Math 253N ASSIGNMENT 19 **Due at the start of class Friday, May 26th**

Suppose that we wish to find the definite integral $\int_a^b f(x)\ dx$ for some function $f(x)$. The way that you are used to doing this is by applying the Fundamental Theorem of Calculus. It says that if $F(x)$ is any function such that $F'(x) = f(x)$ (that is, F is an *antiderivative* of f), then

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

Now if f has no antiderivative, we can't do this! This doesn't mean that the integral has no significance, just that it cannot be computed exactly by that manner.

So what do we do? One option is to apply a method like the trapezoidal rule or Simpson's rule. These are methods that take values of the function f at a few evenly spaced points in the interval and compute an approximate value of the integral from them. Here are the formulas for those methods:

Trapezoidal Rule:

$$\int_a^b f(x)\ dx \approx \left(\frac{b-a}{2n}\right)[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Simpson's Rule:

$$\int_a^b f(x)\ dx \approx \left(\frac{b-a}{3n}\right)[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Here n is the number of smaller intervals that the interval $[a, b]$ is divided into; in the case of Simpson's rule, n must be even. Additionally, $x_0 = a$ and $x_n = b$.

Give all answers as decimals rounded to three places past the decimal.

1. In this exercise we consider $\int_0^\pi \sin x\ dx$.

   (a) Find the exact value of the integral, using the Fundamental Theorem of Calculus.

   (b) Divide the interval $[0, \pi]$ into four equal subintervals. What are the values of x_0, x_1, x_2, x_3 and x_4?

   (c) Approximate the integral with the trapezoidal rule. **Show your setup, then just compute the answer from that.** What is the percent error in this approximation? (Recall that the percent error is given by the absolute value of the difference between the approximation and the actual value, divided by the absolute value of the actual value.

   (d) Approximate the integral with Simpson's rule. **Show your setup, then just compute the answer from that.** What is the percent error in this approximation?

You should know that

$$\int_a^b [f(x) + g(x)]\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$$

It would be reasonable to think that if we have a function $f(x)$ with a power series representation $f(x) = \sum_{k=0}^{\infty} c_k(x - x_0)^k$ that we would have

$$\int_a^b f(x)\ dx = \int_a^b \left[\sum_{k=0}^{\infty} c_k(x - x_0)^k\right] dx = \sum_{k=0}^{\infty} \left[\int_a^b c_k(x - x_0)^k\, dx\right]$$

This is indeed true *as long as the interval* $[a, b]$ *lies within the interval of convergence of the power series.*

2. Recall that $f(x) = \sin x$ has the power series representation about zero

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

   (a) Find the interval of convergence for this series.

   (b) Find an infinite numerical series whose sum represents the integral $\int_0^{\pi} \sin x\ dx$ by integrating each term of the series for sine. Write the series in summation notation.

   (c) Approximate the integral $\int_0^{\pi} \sin x\ dx$ by adding the first three terms of the series you obtained in (b). (Note, of course, that you could get a better approximation if you used more terms.)

   (d) Find the percent error in your approximation.

3. In statistics, we often need to find the value of an integral that is essentially of the form $\int_0^z e^{-x^2}\ dx$. Since the function $f(x) = e^{-x^2}$ has no antiderivative $F(x)$, it is impossible to find the exact value of the integral.

   (a) Substitute $-x^2$ for x in the power series representation of e^x to find the power series for e^{-x^2}. Simplify your series and give it in summation form.

   (b) Your series from (a) converges for all x. Use it to find a numerical series representing $\int_0^1 e^{-x^2}\ dx$. Give your answer in summation form.

   (c) Approximate $\int_0^1 e^{-x^2}\ dx$ by adding the first three terms of your series for the integral.