation, the outside thermometer of your car registered a temperature of 28°F. In Medford, at 1400 feet of elevation, the temperature was 47°. What was the average rate of change of temperature with respect to elevation?

The words "temperature with respect to elevation" implies that temperature is the dependent variable and elevation is the independent variable. From the top of the pass to Medford we have

$$\text{average rate of change} = \frac{28°\text{F} - 47°\text{F}}{5000 \text{ feet} - 1400 \text{ feet}} = \frac{-19°\text{F}}{3600 \text{ feet}} = -0.0053°\text{F per foot}$$

Note that if instead we were to consider going from Medford to the top of the pass we would have

$$\text{average rate of change} = \frac{47°\text{F} - 28°\text{F}}{1400 \text{ feet} - 5000 \text{ feet}} = \frac{19°\text{F}}{-3600 \text{ feet}} = -0.0053°\text{F per foot}$$

How do we interpret the negative sign with our answer? Usually, when interpreting a rate of change, its value is *the change in the dependent variable for each INCREASE in one unit of the independent variable.* So the above result tells us that the temperature (on that particular day, at that time and in that place) *decreases* by 0.0053 degrees Fahrenheit for each foot of elevation *gained.*

Average Rate of Change of a Function

You probably recognized that what we have been calling independent and dependent variables are the "inputs" and "outputs" of a function. This leads us to the following:

> **Average Rate of Change of a Function**
>
> For a function $f(x)$ and two values a and b of x with $a < b$, the **average rate of change of** f **with respect to** x over the interval $[a, b]$ is
>
> $$\left.\frac{\Delta f(x)}{\Delta x}\right|_{[a,b]} = \frac{f(b) - f(a)}{b - a} = \frac{f(a) - f(b)}{a - b}$$

There is no standard notation for average rate of change, but it is standard to use the capital Greek letter delta (Δ) for "change." The notation $\left.\frac{\Delta f(x)}{\Delta x}\right|_{[a,b]}$ could be understood by any mathematician as change in f over change in x, over the interval $[a, b]$. Because it is over an interval it must necessarily be an average, and the fact that it is one change divided by another indicates that it is a *rate* of change.

◇ **Example 3:** For the function $h(t) = -16t^2 + 48t$, find the average rate of change of h with respect to t from $t = 0$ to $t = 2.5$.

$$\left.\frac{\Delta h}{\Delta t}\right|_{[0,2.5]} = \frac{h(2.5) - h(0)}{2.5 - 0} = \frac{20 - 0}{2.5 - 0} = 8$$

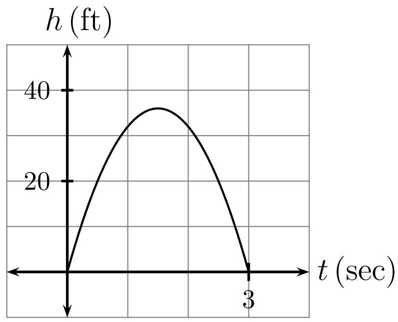
You might recognize $h(t) = -16t^2 + 48t$ as the height h of a projectile at any time t, with h in feet and t in seconds. The units for our answer are then feet per second, indicating that over the first 2.5 seconds of its flight, the height of the projectile increases at an average rate of 8 feet per second.

Secant Lines

The graph below and to the left is for the function $h(t) = -16t^2 + 48t$ of Example 3.7(c). To the right is the same graph, with a line drawn through the two points $(0,0)$ and $(2.5, 20)$. Recall that the average rate of change in height with respect to time from $t = 0$ to $t = 2.5$ was determined by

$$\left.\frac{\Delta h}{\Delta t}\right|_{[0,2.5]} = \frac{h(2.5) - h(0)}{2.5 - 0} = \frac{20 - 0}{2.5 - 0} = 8 \text{ feet per second}$$

