$$\boxed{\sinh x = \frac{e^x - e^{-x}}{2}}$$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)\right]$$

$$= \frac{1}{2}\left(2x + 2\left(\frac{x^3}{3!}\right) + 2\left(\frac{x^5}{5!}\right) + \cdots\right)$$

$$\boxed{\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}$$

Like series for $\sin x$ but not alternating.

$$(\sinh x)' = \left[\frac{1}{2}(e^x - e^{-x})\right]'$$
$$= \frac{1}{2}(e^x + (+e^{-x})) = \cosh x$$

$$(\sinh x)' = \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$
$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \cosh x$$

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2$$

$$(e^x+e^{-x})(e^x+e^{-x})$$

$$e^{2x}+e^0+e^0+e^{-2x}$$

$$e^{2x}+e^{-2x}+2$$

$$= \frac{e^{2x}+e^{-2x}+2}{4} - \frac{e^{2x}+e^{-2}-2}{4}$$

$$= \frac{\cancel{e^{2x}}+\cancel{e^{-2x}}+2-\cancel{e^{2x}}-\cancel{e^{-2x}}+2}{4}$$

$$=1$$

$a_{n+1} = 2a_n$, $a_0 = 1$ ← Recursive

$a_0 = 1$, $a_1 = 2$, $a_2 = 4$, $a_3 = 8$, $a_4 = 16$

each term is twice the term before it.

Explicit

$a_n = 2^n$, $n = 0, 1, 2, \ldots$

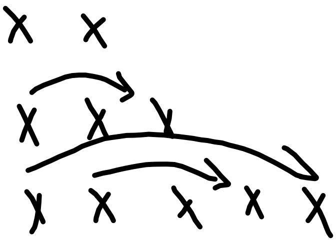
$a_{n+2} = a_{n+1} + a_n, \quad a_0 = 1, \; a_1 = 1$

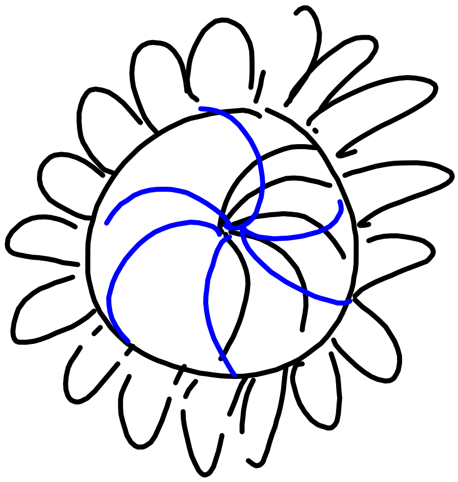
$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ 1, & 1, & 2, & 3, & 5, & 8, & 13, & 21, & \ldots \end{array}$$

Fibonacci sequence

1 pair

1 pair





$$y'' + 5y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Guess $y = e^{rt}$

$y' = re^{rt}$

$y'' = r^2e^{rt}$

$$r^2e^{rt} + 5re^{rt} + 4e^{rt} = 0$$

$$\underbrace{e^{rt}}_{\neq 0}\underbrace{(r^2 + 5r + 4)}_{= 0} = 0$$

$$(r+4)(r+1) = 0$$

$r = -1, -4$ $\quad y = e^{-t}$ and $y = e^{-4t}$ are both solutions

General solution is

$$\boxed{y = C_1e^{-t} + C_2e^{-4t}}$$

$$y' = -C_1e^{-t} - 4C_2e^{-4t}$$

$C_1 + C_2 = 1 \xleftarrow{y(0)=1}$

$-C_1 - 4C_2 = 0 \xleftarrow{y'(0)=0}$

$-3C_2 = 1$

$C_2 = -\frac{1}{3} \Rightarrow C_1 = \frac{4}{3}$

$$\boxed{y = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}}$$

$$a_{n+2} = a_{n+1} + a_n$$

$$\boxed{a_0 = 1, \quad a_1 = 1}$$

$$a_{n+2} - a_{n+1} - a_n = 0$$

Guess $a_n = \lambda^n$

λ is a constant

$a_{n+1} = \lambda^{n+1}$

$$\lambda^{n+2} - \lambda^{n+1} - \lambda^n = 0$$

$$\lambda^n(\underbrace{\lambda^2 - \lambda - 1}_{=0}) = 0$$

$a = 1$
$b = -1$
$c = -1$

Quad fmla

$$\lambda = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2}$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

$$a_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$1 = C_1 + C_2 \quad (a_0 = 1)$$

$$1 = C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right)$$

$$C_1 + C_2 = 1 \rightsquigarrow C_2 = 1 - C_1$$

$$1 = C_1 + C_2$$

$$(1+\sqrt{5})C_1 + (1-\sqrt{5})C_2 = 2$$

$$1 = C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right)$$

$$(1+\sqrt{5})C_1 + (1-\sqrt{5})(1-C_1) = 2$$

$$\cancel{C_1} + \sqrt{5}C_1 \cancel{+1} - \cancel{C_1} - \cancel{\sqrt{5}} + \sqrt{5}C_1 = 2$$

$$C_2 = 1 - C_1$$

$$= \frac{2\sqrt{5}}{2\sqrt{5}} - \frac{1+\sqrt{5}}{2\sqrt{5}}$$

$$= \frac{-1+\sqrt{5}}{2\sqrt{5}}$$

$$= -\frac{1-\sqrt{5}}{2\sqrt{5}}$$

$$2\sqrt{5}C_1 = 1+\sqrt{5}$$

$$C_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

$$a_n = \frac{1+\sqrt{5}}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1-\sqrt{5}}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\boxed{a_n = \frac{(1+\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}} - \frac{(1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}}$$