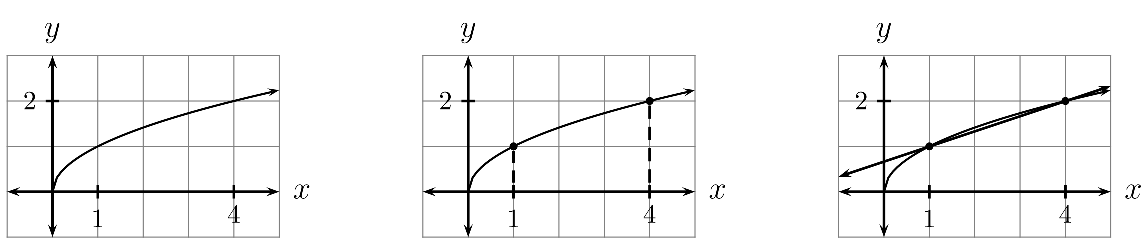
With a little thought, one can see that the average rate of change is simply the slope of the line in the picture to the right below.



A line drawn through two points on the graph of a function is called a **secant line**, and the slope of a secant line represents the rate of change of the function between the values of the independent variable where the line intersects the graph of the function.

◇ **Example 4:** Sketch the secant line through the points on the graph of the function $y = \sqrt{x}$ where $x = 1$ and $x = 4$. Then compute the average rate of change of the function between those two x values.

The graph below and to the left is that of the function. In the second graph we can see how we find the two points on the graph that correspond to $x = 1$ and $x = 4$. The last graph shows the secant line drawn in through those two points.



The average rate of change is $\frac{2-1}{4-1} = \frac{1}{3}$, the slope of the secant line.

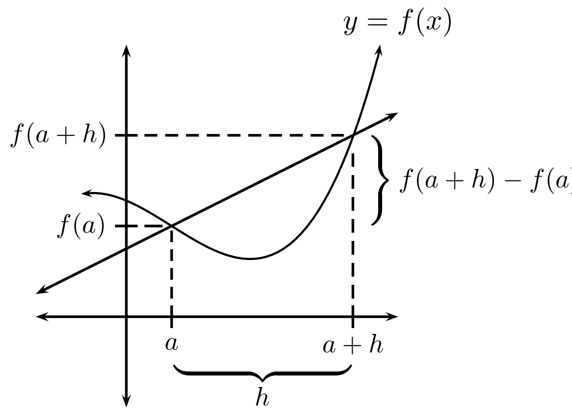
Difference Quotients

The following quantity is the basis of a large part of the subject of calculus:

> **Difference Quotient for a Function**
>
> For a function $f(x)$ and two values a and h of x, the **difference quotient** for f is
> $$\frac{f(a+h)-f(a)}{h}$$

Let's rewrite the above expression to get a better understanding of what it means. With a little thought you should agree that

$$\frac{f(a+h)-f(a)}{h} = \frac{f(a+h)-f(a)}{(a+h)-a}$$



This looks a lot like a slope, and we can see from the picture to the right that it is. The quantity $f(a+h)-f(a)$ is a "rise" and $(a+h)-a=h$ is the corresponding "run." The difference quotient is then the slope of the tangent line through the two points $(a, f(a))$ and $(a+h, f(a+h))$.

◇ **Example 5:** For the function $f(x) = 3x-5$, find and simplify the difference quotient $\dfrac{f(2+h)-f(2)}{h}$.

$$\frac{f(2+h)-f(2)}{h} = \frac{[3(2+h)-5]-[3(2)-5]}{h} = \frac{[6+3h-5]-[6-5]}{h} = \frac{3h}{h} = 3$$

Note that in the above example, the h in the denominator eventually canceled with one in the numerator. *This will always happen if you carry out all computations correctly!*

◇ **Example 6:** Find and simplify the difference quotient $\dfrac{g(x+h)-g(x)}{h}$ for the function $g(x) = 3x-x^2$.

With more complicated difference quotients like this one, it is sometimes best to compute $g(x+h)$, then $g(x+h)-g(x)$, before computing the full difference quotient:

$$g(x+h) = 3(x+h)-(x+h)^2 = 3x+3h-(x^2+2xh+h^2) = 3x+3h-x^2-2xh-h^2$$

$$g(x+h)-g(x) = (3x+3h-x^2-2xh-h^2)-(3x-x^2) = 3h-2xh-h^2$$

$$\frac{g(x+h)-g(x)}{h} = \frac{3h-2xh-h^2}{h} = \frac{h(3-2x-h)}{h} = 3-2x-h$$

Practice Exercises

1. Consider a rectangle that started with a width of 5 inches and a length of 8 inches. At time zero the width started growing at a constant rate of 2 inches per minute, and the length began growing by 3 inches per minute.

   (a) Determine the average rate of increase in area with respect to time from time 5 minutes to time 15 minutes. Give units with your answer!

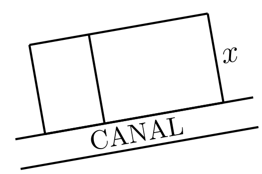
   (b) Find an equation for the area A (in square inches) as a function of time t (in minutes). Use this equation to compute the answer to (a) again - you should of course get the same result!

2. The height h (feet) of a projectile at time t (seconds) is given by $h = -16t^2 + 144t$.

   (a) Find the average rate of change in height, with respect to time, from 4 seconds to 7 seconds. Give your answer as a sentence that includes these two times, your answer, and whichever of the words *increasing* or *decreasing* that is appropriate.

   (b) Find the average rate of change in height, with respect to time, from time 2 seconds to time 7 seconds. Explain your result *in terms of the physical situation.*

3. A farmer is going to create a rectangular field with two compartments against a straight canal, as shown to the right, using 1000 feet of fence. No fence is needed along the side formed by the canal.



   (a) Find the total area of the field when x is 200 feet and again when x is 300 feet.

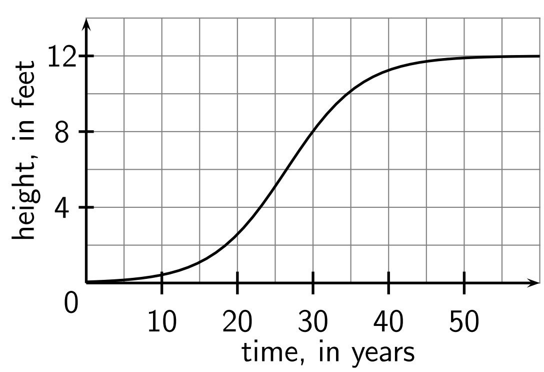
   (b) Find the average rate of change in area, with respect to x, between when x is 200 feet and when it is 300 feet. Give units with your answer.

   (c) Determine an equation for the area A of the field as a function of x. Use it to answer part (a) again to check your equation.

4. The graph to the right shows the heights of a certain kind of tree as it grows. Use it to find the average rate of change of height with respect to time from

   (a) 10 years to 50 years

   (b) 25 years to 30 years



5. In a previous exercise you found the revenue equation $R = 20000p - 100p^2$ for sales of Widgets, where p is the price of a widget and R is the revenue obtained at that price. (Both are in dollars.)

   (a) Find the average change in revenue, with respect to price, from a price of \$50 to \$110. Include units with your answer.

   (b) Sketch a graph of the revenue function and draw in the secant line whose slope represents your answer to (a). Label the relevant values on the horizontal and vertical axes.

6. (a) Sketch the graph of the function $y = 2^x$. Then draw in the secant line whose slope represents the average rate of change of the function from $x = -2$ to $x = 1$.

   (b) Determine the average rate of change of the function from $x = -2$ to $x = 1$. Give your answer in (reduced) fraction form.

7. Find the average rate of change of $f(x) = x^3 - 5x + 1$ from $x = 1$ to $x = 4$.

8. Find the average rate of change of $g(x) = \dfrac{3}{x-2}$ from $x = 4$ to $x = 7$.

9. Consider the function $h(x) = \frac{2}{3}x - 1$.

   (a) Find the average rate of change from $x = -6$ to $x = 0$.

   (b) Find the average rate of change from $x = 1$ to $x = 8$.

   (c) You should notice two things about your answers to (a) and (b). What are they?

10. For the function $h(x) = \frac{2}{3}x - 1$ from the previous exercise, find and simplify the difference quotient $\dfrac{h(x+s) - h(s)}{s}$.

11. For the function $f(x) = 3x^2$, find and simplify the difference quotient $\dfrac{f(5+h) - f(5)}{h}$.

12. (a) Compute and simplify $(x+h)^3$

   (b) For the function $g(x) = x^3 - 5x + 1$, find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$.

   (c) In Exercise 7 you found the average rate of change of $f(x) = x^3 - 5x + 1$ from $x = 1$ to $x = 4$. That is equivalent to computing the difference quotient with $x = 1$ and $x + h = 4$. Determine the value of h, then substitute it and $x = 1$ into your answer to part (b) of this exercise. You should get the same thing as you did for Exercise 7!

13. For this exercise you will be working with the function $f(x) = \dfrac{1}{x}$.

(a) Find and simplify $f(x+h) - f(x)$ by carefully obtaining a common denominator and combining the two fractions.

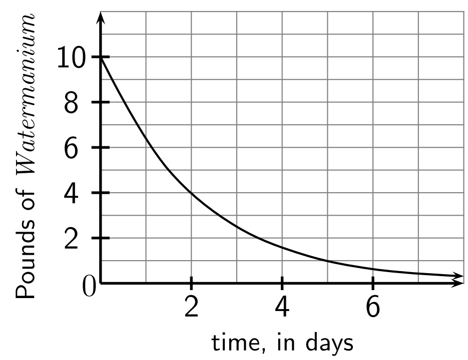
(b) Using the fact that dividing by h is the same as multiplying by $\frac{1}{h}$, find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$.

14. Let $f(t) = 150(2)^{-0.3t}$. Find the average rate of change of f with respect to t from $t = 2$ to $t = 8$.

15. The graph below and to the right shows the decay of 10 pounds of the newly discovered radioactive element *Watermanium*. Use it to answer the following.

(a) Find the average rate of change in the amount from zero days to five days.

(b) Find the average rate of change in the amount from two days to five days.



Compound Interest Formula and Continuous Compound Interest Formula

When a principal amount of P is invested for t years at an annual interest rate (in decimal form) of r, compounded n times per year, the total amount A is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

When P dollars are invested for t years at an annual interest rate of r, compounded continuously, the amount A of money that results is given by

$$A = Pe^{rt}$$

16. You invest $\$10{,}000$ at an annual interest rate of 6.25%, compounded semi-annually (twice a year). What is the average rate of increase in your investment over the 30 years?

17. $\$3000$ is invested for 9 years, compounded continuously, at $5\frac{1}{4}\%$ annual interest. What is the average rate of increase of the investment over those nine years?

18. In 1985 the growth rate of the population of India was 2.2% per year, and the population was 762 million. Assuming continuous exponential growth, the number of people t years after 1985 is given by the equation $N(t) = 762e^{0.022t}$. Based on this model, what is the average rate of change in the population, with respect to time, from 1985 to 2010?

19. The height h (in feet) of a certain kind of tree that is t years old is given by

$$h = \frac{120}{1 + 200e^{-0.2t}} \, .$$

   (a) Use this to find the height of a tree that is 30 years old.

   (b) This type of equation is called a *logistic equation*. It models the growth of things that start growing slowly, then experience a period of rapid growth, then level off. Sketch the graph of the function for the first 60 years of growth of a tree. Use your graphing calculator, or find values of h for $t = 0, 10, 20, ..., 60$.

   (c) Around what age does the tree seem to be growing most rapidly?

20. For the variety of tree from the previous exercise,

   (a) What is the average rate of growth for the first 30 years?

   (b) What is the average rate of growth from 20 years to 30 years?

Answers to Exercises

1. (a) 151 square inches per minute (b) $A = 6t^2 + 31t + 40$

2. (a) From 4 seconds to 7 seconds the height decreased at an average rate of 26.7 feet per second.

   (b) The average rate of change is 0 feet per second. This is because at 2 seconds the projectile is at a height of 224 feet, on the way up, and at 7 seconds it is at 224 feet again, on the way down.

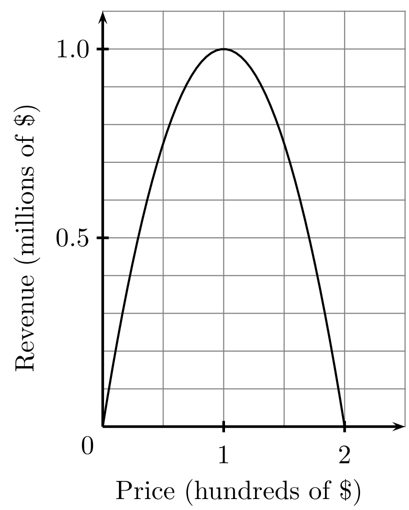
3. (a) When $x = 200$ feet, the area is 80,000 square feet, and when $x = 300$ feet, the area is 30,000 square feet.

   (b) -500 square feet per foot (c) $A = x(1000 - 3x) = 1000x - 3x^2$

4. (a) $\frac{11.5}{40} = 0.29$ feet per year (b) $\frac{3}{5} = 0.6$ feet per year

5. (a) \$4000 of revenue per dollar of price

   (b) See graph to the right.



Exercise 5(b)

6. (a) See graph below and to the right.

   (b) $\frac{7}{12}$

7. 16 8. $-\frac{9}{10} = -0.9$

9. (a) $\frac{2}{3}$ (b) $\frac{2}{3}$

   (c) The answers are the same, and they are both equal to the slope of the line.

10. $\frac{2}{3}$ 11. $30 + 3h$

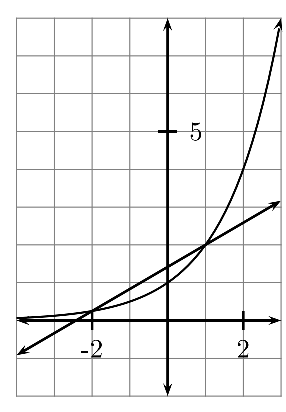
12. (a) $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

    (b) $f(x+h) = x^3 + 3x^2h + 3xh^2 + h^3 - 5x - 5h + 1$

    $f(x+h) - f(x) = 3x^2h + 3xh^2 + h^3 - 5h$

    $$\frac{f(x+h) - f(x)}{h} = 3x^2 + 3xh + h^2 - 5$$

    (c) $3(1)^2 + 3(1)(3) + 3^2 - 5 = 16$



Exercise 6(a)

13. (a) $f(x+h) - f(x) = \dfrac{-h}{x(x+h)}$ (b) $\dfrac{f(x+h) - f(x)}{h} = \dfrac{-1}{x(x+h)}$

14. -11.76

15. (a) The average rate of change from 0 days to 5 days is -1.8 pounds per day.

(b) The average rate of change from 2 days to 5 days is -1 pounds per day.

16. \$1778.78 per year

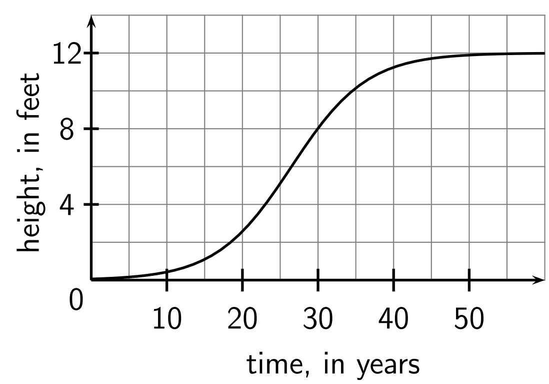
17. \$201.33 per year

18. 22.36 million people per year

19. (a) 80.2 feet

(b) See graph to the right.

(c) The tree seems to be growing most rapidly when it is about 25 to 30 years old.



20. (a) 2.65 feet per year (b) 25.7 feet per year